DOI: 10.24996/ijs.2020.61.3.21

ISSN: 0067-2904

Duo Gamma Modules and Full Stability

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Received: 18/6/ 2019 Accepted: 21/ 9/2019

Abstract

In this work we study gamma modules which are implying full stability or implying by full stability. A gamma module M is fully stable if $\theta(N) \subseteq N$ for each gamma submodule N of M and each $R_{\Gamma}-$ homomorphism θ of N into M. Many properties and characterizations of these classes of gamma modules are considered. We extend some results from the module to the gamma module theories.

Keywords: Gamma modules, fully stable gamma modules, duo gamma modules, uniserial gamma module, Γ −Hopfian and Γ −coHopfian gamma modules.

مقاسات كاما الاثنائية والاستقرارية التامة

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الخلاصه

في هذا العمل ندرس مقاسات من نمط كاما والتي تكون تامة الاستقرار ونقول عن M انها تامة الاستقرار إذا كان $\theta(N) \subseteq N$ لكل مقاس جزئي N الى M ولكل تشاكل موديولي θ من N الى M. العديد من الصفات والخصائص ممكن اعتبارها من مقاسات كاما. نحن وسعنا بعض النتائج من المقاسات الى نظرية المقاسات من نمط كاما.

1- Introduction:

In 1964, Nobusawa introduced the idea of gamma rings as a generalization of the idea of rings [1]. In 1966, Barnes summed up this idea and obtained entirety fundamental properties of gamma rings [2].

Let R and Γ be two additive abelian groups. R is called a Γ −ring if there is a mapping $R \times \Gamma \times R \rightarrow R$, $(r, \alpha, \bar{r}) \rightarrow r\alpha\bar{r}$ such that the followings hold:

(i) $(r_1 + r_2)\, \alpha\, r_3 = r_1\alpha\, r_3 + r_2 \propto r_3$,

(ii) $r_1(\alpha + \beta)r_2 = r_1\, \alpha\, r_2 + r_1\beta\, r_2$,

(iii) $r_1\alpha(r_2 + r_3) = r_1\alpha\, r_2 + r_1\alpha\, r_3$ and

(iv) $(r_1\alpha\, r_2)\beta\, r_3 = r_1\alpha(r_2\, \beta\, r_3)$, for all $r_1, r_2, r_3 \in R, \alpha, \beta \in \Gamma$.

In 2010, Ameri and Sadeqhi extended the idea of modules to gamma modules [3].

Let R be a Γ −ring. An additive abelian group M is called a left $R_{\Gamma} - m$odule, if there exists a mapping : $R \times \Gamma \times M \rightarrow M$, $r\alpha m$ denote the image of (r, α, m) such that the followings hold:

(i) $r\alpha(m_1 + m_2) = r\alpha\, m_1 + r\alpha\, m_2$,

(ii) $(r_1 + r_2)\alpha\, m = r_1\alpha\, m + r_2\alpha\, m$,

(iii) $r\,(\alpha_1 + \alpha_2)m = r\alpha_1\, m + r\alpha_2\, m$ and

(iv) $r_1\alpha_1(r_2\alpha_2 m) = (r_1\alpha_1\, r_2)\, \alpha_2\, m$, for all $m, m_1, m_2 \in M$, $\alpha, \alpha_1, \alpha_2 \in \Gamma$ and $r, r_1, r_2, \in R$.

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An R_Γ −module M is called unitary if there is $1 \in R$, $\alpha_0 \in \Gamma$ such that $1\alpha_0 m = m$ for all m in M. A previous article provided more details of gamma modules [3].

In 1973, Faith introduced the definition of duo modules. Let M be an R −module, a submodule N of M is said to be fully invariant if $\theta(N) \subseteq N$ for each R −endomorphism of M [4]. In the case that each submodule of M is fully invariant, then M is called duo.

In 1991, Abbas studied the relationship between the fully stable modules and the duo modules; an R −module M is fully stable if for each submodule N of M, $\theta(N) \subseteq N$ for each R −homomorphism θ from N into M [5].

In this paper, we consider the duo property in the category of gamma modules. A left R_Γ −module M is called duo if $\theta(N) \subseteq N$ for each R_Γ −submodule N of M and R_Γ −endomorphism of M. For an arbitrary fixed α in Γ, a subset A of R and a subset L of M, we define:
$l_R^\alpha(L) = \{r \in R | r\alpha L = o\}$ and $r_M^\alpha(A) = \{m \in M | A\alpha\, m = o\}$.

We give many properties and characterizations of this class of gamma modules. A left R_Γ –module M is a duo if and only if every α −cyclic R_Γ −submodule $R\alpha x$ of M is fully invariant where $x \in M$.We study the relationship between the duo and the multiplication gamma modules, while every fully stable gamma module is duo and the convers is true in principally quasi-injective gamma modules. We consider direct summand and sum of duo gamma modules. Finally, we consider some generalizations of full stability which are related to the duo property.

2. Basics of duo gamma modules

Let M be an R_Γ −module. An R_Γ −submodule N of M is called fully invariant if $f(N) \subseteq N$ for each R_Γ −endomorphism f of M. In case that each R_Γ −submodule of M is fully invariant, then M is called a duo. Clearly, (0) and M are fully invariant R_Γ −submodules, and hence, simple R_Γ −modules are duo. Let M be an R_Γ −module, $\alpha \in \Gamma$ an arbitrary fixed element and $m \in M$. Then the set $R\alpha m = \{r\alpha m | r \in R\}$ is an R_Γ −submodule of M and it is called an α −cyclic. It is easy to see that an R_Γ −module M is a duo if and only if every α −cyclic R_Γ −submodule of M is fully invariant, that is for each x in M and R_Γ −endomorphism θ of M, there exists $r \in R$ such that $\theta(x) = r\alpha\, x$.

In general, R_Γ −submodules of duo gamma modules may not be duo. However, every direct summand of duo gamma modules is a duo, for if K is an R_Γ −submodule of a direct summand N of an R_Γ −module M and θ is an R_Γ −endomorphism of N, then θ can be extended in the usual way to an R_Γ −endomorphism $\bar{\theta}$ of M, $\theta(K) = \bar{\theta}(K) \subseteq K$.

It is clear that any fully stable R_Γ −module is a duo, but the converse is not true generally. For example, the Z_Z −module Z is a duo, but not fully stable.

In the following, we consider conditions under which every gamma submodule of a duo module is a duo, as well as the homomorphic image, but first we introduce the following.

An R_Γ −module M is said to be Γ −poorly injective, if each R_Γ −endomorphism of an R_Γ −submodule of M can be extended to an R_Γ −endomorphism of M.

We call an R_Γ −module M an Γ −quasi projective if, for any R_Γ −module W and R_Γ −homo morphisms f, g: $M \to W$ with f is surjective, there is an R_Γ −endomorphism h of M such that $g = fh$. Then we have the following.

Proposition (2.1): Let M be a duo gamma module. Then:

i) If M is Γ −poorly injective, then every gamma submodule of M is a duo.

ii) If M is Γ −quasi projective, then every R_Γ −homomorphic image of M is a duo.

Proof (i): Let K be an R_Γ −submodule of M, N an R_Γ −submodule of K, and θ an R_Γ −endomorphism of K . Γ −poor injectivity of M implies that θ can be extended to an R_Γ −endomorphism $\bar{\theta}$ of M. Then $\theta(N) = \bar{\theta}(N) \subseteq N$.

(ii): Let K be an R_Γ −submodule of a duo R_Γ −module M, and f be an R_Γ −endomorphism of M/K. For each R_Γ −submodule L/K of M/K where L is an R_Γ −submodule of M containing K. Γ −quasi projectivity of M implies that there is an R_Γ −endomorphism g of M such that $g(m + K) = f(m) + K$ for each m in M. Duo property of M implies that $g(L) \subseteq L$ and hence $f(L/K) \subseteq L/K$. This shows that M/K is a duo.

An R_Γ −submodule K is called countably α −generated of an R_Γ −module M, where α is an arbitrary fixed element in Γ, if there a countable subset $\{K_i | i \in \mathbb{N}\}$ of M such that $K = \sum_{i \in \mathbb{N}} R\,\alpha n_i$. Then we have the following result:

Proposition (2.2): Let M be an R_Γ −module in which every countably α −generated R_Γ −submodule of M is a duo. Then M is a duo.

Proof: Suppose that m is any element of $M, \alpha \in \Gamma$ and f is an R_Γ −endomorphism of M. Let $N = R\alpha m + R\alpha f(m) + R\alpha f^2(m) + \cdots$. It is clear that N is a countably α −generated R_Γ −submodule of M and $f\big|_K : K \to K$. So, $f(m) = r\alpha m$ for some $r \in R$. This implies that M is a duo.

In the following, we show that there are a lot of gamma modules which are not duo. Let S be a Γ −ring. A nonempty subset R of S is called a Γ −subring of S, if R is itself a Γ −ring.

Proposition (2.3): Let R be a proper Γ −subring of a Γ −ring S. Then the R_Γ −module S is not a duo.

Proof: Let t be any element of a Γ −ring S such that $t \notin R$. Then the mapping $f: S \to S$, defined by $f(a) = t\alpha_0 a$ for all $a \in S$, is an R_Γ −homomorphism. If S is a duo, then $t = t\alpha_0 1 = f(1) \in R$, which is a contradiction.

In the following, we give some sources of duo gamma modules.

An R_Γ −module M is called a Γ −multiplication if, for any R_Γ −submodule N of M, there exists a two-sided Γ −ideal I of R such that $N = I\Gamma M$. It is easy to see that $N = [N: M]\Gamma M$ where $[N: M] = \{r \in R \mid r\Gamma M = 0\}$ [6].

Proposition (2.4): Every Γ −multiplication R_Γ −module is a duo.

Proof: If N is an R_Γ −submodule of an Γ −multiplication R_Γ −module M, then $N = I\Gamma M$ for some two-sided Γ −ideal I of R, and so for every R_Γ −endomorphism f of M, $f(N) = f(I\Gamma M) = I\Gamma f(M) \subseteq I\Gamma M = N$.

The converse of proposition (2.4) is not true generally.

Let M be an R_Γ −module and α be an arbitrary fixed element of Γ. A previous article [7] introduced the concept of the α −free gamma module. An R_Γ −module P is called an α −projective if P is an α −direct summand of an α −free R_Γ −module for an arbitrary fixed $\alpha \in \Gamma$. This is equivalent to saying that, for every α −generating set $\{x_i | i \in I\}$ of P, there exists a family $\{\varphi_i | i \in I\}$ of $P^* = Hom_{R_\Gamma}(P, R)$, such that for each $x \in P, \varphi_i(x) \neq 0$ for finitely many $i \in I$ and $x = \sum_{i \in I} \varphi_i(x)\alpha x_i$. With regard to these concepts, we have the following theorem.

Theorem (2.5): The followings are equivalent for an α −projective R_Γ −module M where α is an arbitrary fixed element in Γ.

(1) M is a duo.

(2) M is a Γ −multiplication .

Proof: Assume that M is a duo and N is an R_Γ −submodule of M. α −projective of M implies that, for every α −generators $\{x_i | i \in I\}$ of M, there exists a family $\{\varphi_i | i \in I\}$ of elements $\varphi_i \in Hom(P, R)$, such that for every $m \in M, \varphi_i(m) \neq 0$ for finitely many $i \in I$ and $m = \sum_{i \in I} \varphi_i(m)\alpha x_i$. Let A be the Γ −ideal of R, α −generated by $\{\varphi_i | i \in I\}$ for $x \in N$ and $i \in I$. We show that $N = A\Gamma M$. If $x \in N$, then $x = \sum_{i \in I} \varphi_i(m)\alpha x_i$ and hence $N \subseteq A\Gamma M$. For other inclusions, suppose that $x \in N$ and $m \in M$, define $\theta_\alpha: R \to M$ by $\theta_\alpha(r) = r\alpha m$ for all r in R. Then $\theta_\alpha o \varphi_i$ is an R_Γ −endomorphism of M and $\varphi_i(x)\alpha m = \theta_\alpha(\varphi_i(x)) \in (\theta_\alpha o \varphi_i)(R\alpha x) \subseteq R\alpha x \subseteq N$,

Since M is a duo, then $A\Gamma M \subseteq N$ and so $N = A\Gamma M$.

A Γ −ideal I of a Γ −ring R is called a Γ −idempotent if $I = I\Gamma I$, [8]. We call an R_Γ −module M as a ΓI −multiplication if for each R_Γ −submodule N, there is a Γ −idempotent Γ −ideal I of R, such that $N = I\Gamma M$. We define that a Γ −ring R is called regular if all its Γ −ideals are Γ −idempotent.

Then we have the following result:

Corollary (2.6): Let M be an α −projective gamma module over a regular Γ −ring R. Then the following statements are equivalent:

1- M is a ΓI −multiplication.

2- M is fully stable.

3- M is a duo.

4- M is a Γ −multiplication.

Proof: **(1)⇒(2)** Let N be an R_Γ −submodule of M and $\theta: N \to M$ an R_Γ −homomorphism. By (1), there is a Γ −idempotent Γ −ideal A of R such that $N = A\Gamma M$. Now, $\theta(N) = \theta(A\Gamma M) = \theta(A\Gamma A\Gamma M) = A\Gamma\theta(A\Gamma M) = A\Gamma\theta(N) \subseteq A\Gamma M = N$.

(2)⇒(3) is clear.

(3)⇒(4) follows from theorem (2.5).

(4)⇒(1) is clear.

A gamma module M is called uniserial if, for all gamma submodules K and N of M, either $K \subseteq N$ or $N \subseteq K$ [9].

A Γ −ring R is with a supper identity, if there is $1 \in R$ such that $r\alpha 1 = 1\alpha r = r$ for all $\in R$, $\alpha \in \Gamma$. And an R_Γ −module M is supper unitary if there is $1 \in R$ such that $1\alpha m = m$ for all m in M and $\alpha \in \Gamma$ [7].

Proposition (2.7): Let R be a Γ−ring and M a supper unitary R_Γ −module. If M is a uniserial satisfying the a. c. c. on α −cyclic R_Γ −submodules, then M is a duo.

Proof: Let $m(\neq 0) \in M$ and f an R_Γ −endomorphism of M. Suppose that $f(m) \notin R\alpha m$. Then $m \in R\alpha f(m)$ and hence $m = r\alpha f(m)$ for some $r \in R$. It follows that $f^n(m) = f^n(r\alpha f(m)) = r\alpha f^{n+1}(m)$ for each positive integer n. Consider the a. c:

$$R\alpha m \subseteq R\alpha f(m) \subseteq R\alpha f^2(m) \subseteq \cdots$$

The hypothesis implies that there is a positive integer n_0 such that $R\alpha f^t(m) = R\alpha f^{t+1}(m)$, for all $t \geq n_0$ and there is $z \in R$ such that $f^{t+1}(m) = z\alpha f^t(m) = f^t(z\alpha m)$. Hence $f(m) - z\alpha m \in \ker(f^t)$. If $R\alpha m \subseteq \ker(f^t)$, then $f^t(m) = 0$ and hence $m = 0$ which is a contradiction. Thus $\ker(f^t) \subseteq R\alpha m$ and hence $f(m) - z\alpha m \in R\alpha m$. $f(m) \in R\alpha m$ is a contradiction. Therefore M is a duo.

It was previously proved [9] that a fully stable R_Γ −module M satisfies for every pair of R_Γ −submodules N_1, N_2 of M with $N_1 \cap N_2 = 0$. We have $Hom_{R_\Gamma}(N_1, N_2) = 0 = Hom_{R_\Gamma}(N_2, N_1)$, but the converse may not be true. However, the converse is true in case that M is fully essential stable [9].

In the following Lemma we have the following:

Lemma (2.8): Let an R_Γ −module $M = N_1 \oplus N_2$ be a direct sum of R_Γ −submodules N_1, N_2. Then N_1 is a fully invariant if and only if $Hom_{R_\Gamma}(N_1, N_2) = 0$.

Proof: Denote ρ_1(resp. ρ_2): $M \to N_1$ (resp. N_2) the canonical projection onto N_1 (resp. N_2) and $i_1(resp. i_2)$: N_1 (resp. N_2) $\to$ M denote the injection mapping of N_1 (resp. N_2) .

Suppose that N_1 is a fully invariant R_Γ −submodule of M and $f: N_1 \to N_2$ is an R_Γ −homomorphism. Then $f' = i_2 o f o \rho_1$ is an R_Γ −endemorphism of , and hence $f'(N_1) \subseteq N_1$, so that $f(N_1) \subseteq N_1 \cap N_2 = 0$. It follows that $f = 0$.

For any R_Γ −endemorphism g of M, $g(N_1) \subseteq \rho_1 o g o i_2(N_1) + \rho_2 o g o i_1(N_1) = \rho_1 o g(N_1) \subseteq N_1$, because $\rho_2 o g o i_1 \in Hom_{R_\Gamma}(N_1, N_2) = 0$. It follows that N_1 is a fully invariant R_Γ −submodule of M.

Lemma (2.9): Let an R_Γ −module $M = \oplus_{i\in I} M_i$ be a direct sum of R_Γ −submodules M_i $(i \in I)$ and N be a fully invariant R_Γ −submodule of M. Then $N = \oplus_{i\in J}(N \cap M_i)$.

Proof: Suppose that $\rho_i: M \to M_i$ is the canonical projection for each $i \in I$, and that $j_i: M_i \to M$ is the injection, then $j_i o p_i: M \to M$, and hence $j_i o \rho_i(N) \subseteq N$ for each $j \in I$. It follows that $N \subseteq \oplus_{i\in J} j_i o \rho_i(N) \subseteq \oplus_{i\in J}(N \cap M_i) \subseteq N$ so that $N = \oplus_{i\in J}(N \cap M_i)$.

Lemma (2.10): Let an R_Γ −module $M = \oplus_{i\in I} M_i$ be a direct sum of R_Γ −submodules M_i $(i \in I)$, and it is supper unitary. Then the following statements are equivalent.

(1) $R = l^\alpha_{R_\Gamma}(m_i) + l^\alpha_{R_\Gamma}(m_j)$ for all $m_i \in M_i, m_j \in M_j$ with $i \neq j$ in I.

(2) $N = \oplus_{i\in I}(N \cap M_i)$ for every (α −cyclic) R_Γ −submodules N of M. Moreover, in this case $Hom(M_i, M_j) = 0$ for all distinct i, j in I.

Proof: (1)⇒(2): Let N be any α −cyclic R_Γ −submodule of M, and $m \in N$. Then there exists a positive integer n, distinct elements $i_j \in I (1 \leq j \leq n)$, and elements $m_j \in M_{ij} (1 \leq j \leq n)$, such that $m = m_1 + m_2 + \cdots + m_n$. For $n = 1$, then $m = m_1 \in N \cap M_{i1}$, and hence $N = \oplus(N \cap M_i)$ Suppose that $n \geq 2$. By the hypothesis, there exists elements r, s in R, such that

$1 = r + s, r\alpha m_1 = 0$ and $s\alpha m_n = 0$. Then:

$s\alpha m = s\alpha(m_1 + m_2 + \cdots + m_n) = s\alpha m_1 + s\alpha m_2 + \cdots + s\alpha m_n$

$= s\alpha m_1 + s\alpha m_2 + \cdots + s\alpha m_{n-1} = 1\alpha m_1 - r\alpha m_1 + \cdots + s\alpha m_{n-1}$

$= 1\alpha m_1 + s\alpha m_2 + \cdots + s\alpha m_{n-1}$

Note that $s\alpha m_j \in M_{ij} (2 \leq j \leq n-1)$ and $s\alpha m \in N$. By induction on $n, m_1 \in N \cap M_{i1}$.

Similarly $m_j \in N \cap M_{ij} (2 \leq j \leq n)$.

(2)⇒(1): Let i,j be distinct elements of I, let $x \in M_i$ and let $y \in M_j$. If $K = R\alpha(x+y)$, then $K = \oplus_{i\in I}(K \cap M_i)$ and hence $(x+y) \in (K \cap M_i) \oplus (K \cap M_j)$. There exists $a,b \in R$, such that $x+y = a\alpha(x+y) + b\alpha(x+y)$, where $a\alpha(x+y) \in M_i$ and $b\alpha(x+y) \in M_j$. Then $x = a\alpha(x+y)$, so that: $x = a\alpha x + a\alpha y \Rightarrow x - a\alpha x = a\alpha y \Rightarrow x(1 - a\alpha 1) = a\alpha y$. So that $x(1-a\alpha 1) = 0$ and $a\alpha y = 0$. Thus $a\alpha 1 \in l_{R_\Gamma}(y), (1 - a\alpha 1) \in l_{R_\Gamma}(x), 1 = (1-a\alpha 1) + a\alpha 1 \in l^{\alpha}_{R_\Gamma}(x) + l^{\alpha}_{R_\Gamma}(y)$.

Finally, let i,j be distinct elements of I. Let $f: M_i \to M_j$ be any $R_\Gamma -$homomorphism. Let $n \in M_i$. By (1), $R = l_{R_\Gamma}(n) + l_{R_\Gamma}(f(n))$ so that $1 = c + d$ for some c, d in R, α in Γ with $c\alpha n = 0, d\alpha f(n) = 0$. It follows that $f(n) = c\alpha f(n) + d\alpha f(n) = f(c\alpha n) + f(d\alpha n) = 0$, thus $f = 0$.

The following corollary follows from (2.10)and (2.9).

Corollary (2.11): Let a supper unitary gamma module $M = \oplus_{i\in I} M_i$ be a direct sum of gamma submodules M_i $(i \in I)$, $\alpha \in \Gamma$ be an arbitrary fixed element, and N be a fully invariant gamma submodule of M. Then $l^{\alpha}_{R_\Gamma}(m_i) + l^{\alpha}_{R_\Gamma}(m_j) = R$ for all $m_i \in M_i, m_j \in M_j$ for all $i \neq j$ in I.

Theorem (2.12): Let an $R_\Gamma -$module $M = \oplus_{i\in I} M_i$ be a direct sum of $R_\Gamma -$submodules $M_i (i \in I)$. Then M is a duo $R_\Gamma -$ module if and only if:

(a) M_i is a duo gamma module for all $i \in I$ and

(b) $N = \oplus_{i\in I}(N \cap M_i)$ for every $R_\Gamma -$submodule N of M.

Proof: ⇒ follows by Lemma (2.9).

⇐ Suppose that M satisfies the above conditions. Let L be an $R_\Gamma -$submodule of M and f any $R_\Gamma -$endomorphism of M. For $i \in I$ let $\rho_i: M \to M_i$ denotes the canonical projection and let $i_i: M_i \to M$ denotes the injection. By (a), $\rho_i \circ f \circ i_i(L \cap M_i) \subseteq L \cap M_i$ for $i \in I$. Now (b) gives $f(L) = \sum_{i\in I} f(L \cap M_i) \subseteq \sum_{i\in I} \rho_i \circ f \circ i_i(L \cap M_i)) \subseteq \sum_{i\in I}(L \cap M_i) \subseteq L$. Thus M is a duo.

Corollary (2.13): Let a supper unitary gamma module $M = \oplus_{i\in I} M_i$ be a direct sum of $R_\Gamma -$submodules $M_i (i \in I)$. Then M is a duo gamma module if and only if $M_i \oplus M_j$ is a duo gamma module for all $i \neq j \in I$.

Proof: ⇒ The assumption that any direct summand of a duo gamma module is a duo proves the first direction.

Conversely, suppose that $M_i \oplus M_j$ is a duo gamma module for all $i \neq j$ in I. Then M_i is a duo gamma module for all $i \in I$. Furthermore, for all $i \neq j$ in I , $R = l^{\alpha}_{R_\Gamma}(m_i) + l^{\alpha}_{R_\Gamma}(m_j)$ for all $m_i \in M_i, m_j \in M_j$. By Lemma (2.9), Lemma (2.10) and Theorem (2.12), we get that M is a duo gamma module.

We introduce the following generalization of fully stable gamma modules.

An $R_\Gamma -$module M is called fully direct-summand stable (for short, fully ds-stable) if every direct summand of M is stable.

It is clear that a direct summand of a fully ds-stable is fully ds-stable.

Theorem (2.14): Let a gamma module $M = \oplus_{i\in I} M_i$ be a direct sum of $R_\Gamma -$submodules $M_i (i \in I)$. Then M is a fully ds-stable if and only if:

(1) M_i is a fully ds-stable for all $i \in I$,

(2) $N = \oplus_{i\in I}(N \cap M_i)$ for every direct summand N of M.

Proof: Assume that M is a fully ds-stable $R_\Gamma -$module. Then, clearly, M_i is a fully ds-stable for all $i \in I$ and hence we get (1). Lemma (2.9) gives (2).

Conversely, suppose that M satisfies the above conditions. Let L be a direct summand of M and $g: L \to M$ an $R_\Gamma -$homomorphism. By (2), $L = \oplus_{i\in I}(L \cap M_i)$ and from this we get $g: \oplus_{i\in I}(L \cap M_i) \to \oplus_{i\in I} M_i$ for each i in I. Let $\rho_i: \oplus_{i\in I} M_i \to M_i$ denotes the canonical projection and let $i_i: L \cap M_i \to L$ denotes the inclusion. Hence, $\rho_i \circ g \circ i_i: L \cap M_i \to M_i$, by (1), $\rho_i \circ g \circ i_i(L \cap M_i) \subseteq L \cap M_i$ for all $i \in I$. Now (2) gives $g(L) = \sum_{i\in I} g(L \cap M_i) \subseteq \sum_{i\in I} \rho_i \circ g \circ i_i(L \cap M_i) \subseteq \sum_{i\in I}(L \cap M_i) \subseteq L$. Thus M is fully ds-stable.

An element $r\,(\neq 0) \in R$ is called $\Gamma -$zero divisor if there exists $\alpha\,(\neq 0) \in \Gamma$ and $s\,(\neq 0) \in R$ such that $s\alpha r = 0$ [10]. Let M be an $R_\Gamma -$module, an element m in M is called a $\Gamma -$torsion if there is a non-zero divisor r in R, and a non-zero element $\alpha \in \Gamma$ such that $r\alpha m = 0$.

Denote the set of all $\Gamma -$torsion elements in M by $T_\Gamma(M)$, if $T_\Gamma(M) = M$ (resp. 0), then M is called $\Gamma -$torsion (resp. $\Gamma -$torsion free).

It is a matter of checking that $T_\Gamma(M)$ is an $R_\Gamma -$submodule of M.

In example, let R be a Γ −ring, $M = \left\{\binom{m_1}{m_2}; m_1, m_2 \in R\right\}$ and $\Gamma = \left\{\binom{\alpha}{0}; \alpha \in R\right\}$, let $r = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \neq 0$, take $\alpha = \binom{1}{0} \neq 0$. Then $r\,\alpha \binom{m_1}{m_2} = \binom{0}{0}$ for any $\binom{m_1}{m_2} \in M$. So, M is a Γ −torsion.

Lemma (2.15): Let a supper unitary R_Γ −module $M = M_i \oplus M_j$ be a direct sum of a non-zero torsion free R_Γ −submodule M_i and a non-zero R_Γ −submodule M. Then M is not a duo gamma module.

Proof: Let m_i and m_j be non-zero elements of M_i and M_j**,** respectively. Then $l_{R_\Gamma}(m_i) = 0$, hence $l_{R_\Gamma}(m_i) + l_{R_\Gamma}(m_j) = l_{R_\Gamma}(m_j) \neq R$. By Lemma (2.10) and Theorem (2.12), M is not a duo gamma module.

Let M be an R_Γ −module. An R_Γ −submodule N of M is called a Γ −essential if N has a nontrivial intersection with every nonzero R_Γ −submodule of M [10].

Dually, we say that an R_Γ −submodule N of M is called small if $N + K$ is a proper R_Γ −submodule of M for each proper R_Γ −submodule K of M.

An R_Γ −module M is called Γ −Hopfian (resp. generalized Γ −Hopfian) if every surjective R_Γ −endomorphism of M is an isomorphism (resp. has a small kernel).

An R_Γ −module M is called Γ −coHopfian (resp. weakly Γ −coHopfian) if every injective R_Γ −endomorphism of M is an isomorphism (resp. has an Γ −essential image of M).

Proposition (2.16): Every fully stable gamma module is a Γ –coHopfian, and hence is a weakly Γ –coHopfian.

Proof: Let M be a fully stable R_Γ −module and $f: M \to M$ is an R_Γ −monomorphism, then $M \cong f(M)$. Hence, we have $M = f(M)$ so that f is an R_Γ −epimorphism. By Corollary (2.4) in a previous study [9], we have $M = f(M)$.

Proposition (2.17): Every duo gamma module is a generalized Γ −Hopfian and a weakly Γ −coHopfian.

Proof: Let f be any surjective R_Γ −endomorphism of M. Let $K \leq M$ such that $M = \ker(f) + K$. Then $M = f(M) = f(\ker(f) + K) = f(K) \subseteq K$. It follows that $\ker(f)$ is a small R_Γ −submodule of M. Let g be an injective R_Γ −endomorphism of M, let $N \leq M$ such that $N \cap g(M) = 0$. Since N is fully invariant, we get $g(N) = 0$ and hence $N = 0$. It follows that $g(N)$ is an essential R_Γ −submodule of M.

Duo gamma modules are neither Γ −Hopfian nor Γ −coHopfian in general.

We have seen in a previous article [9] that Z_{p^∞} is a fully stable Z_S-module where S is an arbitrary subring of Z. Let s_0 be an arbitrary fixed element in S. The mapping $f: Z_{p^\infty} \to Z_{p^\infty}$, defined by $f(x) = ps_0x$ for all x in Z_{p^∞}, is a surjective which is not an isomorphism, and hence Z_{p^∞} is a duo which is not a Γ −Hopfian. On the other hand, it is clear that Z is a duo Z_S-module. We define $h: Z \to Z$ by $h(z) = 2s_0z$ for all $z \in Z$ is an injective which is not an isomorphism.

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