DOI: 10.24996/ijs.2020.61.2.20

ISSN: 0067-2904

Resolution for the two-rowed weyl module inThe cases of (6,5) / (1,0) and (6,5) / (2,0)

Annam Ali Abbas*, Haytham R. Hassan
Department of Mathematic, College of Science, Mustansiriyah University, Baghdad, Iraq

Received: 18/6/ 2019 Accepted: 28/ 8/2019

Abstract
The main purpose of this paper is to study the application of weyl module and resolution in the case skew- shapes (6, 5) / (1, 0) and (6, 5) / (2, 0) by using contracting homotopy and the place polarization.

Key words: weyl module, skew- shape, graded contracting, homotopy.

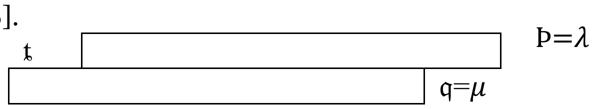
تحلل مقاس وايل لصفين في حالة (6،5)/(1،0) و (6،5)/(2،0)

انعام علي عباس*, هيثم رزوقي حسن
قسم الرياضيات، كلية العلوم، الجامعة المستنصرية، بغداد ،العراق

الخلاصة
الغرض الرئسي من هذا البحث هو دراسة تطبيق وحدة ويل والدقة في انحراف الحالة (6 ، 5) / (1 ، 0) و (6 ، 5) / (2 ، 0) باستخدام هوموتوبي المتعاقد مع وضع الاستقطاب .

1: Introduction

Let Ʀ be a commutative ring with identity (1) and Ƒ be a free Ʀ-module. In this work we give resolution of weyl modules in the cases of skew partitions (6,5)/(1,0) and (6,5)/(2,0) which are used by D. A . Buchsbaum in the case partition (2,2), Haytham R. Hassan in the case (3,3) and Niran Sabah Jasim in the case (8,7) [1, 2, 3].

$\lambda, \mu =$ 

Let Z_{21}be the free generator of a divided power algebra
ĐF (Z_{21}) is one generator of the divided element power Z_{21} of degree k of free generator. Z_{21} is $Đ_{Þ+\kappa}F \otimes Đ_{q-\kappa}F$ by place polarization of degree κ from place 1 to place 2.
"The graded algebra with identity" Ʀ = ĐF (Z_{21}) is the graded module that acts of the graded module $\mathcal{N} = \sum Đ_{Þ+\kappa}F \otimes Đ_{q-\kappa}F$ i.e. $\boldsymbol{\mathcal{N}}$ is a (graded) left Ʀ- module;

$$\mathcal{N}\bullet: 0 \longrightarrow \mathcal{N}_{q-ŧ} \xrightarrow{\partial_S} \longrightarrow \mathcal{N}_{ŧ} \xrightarrow{\partial_S} \mathcal{N}_1 \xrightarrow{\partial_S} \mathcal{N}_0 \quad (1)$$

is the weyl module By definition $\boldsymbol{\mathcal{N}}\bullet$ is the resolution of $\kappa_{\lambda/\mu}(F)$; *where* $\kappa_{\lambda/\mu}(F)$

$$\sum_{\kappa_i \geq 0} Z_{21}^{(\kappa_1+ŧ)}\chi \; Z_{21}^{(\kappa_2)}\chi \ldots\chi \; Z_{21}^{(\kappa_1)}\chi \; Đ_{Þ+(ŧ+|\kappa|)}F \otimes Đ_{q-(ŧ+|\kappa|)}F \xrightarrow{d_1} \quad (2)$$

*Email: anaamaban@gmail.com

$$\sum_{\kappa_i \geq 0} Z_{21\ \chi}^{(t+\kappa_1)}\ Z_{21\ \chi}^{(\kappa_2)} \ldots_\chi\ Z_{21\ \chi}^{(\kappa_l-1)}\ D_{p+(t+|\kappa|)}F \otimes D_{q-(t+|\kappa|)} \xrightarrow{d_{l-1}}$$

$$\ldots \longrightarrow \sum_{\kappa_i \geq 0} Z_{21\ \chi}^{(t+\kappa)}\ D_{p+(t+|\kappa|)}F \otimes D_{q-(t+\kappa)}F \xrightarrow{d_0} D_pF \otimes D_q$$

Where $|\kappa| = \sum \kappa i$ and d_l is the "boundary operator" $\boldsymbol{\partial}_\chi$

definition of is $\mathcal{S}_i$ *see* [1] and the

$$\mathcal{S}_0 : D_pF \otimes D_qF \longrightarrow \sum_{\kappa_i \geq 0} Z_{21\ \chi}^{(t+\kappa)}\ D_{p+(t+\kappa)}F \otimes D_{q-(t+\kappa)}F$$

$$\begin{pmatrix} w \\ w' \end{pmatrix}\!\!\begin{vmatrix} 1^{(p)}2^{(\kappa)} \\ 2^{(q-\kappa)} \end{vmatrix} \longrightarrow \begin{cases} Z_{21\ \chi}^{(\kappa)} \begin{pmatrix} w \\ w' \end{pmatrix}\!\!\begin{vmatrix} 1^{(5+\kappa)} \\ 2^{(5-\kappa)} \end{vmatrix} & ; \quad \text{where} \quad \kappa \leq t \\ o & ; \quad \text{where} \quad \kappa > t \end{cases} \longrightarrow \tag{3}$$

And for higher dimensions as :

$$\mathcal{S}_{l-1} : \sum_{\kappa_i \geq 0} Z_{21\ \chi}^{(t+\kappa_1)}\ Z_{21\ \chi}^{(\kappa_2)} \ldots Z_{21\ \chi}^{(\kappa_l-1)}\ D_{p+(t+|\kappa|)}F \otimes D_{q-(t+|\kappa|)}F \longrightarrow$$

$$Z_{21\ \chi}^{(t+\kappa_1)}\ Z_{21\ \chi}^{(\kappa_2)} \ldots Z_{21\ \chi}^{(\kappa_l-1)}\ Z_{21\ \chi}^{(\kappa_l)}\ D_{p+(t+|\kappa|)}F \otimes D_{q-(t+|\kappa|)}F$$

$$Z_{21\ \chi}^{(t+\kappa_1)}\ Z_{21\ \chi}^{(\kappa_2)} \ldots Z_{21\ \chi}^{(\kappa_l-1)} \begin{pmatrix} w \\ w' \end{pmatrix}\!\!\begin{vmatrix} 1^{(p+t+|\kappa|} \\ 2^{(p-(t+|\kappa|+m)} \end{vmatrix} \longrightarrow \tag{4}$$

$$\begin{pmatrix} Z_{21\ \chi}^{(t+\kappa_1)}\ Z_{21\ \chi}^{(\kappa_2)} \ldots Z_{21\ \chi}^{(\kappa_l-1)}\ Z_{21\ \chi}^{(m)} \begin{pmatrix} w \\ w' \end{pmatrix}\!\!\begin{vmatrix} 1^{(p+t+|\kappa|+m)} \\ 2^{q-(t+|\kappa|+m)} \end{vmatrix} & ; where\ m = 0 \\ o & ; where\ m > 0 \end{pmatrix}$$

As in 1, 4, 5, and 6 that follows, we write the moudules of the resolution as $\boldsymbol{\mathcal{N}}$i , in which i=0,1, ….q-t ; with $\mathcal{N}_0 = D_pF \otimes D_q$ *and*

$$\mathcal{N}_i = Z_{21\ \chi}^{(\kappa_l-1)} Z_{21\ \chi}^{(\kappa 2)} \ldots Z_{21\ \chi}^{(\kappa_i)} D_{p+t+|\kappa|}F \otimes D_{q-(t+|\kappa|)}F \qquad i \geq 1$$

2. Definitions

(1) Let R be a commutative ring. A graded R-algebra is a graded R-module M=$\bigoplus_{i\geq 0} M_i$ together with a "multiplication" homogenous $m: M \otimes M \longrightarrow M$ and a unit η:R⟶M .

(2) The weyl module shape $\propto$ is the image of the map $d_\propto(F)$, and it is denoted by $l_\propto(F)$.

(3) A contracting homotopy:- $\{\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots\ldots\}$ is a contracting homotopy if $\mathcal{S}_{i-1}\,\partial_\chi + \partial_\chi\,\mathcal{S}_i = \text{id}$.

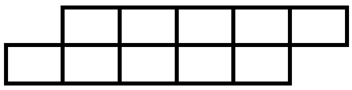
(4) The divided power algebra $\mathrm{D\,F} = \sum_{\kappa_i \geq 0} \mathrm{D_i\ F}$ can be defned as the graded commutative a lgebra generated by element χ^((i)) in degree where χ∈ F and I is a non negative integer.

3- Application of weyl modules and resolution in the case skew- shape (6, 5)/(1, 0).

In this section we study the application of weyl module and resolution in the case (6, 5)/(1, 0) and find the terms of resolution of the skew-shape in the case (6, 5)/(1, 0) as well as the proof of exactness.

"Let skew- Shape (6, 5)/ (1, 0), in this case we have the following terms of characteristic free resolution"

The characteristic – free resolutions are:



$\mathcal{N}_0 : = D_5F \otimes D_4F$

$\mathcal{N}_1 : = Z_{21\chi}^{(1)} D_6F \otimes D_3F \oplus Z_{21\chi}^{(2)} D_7F \otimes D_2F \oplus Z_{21\chi}^{(3)} D_8F \otimes D_1 F \oplus$

$\mathcal{N}_2 : = Z_{21\chi}^{(2)} Z_{21\chi} D_9F \otimes D_1F \oplus Z_{21\chi}^{(3)} Z_{21\chi} D_{10}F \otimes D_0 F$

$\mathcal{N}_3 := Z_{21\chi}^{(2)} Z_{21\chi} Z_{21\chi} D_{10}F \otimes D_0F$

Then we have

$\mathcal{S}_0 : \mathcal{N}_0 \longrightarrow \mathcal{N}_1$

$$\mathcal{S}_0 \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix}1^{(5)}2^{(\kappa)} \\ 2^{(5-\kappa)}\end{matrix}\right) = \begin{cases} Z_{21\,\chi}^{(\kappa)} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(5+\kappa)}\\2^{(5-\kappa)}\end{matrix}\right) & if \quad \kappa>2 \\ o & if \quad \kappa \le 2\end{cases}$$

and $\mathcal{S}_1 : \mathcal{N}_1 \longrightarrow \mathcal{N}_2$ where

$$\mathcal{S}_1 \left(Z_{21\,\chi}^{(\kappa+1)} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+\kappa)}\\2^{(1-\kappa-\mathrm{m})}\end{matrix}\right)\right) = \begin{cases} Z_{21\,\chi}^{(\kappa+1)} Z_{21\,\chi}^{(\mathrm{m})} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+\kappa+\mathrm{m})}\\2^{(1-\kappa-\mathrm{m})}\end{matrix}\right) & if\ \mathrm{m}=1,2 \\ o & if\ \mathrm{m}=0\end{cases}$$

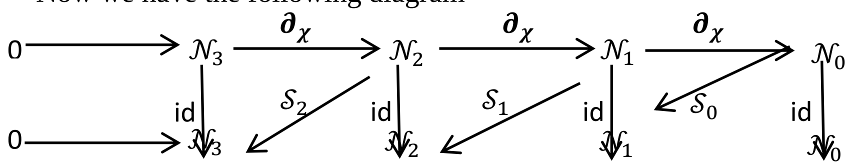
and $\mathcal{S}_2 : \mathcal{N}_2 \longrightarrow \mathcal{N}_3$ where

$$\mathcal{S}_2 \left(Z_{21\,\chi}^{(\kappa_1+1)} \quad Z_{21\,\chi}^{(\kappa_2)} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+|\kappa|} \quad 2^{(\mathrm{m})}\\2^{(1-|\kappa|-\mathrm{m})}\end{matrix}\right)\right)$$

$$= \begin{cases} Z_{21\,\chi}^{(\kappa+1)} Z_{21\,\chi}^{(\kappa_2)} Z_{21\,\chi}^{(\mathrm{m})} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+|\kappa|+\mathrm{m})}\\2^{(1-|\kappa|-\mathrm{m})}\end{matrix}\right) & if\ \mathrm{m}=1 \\ \mathrm{o} & if\ \mathrm{m}=0\end{cases}$$

Since $|\kappa| = \kappa_1 + \kappa_2$

"Now we have the following diagram"



(5)

in diagram (5) we can see that

$$\mathcal{S}_0\, \partial_\chi \left(Z_{21\,\chi}^{(\kappa+1)} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+\kappa)} \quad 2^{(\mathrm{m})}\\2^{(1-\kappa-\mathrm{m})}\end{matrix}\right)\right)$$

$$= \mathcal{S}_0 \left(Z_{21\,\chi}^{(\kappa+1)} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+\kappa)} \quad 2^{(\mathrm{m})}\\2^{(1-\kappa-\mathrm{m})}\end{matrix}\right)\right)$$

$$= \mathcal{S}_0 \left(\left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(5)} \quad 2^{(\kappa+1)} \quad 2^{(\mathrm{m})}\\2^{(1-\kappa-\mathrm{m})}\end{matrix}\right)\right)$$

$$= \mathcal{S}_0 \left(\binom{\kappa+1+\mathrm{m}}{\mathrm{m}} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(5)}2^{(\kappa+1+\mathrm{m})}\\2^{(1-\kappa-\mathrm{m})}\end{matrix}\right)\right)$$

$$= \binom{k+1+\mathrm{m}}{\mathrm{m}} \mathcal{S}_0 \left(\left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(5)} \quad 2^{(\kappa+1+\mathrm{m})}\\2^{(1-\kappa-\mathrm{m})}\end{matrix}\right)\right)$$

$$= \binom{\kappa+1+\mathrm{m}}{\mathrm{m}} Z_{21\,\chi}^{(\kappa+1+\mathrm{m})} \left(\left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+\kappa+\mathrm{m})}\\2^{(1-\kappa-\mathrm{m}}\end{matrix}\right)\right)$$

So that

$$\partial_\chi\, \mathcal{S}_1 \left(Z_{21\,\chi}^{(\kappa+1)} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+\kappa)} \quad 2^{(\mathrm{m})}\\2^{(1-\kappa-\mathrm{m})}\end{matrix}\right)\right)$$

$$= \partial_\chi \left(Z_{21\,\chi}^{(\kappa+1)} \quad Z_{21\,\chi}^{(\mathrm{m})} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+\kappa+\mathrm{m})}\\2^{(1-\kappa-\mathrm{m})}\end{matrix}\right)\right)$$

$$= -\binom{\kappa+1+\mathrm{m}}{\mathrm{m}} Z_{21\,\chi}^{(\kappa+1+\mathrm{m})} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+\kappa+\mathrm{m})}\\2^{(1-\kappa-\mathrm{m})}\end{matrix}\right) + Z_{21\,\chi}^{(\kappa+1)} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+\kappa)} \quad 2^{(\mathrm{m})}\\2^{(1-\kappa-\mathrm{m})}\end{matrix}\right)$$

It is clear that: $S_0\, \partial_\chi + \partial_\chi\, S_1 = id$.

Now

$$S_1\, \partial_\chi \left(Z_{21\,\chi}^{(\kappa_1+1)} \quad Z_{21\,\chi}^{(\kappa_2)} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+\kappa)} \quad 2^{(\mathrm{m})}\\2^{(1-\kappa-\mathrm{m})}\end{matrix}\right)\right) \text{ where } |\kappa| = \kappa_1 + \kappa_2$$

$$= \mathcal{S}_1 \left(-\binom{|\kappa|}{\kappa_2} Z_{21\,\chi}^{|\kappa|+1} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+\kappa)} \quad 2^{(\mathrm{m})}\\2^{(1-\kappa-\mathrm{m})}\end{matrix}\right) + Z_{21\,\chi}^{(\kappa+1)} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+\kappa)} \quad 2^{(\kappa_2)} \quad 2^{(\mathrm{m})}\\2^{(1-\kappa-\mathrm{m})}\end{matrix}\right)\right)$$

$$\mathcal{S}_1 \left(-\binom{|\kappa|+1}{\kappa_2} Z_{21\,\chi}^{(|\kappa|+1)} \left(\begin{matrix} w\\ w'\end{matrix}\middle|\begin{matrix}1^{(6+|\kappa|)} \quad 2^{(\mathrm{m})}\\2^{(1-|\kappa|-\mathrm{m})}\end{matrix}\right)\right)$$

$+\left(\binom{\kappa_2+m}{m} Z_{21\ \chi}^{(\kappa+1)} \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(6+\kappa_1)} & 2^{(\kappa_2+m)} \\ 2^{(1-|\kappa|-m)} & \end{matrix}\right)\right)$

$= -\binom{|\kappa|+m}{\kappa_m} Z_{21\ \chi}^{(|\kappa|+1)} Z_{21\ \chi}^{(m)} \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(6+|\kappa|+m} \\ 2^{(1-|\kappa|-m} \end{matrix}\right)$

$+\binom{\kappa_2+m}{m} Z_{21\ \chi}^{(\kappa_1+1)} Z_{21\ \chi}^{(\kappa_2+m)} \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(6+|\kappa|+m} \\ 2^{(1-|\kappa|-m} \end{matrix}\right)$

So and

$\partial_\chi\ \mathcal{S}_2\left(Z_{21\ \chi}^{(\kappa_1+1)}\ \ Z_{21\ \chi}^{(\kappa_2)} \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(6+|\kappa|)} & 2^{(m)} \\ 2^{(1-|\kappa|-m)} & \end{matrix}\right)\right)$

$\partial_\chi\ \left(Z_{21\ \chi}^{(\kappa_1+1)}\ \ Z_{21\ \chi}^{(\kappa_2)}\ Z_{21\ \chi}^{(m)} \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(6+|\kappa|+m)} \\ 2^{(1-|\kappa|-m} \end{matrix}\right)\right)$

$=\binom{|\kappa|+1}{\kappa_2} Z_{21\ \chi}^{(|\kappa|+1)} Z_{21\ \chi}^{(m)} \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(6+|\kappa|+m)} \\ 2^{(1-|\kappa|-m} \end{matrix}\right)$

$= \partial_\chi\ \left(Z_{21\ \chi}^{(\kappa_1+1)}\ \ Z_{21\ \chi}^{(\kappa_2)}\ Z_{21\ \chi}^{(m)} \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(6+|\kappa|+m)} \\ 2^{(1-|\kappa|-m} \end{matrix}\right)\right)$

$= \binom{|\kappa|+1}{\kappa_2} Z_{21\ \chi}^{(|\kappa|+1)}\ Z_{21\ \chi}^{(m)} \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(6+|\kappa|+m)} \\ 2^{(1-|\kappa|-m} \end{matrix}\right)$

$- \binom{\kappa_2+1}{\kappa_2} Z_{21\ \chi}^{(\kappa_1+1)}\ Z_{21\ \chi}^{(\kappa_2+m)} \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(6+|\kappa|+m)} \\ 2^{(1-|\kappa|-m} \end{matrix}\right)$

$+ Z_{21\ \chi}^{(\kappa_1+1)}\ Z_{21\ \chi}^{(\kappa_2)} \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(6+|\kappa|)} & 2^{(m)} \\ 2^{(1-|\kappa|-m)} & \end{matrix}\right)$

S_0 we have" $S_1\ \partial_\chi + \partial_\chi\ \mathcal{S}_2$= id "

Hence { $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ }is a contracting homotopy ; which implies that the complex

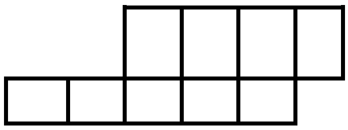
$\mathbf{0} \longrightarrow \mathcal{N}_3 \longrightarrow \mathcal{N}_2 \longrightarrow \mathcal{N}_1 \longrightarrow \mathcal{N}_0$ *is exact.*

4- Application of weyl modules and resolution in the case skew- shape (6, 5)/ (2, 0).

In this section we study the application of weyl module and resolution in the case (6, 5)/(2, 0) and find the terms of resolution of skew-shape in the case (6, 5)/(2, 0) as well as the proof of exactness.

Let skew –shape (6, 5) / (2, 0) = (4,5)

In this case we have the following terms of characteristic free resolution



$\mathcal{N}_0 = Đ_4F \otimes Đ_5F$

$\mathcal{N}_1 = Z_{21\ \chi}^{(3)}\ Đ_7F \otimes Đ_2F \oplus Z_{21\ \chi}^{(4)}\ Đ_8F \otimes Đ_1F \oplus Z_{21\ \chi}^{(5)}\ Đ_9F \otimes Đ_0F$

$\mathcal{N}_2 = Z_{21\ \chi}^{(3)}\ Z_{21\ \chi}\ Đ_8F \otimes Đ_1F \oplus Z_{21\ \chi}^{(4)}\ Z_{21\ \chi}\ Đ_9F \otimes Đ_0F \oplus Z_{21\ \chi}^{(2)}\ Z_{21}^{(2)}\ Đ_9F \otimes Đ_0\ F$

$\mathcal{N}_3 = Z_{21\ \chi}^{(3)}\ Z_{21\ \chi}\ Z_{21\ \chi}\ Đ_9F \otimes Đ_0\ F$

Then we have

□0: □0 ⟶ □1

$□_0 \left(\begin{matrix} □ \\ □' \end{matrix}\middle| \begin{matrix} 1^{(4)} 2^{□} \\ 2^{(5-□)} \end{matrix}\right) = \begin{cases} □_{21\ \chi}^{(□)} \left(\begin{matrix} □ \\ □' \end{matrix}\middle| \begin{matrix} 1^{(4+□)} \\ 2^{(5-□)} \end{matrix}\right) & □□□ = 3,4,5 \\ □ & □□□ \le 2 \end{cases}$

And

□1: □1 ⟶ □2

$□_1 \left(□_{21\ \ □}^{(□+2)} \left(\begin{matrix} □ \\ □' \end{matrix}\middle| \begin{matrix} 1^{(6+□)} & 2^{□} \\ 2^{(2-□-□)} & \end{matrix}\right)\right) = \begin{cases} □_{21}^{(□+2)}\ □_{21}^{□} \left(\begin{matrix} □ \\ □' \end{matrix}\middle| \begin{matrix} 1^{(6+□+□)} \\ 2^{(2-□-□)} \end{matrix}\right) & \\ 0 & □□□ > 0 \end{cases}$

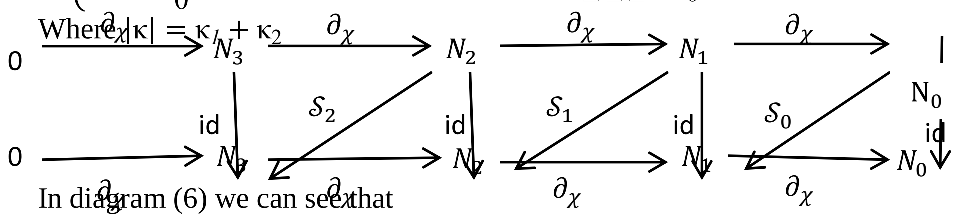
And

S_2: □2 ⟶ □3 where

$□_2 \left(□_{21\ \chi}^{(\kappa_1+2)}\ □_{21\ \chi}^{(\kappa_2)}\ □_{21\ \chi}^{(□)} \left(\begin{matrix} □ \\ □' \end{matrix}\middle| \begin{matrix} 1^{(6+|\kappa|)} & 2^{□} \\ 2^{(2-|\kappa|-□)} & \end{matrix}\right)\right)$

$$= \begin{cases} \square_{21}^{(\kappa_1+2)}{}_{\chi}\, \square_{21}^{(\square)}{}_{\chi}\, \square_{21}^{(\square)}{}_{\chi} \left(\left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+|\kappa|+\square)} \\ 2^{(2-|\kappa|-\square)}\end{matrix}\right)\right) & \square\square\square = 1 \\ 0 & \square\square\square = 0 \end{cases}$$

Where $|\kappa| = \kappa_1 + \kappa_2$



(6)

In diagram (6) we can see that

$\square_0\, \partial_\chi \left(\square_{21}^{(\kappa+2)}{}_{\chi} \left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+\kappa)} & 2^{\square} \\ 2^{(2-\kappa-\square)} & \end{matrix}\right)\right)$

$= \square_0 \left(\left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(4)} & 2^{(\kappa+2)} & 2^{\square} \\ 2^{(2-\kappa-\square)} & & \end{matrix}\right)\right)$

$= \square_0 \left(\binom{\kappa+2+\kappa}{\square} \left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(4)} & 2^{(\kappa+2+\square)} \\ 2^{(2-\kappa-\square)} & \end{matrix}\right)\right)$

$= \binom{\kappa+2+\square}{\square} \square_0 \left(\left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(4)} & 2^{(\square+2+\square)} \\ 2^{(2-\square-\square)} & \end{matrix}\right)\right)$

$= \binom{\kappa+2+\square}{\square} \square_{21}^{(\kappa+2+\square)}{}_{\chi} \left(\left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+\kappa+\square)} \\ 2^{(2-\kappa-\square)}\end{matrix}\right)\right)$

In diagram (6) we can see that

$\partial_\chi\, \square_1 \left(\square_{21}^{(\kappa+2)}{}_{\chi} \left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+\kappa)} & 2^{(\square)} \\ 2^{(2-\kappa-\square)} & \end{matrix}\right)\right)$

$\partial_\chi \left(\square_{21}^{(\kappa+2)}{}_{\chi}\ \square_{21}^{(\square)}{}_{\chi} \left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+\kappa+\square)} \\ 2^{(2-\kappa-\square)}\end{matrix}\right)\right)$

$= -\binom{\square+2+\square}{\square} \square_{21}^{(\square+2+\square)}{}_{\chi} \left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+\square+\square)} \\ 2^{(2-\square-\square)}\end{matrix}\right) + \square_{21}^{(\square+2)}{}_{\chi} \left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+\square)} & 2^{\square} \\ 2^{(2-\square-\square)} & \end{matrix}\right)$

From the preposition in diagram (6) we can see that

$\square_0\, \partial_\chi + \partial_\chi\, \square_1 = \text{id}$

Then

$$\binom{\kappa+2+\square}{\square} \square_{21}^{(\kappa+2+\square)} \left(\left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+\kappa+\square)} \\ 2^{(2-\kappa-\square)}\end{matrix}\right)\right) + \square_{21}^{(\kappa+2)}{}_{\chi} \left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+\kappa)} & 2^{\square} \\ 2^{(2-\kappa-\square)} & \end{matrix}\right)$$

$\Rightarrow \square_{21}^{(\kappa+2)}{}_{\chi} \left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+\kappa)} & 2^{\square} \\ 2^{(2-\kappa-\square)} & \end{matrix}\right)$

Now

$\square_1\, \partial_\chi \left(\square_{21}^{(\kappa_1+2)}{}_{\chi}\ \square_{21}^{(\kappa_2)}{}_{\chi} \left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+\kappa)} & 2^{(\square)} \\ 2^{(2-\kappa-\square)} & \end{matrix}\right)\right)$ where $|\kappa| = \kappa_1 + \kappa_2$

$= \square_1 \left(-\binom{|\kappa|+2}{\kappa_2} \square_{21}^{(|\kappa|+2)}{}_{\chi} \left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+\kappa)} & 2^{\square} \\ 2^{(2-\kappa-\square)} & \end{matrix}\right)\right.$

$\left. + \left(\square_{21}^{(\kappa+2)}{}_{\chi} \left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+\kappa)} & 2^{(\kappa_2)} & 2^{(\square)} \\ 2^{(2-\kappa-\square)} & & \end{matrix}\right)\right)\right)$

$= \square_1 \left(-\binom{|\kappa|+2}{\kappa_2} \square_{21}^{(|\kappa|+2)}{}_{\chi} \left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+|\kappa|)} & 2^{\square} \\ 2^{(2-|\kappa|-\square)} & \end{matrix}\right)\right.$

$\left. + \binom{\kappa_2+\square}{\square} \square_{21}^{(\kappa+2)}{}_{\chi} \left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+\kappa_1)} & 2^{(\kappa_2+\square)} \\ 2^{(2-|\kappa|-\square)} & \end{matrix}\right)\right)$

$= -\binom{|\kappa|+\square}{\kappa_2} \square_{21}^{(|\kappa|+2)}{}_{\chi}\, \square_{21}^{(\square)}{}_{\chi} \left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+|\kappa|+\square} \\ 2^{(2-|\kappa|-\square)}\end{matrix}\right) + \binom{\kappa_2+\square}{\square} \square_{21}^{(\kappa_1+3)}{}_{\chi}\, \square_{21}^{(\kappa_2+\square)}{}_{\chi}$

$\left(\begin{matrix}\square \\ \square'\end{matrix}\middle| \begin{matrix}1^{(6+\kappa+\square)} \\ 2^{(2-\kappa-\square)}\end{matrix}\right)$

and

$$\partial_\chi \square_2 \left(\square_{21}^{(\kappa_1+2)} \chi\, \square_{21}^{(\kappa_2)} \chi \left(\begin{matrix} \square \\ \square' \end{matrix} \middle| \begin{matrix} 1^{(6+|\kappa|)} & 2^{\square} \\ 2^{(2-|\kappa|-\square)} & \end{matrix} \right) \right)$$

$$= \partial_\chi \left(\square_{21}^{(\kappa_1+2)} \chi\, \square_{21}^{(\kappa_2)} \chi \;\; \square_{21}^{(\square)} \chi \left(\begin{matrix} \square \\ \square' \end{matrix} \middle| \begin{matrix} 1^{(6+|\kappa|)} & 2^{\square} \\ 2^{(2-|\kappa|-\square)} & \end{matrix} \right) \right)$$

$$= -\binom{\kappa_2 + \square}{\square} \square_{21}^{(\kappa_1+2)} \chi\, \square_{21}^{(\kappa_2+\square)} \chi \left(\begin{matrix} \square \\ \square' \end{matrix} \middle| \begin{matrix} 1^{(6+|\kappa|+\square)} \\ 2^{(2-|\kappa|-\square)} \end{matrix} \right)$$

$$+ \square_{21}^{(\kappa+2)} \chi\, \square_{21}^{(\kappa_2)} \chi \left(\begin{matrix} \square \\ \square' \end{matrix} \middle| \begin{matrix} 1^{(6+|\kappa|)} & 2^{\square} \\ 2^{(2-|\kappa|-\square)} & \end{matrix} \right)$$

In diagram (6) we can see that $\square_1 \square_{\square} + \square_{\square} \square_2 = \square\square$ hence

$$- \binom{|\kappa| + \square}{\kappa_2} \square_{21}^{(|\kappa|+2)} \chi\, \square_{21}^{(\square)} \chi \left(\begin{matrix} \square \\ \square' \end{matrix} \middle| \begin{matrix} 1^{(6+|\kappa|+\square)} \\ 2^{(2-|\kappa|-\square)} \end{matrix} \right)$$

$$+ \binom{\kappa_2 + \square}{\square} \square_{21}^{(\kappa_1+2)} \chi\, \square_{21}^{(\kappa_2+\square)} \chi \left(\begin{matrix} \square \\ \square' \end{matrix} \middle| \begin{matrix} 1^{(6+|\kappa|+\square)} \\ 2^{(2-|\kappa|-\square)} \end{matrix} \right)$$

$$- \binom{\kappa_2 + \square}{\square} \square_{21}^{(\kappa_1+2)} \chi\, \square_{21}^{(\kappa_2)} \chi \left(\begin{matrix} \square \\ \square' \end{matrix} \middle| \begin{matrix} 1^{(6+|\kappa|)} & 2^{(\square)} \\ 2^{(2-|\kappa|-\square)} & \end{matrix} \right)$$

$$+ \binom{|\kappa_1| + 2}{\kappa_2} \square_{21}^{(|\kappa|+2)} \chi\, \square_{21}^{(\square)} \chi \left(\begin{matrix} \square \\ \square' \end{matrix} \middle| \begin{matrix} 1^{(6+|\kappa|+\square)} \\ 2^{(2-|\kappa|-\square)} \end{matrix} \right)$$

$$= \square_{21}^{(\kappa_1+2)} \chi\, \square_{21}^{(\kappa_2)} \chi \left(\begin{matrix} \square \\ \square' \end{matrix} \middle| \begin{matrix} 1^{(6+|\kappa|)} \\ 2^{(2-|\kappa|-\square)} \end{matrix} \right)$$

Now we get $\square_1 \; \partial_\chi + \partial_\chi \square_2 \; = \text{id}$

Hence $\{\square_0, \square_1, \square_2\}$ is a contracting homotopy

$$0 \longrightarrow \square_3 \longrightarrow \square_2 \longrightarrow \square_1 \longrightarrow \square_0 \qquad \text{is exact}$$

Acknowledgments

The authors thank the Mustansiriyah University / College of Science / Department of Mathematics for supporting this work.

References

1. David A. and Gian C. **1993**. Projective Resolution of weyl Modules, *Natl. Acad. Sci*. USA, **90**: 2448-2450
2. Haytham RH. **2006**. Applicatin of the characteristic-free resolution of Weyl Module to the Lascoux resolution in the case (3, 3, 3). Ph. D. Italia (Tor-vergate-Roma).
3. Haytham RH, Niran SJ. **2018** Application of Weyl module in the Case of Two Rows. *J. Phys. Conf. Ser*. 1003 (012051) (1742-6588, 742-6596): 1-15, doi: 10.1088/1742-6596/1003/1/012051.
4. David A. and Brian D. **2003.** Homotopies for Resolution of Skew-Hook Shapes, *Adv. In Applied Math*. **30**: 26-43.
5. David. A. and Rota, G.C. **2001** Approaches to resolution of weyl modules. *Adv. In applied math*. **27**: 82-191.
6. David A. **2004.** A Characteristic Free Example of Lascaux Resolution, and Letter place Mathods for Intertwining Numbers, *European Journal of Gombinatorics*, **25**: 169-1179.