A STUDY OF THE DYNAMICS OF THE FAMILY $\lambda\frac{\sinh^m z}{z^{2m}}$

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Abstract

In this paper, the dynamical behavior of a family of non- critically finite transcendental meromorphic function $f_\lambda(z) = \lambda\frac{\sinh^m x}{x^{2m}}$, λ >0 and m is an even natural number is described. The Julia set of $f_\lambda(z)$, as the closure of the set of points with orbits escaping to infinity under iteration, is obtained. It is observed that, bifurcation in the dynamics of $f_\lambda(z)$ occurs at two critical parameter values λ = λ1, λ2, where λ1 = $\frac{x_1^{2m+1}}{\sinh^m x_1}$ and λ2 = $\frac{x_2^{2m+1}}{\sinh^m x_2}$ with x1 and x2 are the unique positive real roots of the equations tanh x = $\frac{mx}{2m-1}$ and tanh x = $\frac{mx}{2m+1}$ respectively.

دراسة ديناميكية العائلة $\lambda\frac{\sinh^m z}{z^{2m}}$

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الخلاصة

في هذا البحث وصفنا سلوك ديناميكية العائلة λ $\frac{\sinh^m x}{x^{2m}}$ والتي تضم دوالا" لها عدد غير منته من القـيم الشاذة. أوجدنا مجموعة جوليا لهذه العائلة على أنها أنغلاق مجموعة النقاط والتي تكون فيها غاية الدالة هو المالانهاية. لقد وجدنا أن التفرع لهذه العائلة حصل عند قيم مختلفة للمعلمة λ وهـي λ1 و λ2 حيـث أن λ1 = $\frac{x_1^{2m+1}}{\sinh^m x_1}$ و λ2 = $\frac{x_2^{2m+1}}{\sinh^m x_2}$ وأن x1 وx2 يمثلان الجذور الحقيقية لحل المعادلة tanh x = $\frac{mx}{2m-1}$ و tanh x = $\frac{mx}{2m+1}$ على التوالي.

Introduction

Let f(z) be a non-constant entire function Let $z_o = f^o(z_o)$ and $z_n = f(z_{n-1}) = f^n(z_o)$, n=1,2,…where $f^n = f \circ f \circ \ldots \circ f$ (n-time composition) is the n-th iteration of f. The sequence $\{z_n = f^n(z_o)\}_{n=0}^{\infty}$ is called the sequence of iterations at the point z_o. The investigation of the (discrete) dynamics of a complex function f is the investigation of its iterations at each point z_o in the extended complex plane $\mathbb{C} \cup \{\infty\}$ (denoted by $\hat{\mathbb{C}}$). The main objects studied in the dynamics of a complex function are its Fatou and Julia sets. The Fatou set (or stable set) of a function f, which is denoted by F(f), is defined to be the set of all complex numbers where the family of iterates $\{f^n\}$ of f forms a normal family in the sense of Montel. The Julia set (or chaotic set), denoted by J(f), is the complement of the Fatou set of f. In recent years, the dynamics of transcendental functions has been studied. Devaney [1], [2], [3] and Devaney and Durkin [4] studied the dynamics of certain entire transcendental functions such as λe^z and i λcosz. The singular values of a function play an important role in determining the dynamics of the function. The dynamics of critically finite meromorphic transcendental functions has been studied for several interesting classes during last two decades [5, 6, 7, 8]. Kapoor and Prasad [9], [10] studied the dynamics of certain non-critically finite entire transcendental functions such as $\lambda(e^z-1)/z$. In the present work, an effort is made to fill this gap by studying the dynamics of a one parameter family of non-critically finite even transcendental meromorphic functions. For this purpose, a one parameter family H= $\{ f_\lambda(z)=\lambda\frac{\sinh^m z}{z^{2m}}$, λ >0 m∊ N, and m is even, z ∊ $\mathbb{C}\}$ is considered. Bifurcations in the dynamics on real axis for the functions in our family occur at two parameter values. It is observed that the characterization of the Julia set of a function in H as the closure of the set of all its escaping points continues to hold for function in H. The Julia set of a function in H is found to contain both real and imaginary axes for certain parameter values.

Theorem 1.1([6])

Let f(z) be a transcendental meromorphic function. Suppose z_o lies on an attracting cycle or a parabolic cycle f(z). Then, the orbit of at least one critical value or asymptotic value is attracted to a point in the orbit of z_o.

One parameter family H of non-critically finite functions

Let H be one parameter family of even transcendental meromorphic functions. The following proposition shows that the functions in the family H are indeed non-critically finite.

Proposition 2.1 Let f_λ ∊ H. Then, the function $f_\lambda(z)$ is non-critically finite.

__**Proof**__ The derivative of the function $f_\lambda(z)$ for z ≠ 0 is given by

$$f_\lambda' = \lambda\frac{m\sinh^{m-1} z(z\cosh z - 2\sinh z)}{z^{2m+1}}$$

The critical points of the function $f_\lambda(z)$ are solutions of the equation $f_\lambda'(z) = 0$. and these solutions are z = kπi, where k is a non-zero integer and solutions of the equation

$$z\cosh z - 2\sinh z = 0 \qquad (2.1)$$

are the critical points of $f_\lambda(z)$. The solutions of equation (2.1) are the same as the solutions of the equation tanh z = $\frac{z}{2}$.This equation has a solution z_o if and only if the equation tanw = $\frac{w}{2}$ has a solution iz_o. Now equating the real and imaginary parts of the equation tan w = $\frac{w}{2}$, for a non-zero z = x+iy, $\frac{\sin 2x}{\cos 2x+\cosh 2y} = \frac{1}{2}x$ and $\frac{\sinh 2y}{\cos 2x+\cosh 2y} = \frac{1}{2}y$. This implies to

$$\frac{\sin 2x}{x} = \frac{\sinh 2y}{y} \qquad (2.2)$$

It is easily to show that, for x,y ≠ 0, the $\left|\frac{\sin 2x}{x}\right| < 2$ and $\left|\frac{\sinh 2y}{y}\right| > 2$ so that (2.2) is not possible in this case. Therefore, at least one of x,y must be zero. Hence (2.2) has only real or purely imaginary roots. If x =0, then tan w = $\frac{w}{2}$ implies that tanh y = $\frac{1}{2}$y and this equation has two zeros. Therefore, tan w = $\frac{w}{2}$ has two purely imaginary solutions. If y =0, then tan w = $\frac{w}{2}$ implies that tan x = $\frac{1}{2}$x and this equation has infinitely many purely imaginary solutions. Thus, the function $f_\lambda(z)$ has two real and infinitely many imaginary critical points. To find the critical values of the function $f_\lambda(z)$, we note that $f_\lambda(k\pi i) = 0$, where k is a non-zero integer. Let $\{i\, y_L\}_{L=-\infty}^{\infty}$, y_L be the critical points of $f_\lambda(z)$ other than the critical points $k\pi i$, for k = ± 1, ± 2… since

$$f_\lambda(i\, y_L) = \lambda \frac{\sin^m y_L}{y_L^{2m}}$$

and the values $\frac{\sin^m y_L}{y_L^{2m}}$ are real and distinct , it follows that the values in the set $\{ f_\lambda(i\, y_L)\}_{L=-\infty}^{\infty}$ are real and distinct. Therefore, the function $f_\lambda(z)$ possesses infinitely many real critical values. □

Fixed points and their nature for functions in H

In this section, we find the fixed points of the function $f_\lambda(x) = \lambda \frac{\sinh^m x}{x^{2m}}$ and describe their nature. Let

$$\Phi(x) = \frac{x^{2m+1}}{\sinh^m x} \qquad (3.1)$$

The function Φ(x) has the following properties:-
It follows easily from (3.1) that
1.Φ(x) is continuous in R.
2.Φ(x) is positive in (0, ∞), is negative in (-∞,0).
3.Φ(x) → 0 as x→ -∞ and Φ(x) → 0 as x→ ∞.
Further,
4. Φ'(x) is continuous in R. Since

$$\Phi'(x) = \frac{(2m+1)x^{2m}\sinh x - m x^{2m+1}\cosh x}{\sinh^{m+1} x},$$

it follows easily that $\Phi'(0) = \lim_{x\to o} \Phi'(x)$, so that Φ'(x) is continuous in R.
5.Φ'(x) has a unique positive real zero at x = x_2, where x_2 is a real positive solution of $\tanh x = \frac{mx}{2m+1}$; since Φ'(x) = 0 gives $\tanh x = \frac{mx}{2m+1}$ and by Newton-Raphson's method x_2 is a real positive solution of $\Psi(x) = \tanh x - \frac{mx}{2m+1} = 0$. (Fig.1 (a)), the property (5) follows.
6.Φ(x) is strictly increasing in (0, x_2), is strictly decreasing in (x_2, ∞) and has a maximum at x = x_2, where x_2 is a real positive solution of $\tanh x = \frac{mx}{2m+1}$:
By property (5), Φ'(x_2) = 0, where x_2 is a real positive solution of $\tanh x = \frac{mx}{2m+1}$.

$$\Phi''(x) = \frac{\left[\begin{array}{l}\sinh x[(2m+1)x^{2m}\cosh x + 2m(2m+1)x^{2m-1}\sinh x \\ - m x^{2m+1}\sinh x - m(2m+1)\cosh x] - \\ (m+1)\cosh x[(2m+1)x^{2m}\sinh x - m x^{2m+1}\cosh x]\end{array}\right]}{\sinh^{m+2} x}$$

Since Φ''(x_2) < 0. Therefore, the function Φ(x) has exactly one maxima in (0, ∞) at x = x_2. It therefore follows by property (3) that Φ(x) decreases to 0 in (x_2, ∞) and increases in (0, x_2). The graph of Φ(x) therefore is as shown in Fig.1 (b).

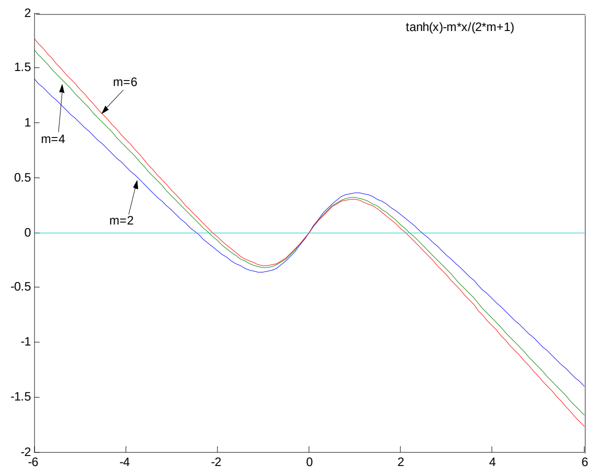

Figure 1: (a) Graph of $\Psi(x)$

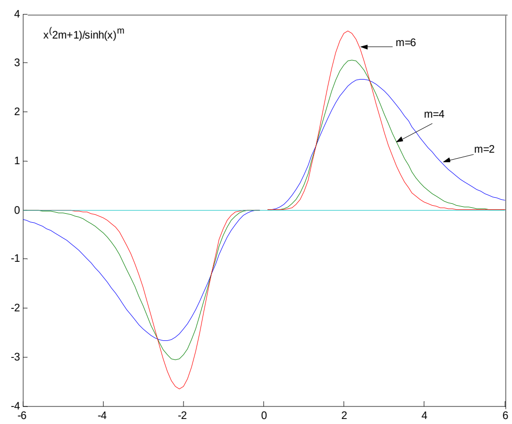

Figure 1. (b) Graph of Φ(x)

Throughout in the sequel, we denote

$$\lambda_2 = \Phi(x_2) \qquad (3.2)$$

Where x_2 is unique positive real solution of the equation $\tanh x = \frac{mx}{2m+1}$.

The following proposition gives the number and locations of real fixed points of the function $f_\lambda(x)$ for $\lambda > 0$:-

Proposition 3.1

Let $f_\lambda \in$ H. Then the locations of real fixed points of the function $f_\lambda(x) = \lambda \frac{\sinh^m x}{x^{2m}}$ are given by the following:-

1-For $0 < \lambda < \lambda_2$, $f_\lambda(x)$ has exactly one fixed point in each of the intervals $(0, x_2)$ and (x_2, ∞), where x_2 is a positive real solution of the equation $\tanh x = \frac{mx}{2m+1}$.

2-For $\lambda = \lambda_2$, the only fixed point of $f_\lambda(x)$ is at $x = x_2$ where x_2 is as in (1).

3- For $\lambda > \lambda_2$, $f_\lambda(x)$ has no fixed points.

Proof The fixed points of the function $f_\lambda(x)$ are the solutions of the equation $\lambda = \Phi(x)$, where $\Phi(x)$ is given by (3.1). We have the following cases:-

1- $0 < \lambda < \lambda_2$

Since $\Phi(x_2) = \lambda_2$ and $\lambda < \lambda_2$, in view of properties (1), (3) and (6) of the function $\Phi(x)$, the line $u = \lambda$ intersects the graph of $\Phi(x)$ at exactly two points. Using properties (2),(3)and (6), it follows in view of $\Phi(x_2) = \lambda_2$ that one of the solutions of $\Phi(x) = \lambda$. For $0 < \lambda < \lambda_2$ lies in the interval $(0, x_2)$. Similarly, since by property (6) $\Phi(x)$ is decreasing in the interval (x_2, ∞) and $\Phi(x_2) = \lambda_2$, the other solution of $\Phi(x) = \lambda$ for $0 < \lambda < \lambda_2$ lies in the interval (x_2, ∞). Thus, $f_\lambda(x)$ has two real fixed points lying in the intervals $(0, x_2)$ and (x_2, ∞).

2- $\lambda = \lambda_2$

The function $\Phi(x)$ has exactly one maxima at $x = x_2$ and the maximum value of $\Phi(x)$ is $\Phi(x_2) = \lambda_2$, the line $u = \lambda_2$ intersects the graph of $\Phi(x)$ at exactly one point at $x = x_2$. Therefore, the equation $\Phi(x) = \lambda_2$ has exactly one solution at $x = x_2$. Thus, $f_\lambda(x)$ has only one real fixed point at $x = x_2$ for $\lambda = \lambda_2$.

3- $\lambda > \lambda_2$

By property (6), the maximum value of $\Phi(x)$ is $\Phi(x_2) = \lambda_2$, therefore, for $\lambda > \lambda_2$, the line $u = \lambda$ does not intersect the graph of $\Phi(x)$. Consequently, the equation $\Phi(x) = \lambda$ has no solution for $\lambda > \lambda_2$. Thus $f_\lambda(x)$ has no fixed point for $\lambda > \lambda_2$.

Now, Let

$$\lambda_1 = \Phi(x_1) \qquad (3.3)$$

where x_1 is a positive solution of the equation $\tanh x = \frac{mx}{2m-1}$. The fixed points of the function $f_\lambda(x)$ found in proposition (3.1) are denoted by $r_1 \in (0, x_1)$, $r_2 \in (x_3, \infty)$, $a_\lambda \in (x_1, x_2)$ and $r_\lambda \in (x_2, x_3)$, where x_2 is a positive solution of $\tanh x = \frac{mx}{2m-1}$ and x_1, x_3 be solutions of $\lambda_1 = \Phi(x)$ lying in the intervals $(0, x_2)$ and (x_2, ∞) respectively.

The nature of these fixed points of the function $f_\lambda(x)$ for different values of parameter λ is described in the following theorem:-

Theorem 3.1

Let $f_\lambda(x) = \lambda \frac{\sinh^m x}{x^{2m}}$ for $x \in R\backslash\{0\}$ and x_1, x_3 be solutions of $\lambda_1 = \Phi(x)$ lying in the intervals $(0, x_2)$ and (x_2, ∞) respectively, where x_2 is a positive solution of the equation $\tanh x = \frac{mx}{2m+1}$.

1- If $0 < \lambda < \lambda_1$, then the fixed points $r_1 \in (0, x_1)$ of $f_\lambda(x)$ and $r_2 \in (x_3, \infty)$ of $f_\lambda(x)$ are repelling.

2- If $\lambda = \lambda_1$, then the fixed point x_1 of $f_\lambda(x)$ is indifferent and the fixed point x_3 of $f_\lambda(x)$ is repelling.

3- If $\lambda_1 < \lambda < \lambda_2$, then the fixed point $a_\lambda \in (x_1, x_2)$ of $f_\lambda(x)$ is attracting and the fixed point $r_\lambda \in (x_2, x_3)$ of $f_\lambda(x)$ is repelling.

4- If $\lambda = \lambda_2$, then the fixed point x_2 of $f_\lambda(x)$ is indifferent.

Proof: since the derivative of the function $f_\lambda(x)$ is given by

$$f'_\lambda(x) = \lambda \frac{\left[m(x \sinh^{m-1} x \cosh x - 2\sinh^m x\right]}{x^{2m+1}}$$

and the fixed points of the function $f_\lambda(x)$ are solution of $\lambda = \dfrac{x^{2m+1}}{\sinh^m x}$, it follows that the multiplier $f'_\lambda(x_f)$ of the fixed point x_f is given by

$$\left|f'_\lambda(x_f)\right| = m\left|x_f \coth(x_f) - 2\right| \qquad (3.4)$$

Let

$$G(x) = \begin{cases} m\,(x \coth x - 2) & \text{for } x \neq 0 \\ -2 & \text{for } x = 0 \end{cases}$$

The function G(x) is differentiable and its derivative is given by

$$G'(x) = \begin{cases} m\,(\coth x - x\,\mathrm{csch}^2 x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Since, $G'(x) \neq 0$, $G'(0) = 0$ and $G''(0) > 0$, the function G(x) has exactly one minima at x = 0 and the minimum value is (-2). Since $G'(x) > 0$ for x ϵ (0, ∞) and $G'(x) < 0$ for x ϵ (-∞, 0), thus G(x) increases for (0, ∞) and decreases for (-∞, 0). Therefore, it follows that the function $|G(x)|$ (Fig. 2) satisfies

$$|G(x)| = \begin{cases} <1 & \text{for } x \in (-x_2, -x_1) \cup (x_1, x_2) \\ 1 & \text{for } x = \pm x_1, \pm x_2 \\ >1 & \text{for } x \in (-\infty, -x_2) \cup (-x_1, 0) \cup (0, x_1) \cup (x_2, \infty) \end{cases}$$

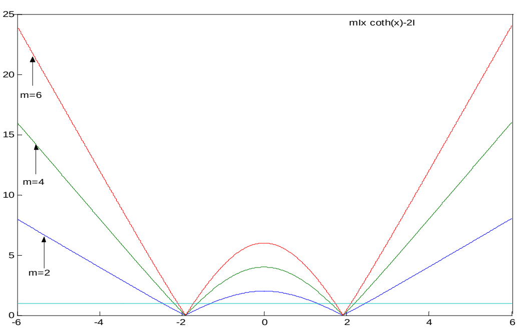

Figure 2: Graph of $|G(x)|$

Consequently, by (3.4), we get that the multiplier $f'_\lambda(x_f)$ of the fixed point x_f satisfies

$$\left|f'_\lambda(x_f)\right| \begin{cases} <1 & \text{for } x \in (-x_2,-x_1) \cup (x_1, x_2) & (3.5) \\ =1 & \text{for } x = \pm x_1, \pm x_2 & (3.6) \\ >1 & \text{for } x \in (-\infty,-x_2) \cup (-x_1,0) \cup (0,x_1) \cup (x_2, \infty) & (3.7) \end{cases}$$

1- $0 < \lambda < \lambda_1$

Since the fixed point r_1 ϵ (0, x_1), by inequality (3.7), $|f'_\lambda(r_1)| > 1$. It therefore follows that r_1 is a repelling fixed point of $f_\lambda(x)$. Similarly, since the fixed point r_2 ϵ (x_3, ∞), by inequality (3.7), $|f'_\lambda(r_2)| > 1$.Consequently, r_2 is a repelling fixed point of $f_\lambda(x)$.

2- $\lambda = \lambda_1$

By equation (3.6), $|f'_\lambda(x_1)| = 1$, therefore, $x = x_1$ is an indifferent fixed point of $f_\lambda(x)$. Further, since $x_3 > x_2 > x_1$, it follows that x_3 ϵ (x_2, ∞). By inequality (3.7), $|f'_\lambda(x_3)| > 1$. It therefore follows that x_3 is a repelling fixed point of $f_\lambda(x)$.

3- $\lambda_1 < \lambda < \lambda_2$

Since the fixed point a_λ ϵ (x_1, x_2), by inequality (3.5), $|f'_\lambda(a_\lambda)| < 1$.Thus a_λ is an attracting fixed point of $f_\lambda(x)$. Further, since the fixed point r_λ ϵ (x_2, x_3), by inequality (3.7) gives that $|f'_\lambda(r_\lambda)| > 1$. It therefore follows that r_λ is a repelling fixed point of $f_\lambda(x)$

4- $\lambda = \lambda_2$

By equation (3.6), $|f'_\lambda(x_2)| = 1$. Consequently, $x = x_2$ is an indifferent fixed point of $f_\lambda(x)$.

Bifurcations of the family H on R\ {0}

In this section, the dynamics of functions f_λ ϵ H on the real line is described. It is proved in the following theorem that there exist parameter values λ_1, $\lambda_2 > 0$ such that bifurcations in the dynamics of the function $f_\lambda(x)$, x ϵ R\T_o occur at $\lambda = \lambda_1$ and $\lambda = \lambda_2$, where T_o is the set of the points that are backward orbits of the pole 0 of the function $f_\lambda(x)$.

<u>**Theorem 4.1.**</u> Let $f_\lambda(x) = \lambda \dfrac{\sinh^m x}{x^{2m}}$ for x ϵ R\ {0}.

a. If $0 < \lambda < \lambda_1$, $f^n_\lambda(x) \to \infty$ as n → ∞ for x ϵ [(-∞, -r_2) ∪ (-α_1, 0) ∪ (0, α_1) ∪ (r_2, ∞)] \$T_o$ and the orbits { $f^n_\lambda(x)$ } are chaotic for x ϵ [(-r_2, -r_1)

$\cup$ ($-r_1,-\alpha_1$) $\cup$ (α_1, r_1) $\cup$ (r_1, r_2)]\T_o, where r_1 and r_2 are repelling fixed points of $f_\lambda(x)$ and α_1 is a positive solution of $f_\lambda(x) = r_2$.

b. If $\lambda = \lambda_1$, $f_\lambda^n(x) \to x_1$ as $n \to \infty$ for $x \in$ [($-x_3,-\alpha_2$) $\cup$ (α_2, x_3)] \$T_o$ and $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in$ [($-\infty$, $-x_3$) $\cup$ ($-\alpha_2$, 0) $\cup$ (0, α_2) $\cup$ (x_3,∞)]\$T_o$, where x_1 is an indifferent fixed point, x_3 is a repelling fixed point of $f_\lambda(x)$ and α_2 is a positive solution of $f_\lambda(x) = x_3$.

c. If $\lambda_1 < \lambda < \lambda_2$, $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $x \in$ [($-r_\lambda$, $-\alpha_3$) $\cup$ (α_3, r_λ)] \$T_o$ and $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in$ [($-\infty$, $-r_\lambda$) $\cup$ ($-\alpha_3$, 0) $\cup$ (0, α_3) $\cup$ (r_λ, ∞)]\$T_o$, where a_λ is an attracting fixed point, r_λ is a repelling fixed point of $f_\lambda(x)$ and α_3 is a positive solution of $f_\lambda(x) = r_\lambda$.

d. If $\lambda = \lambda_2$, $f_\lambda^n(x) \to x_2$ as $n \to \infty$ for $x \in$ [($-x_2$, $-\alpha_4$) $\cup$ (α_4, x_2)]\$T_o$ and $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in$ [($-\infty$, $-x_2$) $\cup$ ($-\alpha_4$, 0) $\cup$ (0, α_4) $\cup$ (x_2, ∞)]\$T_o$, where x_2 is an indifferent fixed point of $f_\lambda(x)$ and α_4 is a positive solution of $f_\lambda(x) = x_2$.

e. If $\lambda > \lambda_2$, $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for all $x \in R$\$T_o$.

Proof. Define the function $t_\lambda(x) = f_\lambda(x) - x$ for $x \in R\setminus\{0\}$. It is easily seen that the function $t_\lambda(x)$ is continuously differentiable for $x \in R\setminus\{0\}$. Note that the fixed points of the function $f_\lambda(x)$ are zeros of the function $t_\lambda(x)$.

a. If $0 < \lambda < \lambda_1$, by theorem 3.1, the function $f_\lambda(x)$ has only two repelling fixed points r_1 and r_2. Since $t'_\lambda(r_1) < 0$ and in a neighborhood of r_1 the function $t'_\lambda(x)$ is continuous, $t'_\lambda(x) < 0$ in some neighborhood of r_1. Therefore, $t_\lambda(x)$ is decreasing in a neighborhood of r_1. By the continuity of the function $t_\lambda(x)$ for sufficiently small $\delta_1 > 0$, $t_\lambda(x) > 0$ in ($r_1 - \delta_1$, r_1) and $t_\lambda(x) < 0$ in (r_1, $r_1+\delta_1$). Further, since $t'_\lambda(r_2) > 0$ and in a neighborhood of r_2 the function $t'_\lambda(x)$ is continuous, $t'_\lambda(x) > 0$ in some neighborhood of r_2. Therefore, $t_\lambda(x)$ is increasing in a neighborhood of r_2. By the continuity of $t_\lambda(x)$, for sufficiently small $\delta_2 > 0$, $t_\lambda(x) > 0$ in (r_2, $r_2+\delta_2$) and $t_\lambda(x) < 0$ in ($r_2-\delta_2$, r_2) since $t_\lambda(x) \neq 0$ in (0, r_1) $\cup$ (r_1, r_2) $\cup$ (r_2, ∞) it now follows that $t_\lambda(x) > 0$ in (0, r_1) $\cup$ (r_2, ∞) and $t_\lambda(x) < 0$ in (r_1, r_2) . Thus,

$$t_\lambda(x) \begin{cases} > 0 & , x \in (0, r_1) \cup (r_2, \infty) \\ < 0 & , x \in (r_1, r_2) \end{cases} \qquad (4.1)$$

The dynamics of the function $f_\lambda(x)$ is now described by the following cases:

Case-i ($x \in$ [($-\infty$, $-r_2$) $\cup$ ($-\alpha_1$, 0) $\cup$ (0, α_1) $\cup$ (r_2, ∞)]\T_o):
By (4.1), it follows that, for $x \in$ (r_2, ∞), $f_\lambda(x) > x$. Since the function $f_\lambda(x)$ is increasing for $x \in$ (r_2, ∞), $f_\lambda^n(x) \to \infty$ as $n \to \infty$. Further, since $f(\alpha_1) = r_2$ and $f_\lambda(x)$ is decreasing in (0, α_1), the function $f_\lambda(x)$ maps the interval (0, α_1) into (r_2, ∞). Now, using the above arguments, we get $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in$ (0, α_1). Next, since $f_\lambda(x)$ is an even function, using the above arguments again, we get $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in$ [($-\infty$, $-r_2$) $\cup$ ($-\alpha_1$, 0)]\$T_o$.

Case-ii ($x \in$ [($-r_2$, $-r_1$) $\cup$ ($-r_1,-\alpha_1$) $\cup$ (α_1, r_1) $\cup$ (r_1, r_2)]\T_o):
Since there is no attractor to attract the system dynamics, the dynamical system will keep moving indefinitely. Therefore, orbits of $f_\lambda(x)$ are chaotic for $x \in$ (α_1, r_1) $\cup$ (r_1, r_2). Further, since $f_\lambda(x)$ is an even function, using the above arguments again, orbits of the function $f_\lambda(x)$ are chaotic for $x \in$ [($-r_2$, $-r_1$) $\cup$ ($-r_1,-\alpha_1$)]\T_o.

b- If $\lambda = \lambda_1$, by Theorem 3.1, the function $f_\lambda(x)$ has an indifferent fixed point x_1 and a repelling fixed point x_3. Since $t'_\lambda(x_1) < 0$ and in a neighborhood of x_1 the function $t'_\lambda(x)$ is continuous, $t'_\lambda(x) < 0$ in some neighborhood of x_1. Therefore, $t_\lambda(x)$ is decreasing in a neighborhood of x_1. By the continuity of $t_\lambda(x)$, for sufficiently small $\delta_1 > 0$, $t_\lambda(x) > 0$ in ($x_1-\delta_1$, x_1) and $t_\lambda(x) < 0$ in (x_1,$x_1+\delta_1$). Further, since $t'_\lambda(x_3) > 0$ and in a neighborhood of x_3 the function $t'_\lambda(x)$ is continuous, $t'_\lambda(x) > 0$ in some neighborhood of x_3. Therefore, $t_\lambda(x)$ is increasing in a neighborhood of x_3. By the continuity of $t_\lambda(x)$, for sufficiently small $\delta_2 > 0$, $t_\lambda(x) > 0$ in $(x_3, x_3+\delta_2)$ and $t_\lambda(x) < 0$ in $(x_3-\delta_2, x_3)$. Since $t_\lambda(x) \neq 0$ in $(0, x_1) \cup (x_1, x_3) \cup (x_3, \infty)$, it now follows that $t_\lambda(x) > 0$ in $(0, x_1) \cup (x_3, \infty)$ and $t_\lambda(x) < 0$ in (x_1, x_3). Thus,

$$t_\lambda(x) \begin{cases} > 0 & , x \in (0, x_1) \cup (x_3, \infty) \\ < 0 & , x \in (x_1, x_3) \end{cases} \quad (4.2)$$

The dynamics of the function $f_\lambda(x)$ is now described by the following cases:

Case-i ($x \in [(-x_3,-\alpha_2) \cup (\alpha_2, x_3)]\backslash T_o$):

By (4.2), it follows that $f'_\lambda(x) < 1$ for $x \in (\alpha_2, x_3)$, $f'_\lambda(x_1) = 1$ and $f'_\lambda(x) > 1$ for $x > x_3$, it follows that, using Mean Value Theorem, $|f'_\lambda(x) - x_1| < |x - x_1|$ for $x \in (\alpha_2, x_3)$. Therefore, $f^n_\lambda(x) \to x_1$ as $n \to \infty$ for $x \in (\alpha_2, x_1)$. Further, since $f_\lambda(x)$ is an even function, using the above arguments again, $f^n_\lambda(x) \to x_1$ as $n \to \infty$ for $x \in [(-x_3,-\alpha_2)\backslash T_o$.

Case-ii ($x \in [(-\infty, -x_3) \cup (-\alpha_2, 0) \cup (0, -\alpha_2) \cup (x_3, \infty)]\backslash T_o$):

By (4.2), it follows that, for $x \in (x_3, \infty)$, $f_\lambda(x) > x$. Since $f_\lambda(x)$ is increasing for $x \in (x_3, \infty)$, so that $f^n_\lambda(x) \to \infty$ as $n \to \infty$. Further, since $f_\lambda(\alpha_2) = x_3$ and $f_\lambda(x)$ is decreasing in $(0, \alpha_2)$, so that $f_\lambda(x)$ maps the interval $(0, \alpha_2)$ into (x_3, ∞). Now, using the above arguments, we get $f^n_\lambda(x) \to \infty$ as $n \to \infty$ for $x \in (0, \alpha_2)$. Next, since $f_\lambda(x)$ is an even function, using the above arguments again, $f^n_\lambda(x) \to \infty$ as $n \to \infty$ for $x \in (-\infty, -x_3)\backslash T_o$.

c. If $\lambda_1 < \lambda < \lambda_2$, by Theorem 3.1, the function $f_\lambda(x)$ has an attracting fixed point a_λ and a repelling fixed point r_λ. Since, $t'_\lambda(a_\lambda) < 0$ and in a neighborhood of a_λ the function $t'_\lambda(x)$ is continuous, $t'_\lambda(x) < 0$ in some neighborhood of a_λ. Therefore, $t_\lambda(x)$ is decreasing in a neighborhood of a_λ. By the continuity of $t_\lambda(x)$, for sufficiently small $\delta_1 > 0$, $t_\lambda(x) > 0$ in $(a_\lambda-\delta_1, a_\lambda)$ and $t_\lambda(x) < 0$ in $(a_\lambda, a_\lambda+\delta_1)$. Further, since $t'_\lambda(r_\lambda) > 0$ and in a neighborhood of r_λ the function $t'_\lambda(x)$ is continuous, $t'_\lambda(x) > 0$ in some neighborhood of r_λ. Therefore, $t_\lambda(x)$ is increasing in a neighborhood of r_λ. By the continuity of $t_\lambda(x)$, for sufficiently small $\delta_2 > 0$, $t_\lambda(x) > 0$ in $(r_\lambda, r_\lambda+\delta_2)$ and $t_\lambda(x) < 0$ in $(r_\lambda-\delta_2, r_\lambda)$. Since $t_\lambda(x) \neq 0$ in $(0, a_\lambda) \cup (a_\lambda, r_\lambda)$, it now follows that $t_\lambda(x) > 0$ in $(0, a_\lambda) \cup (r_\lambda, \infty)$ and $t_\lambda(x) < 0$ in (a_λ, r_λ). Thus

$$t_\lambda(x) \begin{cases} > 0 & , x \in (0, a_\lambda) \cup (r_\lambda, \infty) \\ < 0 & , x \in (a_\lambda, r_\lambda) \end{cases} \quad (4.3)$$

The dynamics of the function $f_\lambda(x)$ is now described by the following cases:

Case-i ($x \in [(-r_\lambda, -\alpha_3) \cup (\alpha_3, r_\lambda)]\backslash T_o$):

Let x_o be a positive solution of the equation $f'(x) = 0$. Thus $f''(x_o) > 0$, hence $f_\lambda(x)$ has a minimum at x_o. Since $|f'_\lambda(a_\lambda)| < 1$, $f'_\lambda(x_o) = 0$, $f'_\lambda(x)$ is increasing for $x > 0$ and $a_\lambda < x_o$, there exists a point $b \in [x_o, r_\lambda]$ such that $|f'_\lambda(c)| < 1$ for all $c \in [a_\lambda, b] \supset [a_\lambda, x_o]$. Using Mean Value Theorem, it follows that $|f_\lambda(x) - f_\lambda(a_\lambda)| < |x - a_\lambda|$ for $x \in [a_\lambda, b]$. Consequently, $f^n_\lambda(x) \to a_\lambda$ as $n \to \infty$ for $x \in [a_\lambda, b]$. For each $x \in [b, r_\lambda)$ the forward orbits contain a point from $[a_\lambda, b]$. Therefore, same as above, $f^n_\lambda(x) \to a_\lambda$ as $n \to \infty$ for all $x \in [b, r_\lambda)$. Hence $f^n_\lambda(x) \to a_\lambda$ as $n \to \infty$ for $x \in [a_\lambda, r_\lambda)$. Again, since $f_\lambda(\alpha_3) = r_\lambda$ and $f_\lambda(x)$ is decreasing in the interval $(\alpha_3, a_\lambda]$, $f_\lambda(x)$ maps the interval (α_3, a_λ) into $[a_\lambda, r_\lambda)$. Therefore, using the above arguments again, $f^n_\lambda(x) \to a_\lambda$ as $n \to \infty$ for $x \in$

(α_3, a_λ]. Thus, $f_\lambda^n(x) \to a_\lambda$ as n → ∞ for x ∈ (α_3, r_λ).Further, since $f_\lambda(x)$ is an even function, using the above arguments again, $f_\lambda^n(x) \to a_\lambda$ as n → ∞ for x ∈ (-r_λ, -α_3)\T$_o$.

Case-ii (x ∈ [(-∞, -r_λ) ∪ (-α_3, 0) ∪ (0, α_3) ∪ (r_λ, ∞)]\T$_o$):

By (4.3), it follows that, for x ∈ (r_λ, ∞), $f_\lambda(x)$ > x and $f_\lambda'(x)$ > 1 for x > r_λ. Therefore, $f_\lambda^n(x)$ → ∞ as n→ ∞. Further, since f_λ(α_3) = r_λ and $f_\lambda(x)$ is decreasing from ∞ to f_λ(α_3) as x increases from 0 to α_3, $f_\lambda(x)$ maps the interval (0,α_3) into the interval (r_λ, ∞).Therefore, using the above arguments again, $f_\lambda^n(x) \to \infty$ as n→ ∞ for x ∈ (0, α_3). Furthermore, since $f_\lambda(x)$ is an even function, using the above arguments, $f_\lambda^n(x)$ → ∞ as n → ∞ for x ∈ [(-∞, -r_λ) ∪ (-α_3, 0)]\T$_o$.

d. If λ = λ_2, by Theorem 3.1, the function $f_\lambda(x)$ has an indifferent fixed point at x_2. Since $t_\lambda'(x_2)$ = 0 and $t_\lambda''(x_2)$ > 0, so that $t_\lambda(x)$ has minima at x_2. Since $t_\lambda(x_2)$ = 0, $t_\lambda(x)$> 0 in a neighborhood of x_2. By the continuity of $t_\lambda(x)$ for sufficiently δ > 0, $t_\lambda(x)$ > 0 in (x_2-δ, x_2) ∪ (x_2, x_2+δ). Since $t_\lambda(x) \neq 0$ in (0, x_2) ∪ (x_2, ∞), it now follows that $t_\lambda(x)$> 0 in (0, x_2) ∪ (x_2, ∞). Thus,

$$t_\lambda(x) > 0 \quad \text{for } x \in (0, x_2) \cup (x_2, \infty) \qquad (4.4)$$

The dynamics of the function $f_\lambda(x)$ is now described by the following cases:

Case-i (x ∈ [(-x_2, -α_4) ∪ (α_4, x_2)]\T$_o$):

By (4.4), it follows that, $f_\lambda'(x)$<1 for x ∈ (α_4,x_2), $f_\lambda'(x_2)$ = 1 and $f_\lambda'(x)$> 1 for x > x_2, it follows that $|f_\lambda(x) - x_2| < |x - x_2|$ for x ∈ (α_4, x_2).

Therefore, $f_\lambda^n(x) \to x_2$ as n→ ∞ for x ∈ (α_4, x_2). Again, since $f_\lambda(x)$ is an even function, $f_\lambda^n(x) \to x_2$ as n → ∞ for x ∈ (-x_2,-α_4)\T$_o$.

Case-ii (x ∈ [(-∞, -x_2) ∪ (-α_4, 0) ∪ (0, α_4) ∪ (x_2, ∞)]\T$_o$):

By (4.4), it follows that, for x ∈ (x_2, ∞), $f_\lambda(x)$>x and $f_\lambda'(x)$> 1 for x > x_2. Therefore, $f_\lambda^n(x) \to \infty$ as n→ ∞.Next, since f_λ(α_4) = x_2 and $f_\lambda(x)$ is decreasing from to ∞ to f_λ(α_4) as x increases from 0 to α_4, $f_\lambda(x)$ maps the interval (0, α_4) into the interval (x_2, ∞). Therefore, using the above arguments again, $f_\lambda^n(x)$ → ∞ as n→ ∞ for x ∈ (0, α_4). Further, since $f_\lambda(x)$ is an even function, $f_\lambda^n(x)$ → ∞ as n→ ∞ for x ∈ [(-∞, -x_2) ∪ (-α_4, 0)]\T$_o$.

e. If λ > λ_2, by proposition 3.1, the function $f_\lambda(x)$ has no fixed points. We observed that $t_\lambda(x)$ > 0 for all x ∈ R\ {0}. Since $f_\lambda(x)$ > x for x > 0, so that $f_\lambda^n(x) \to$ ∞ as n → ∞ for x ∈ (0, ∞). Since $f_\lambda(x)$ is an even function, $f_\lambda^n(x) \to \infty$ as n → ∞ for x ∈ (-∞, 0)\T$_o$. Thus, $f_\lambda^n(x) \to \infty$ as n → ∞, x ∈ [(-∞, 0) ∪ (0, ∞)]\T$_o$.

It follows by Theorem 4.1 that bifurcation in the dynamics of the function $f_\lambda(x)$ for all x ∈ R\{0} occur at the two critical parameter values λ = λ_1, λ_2, where $\lambda_1 = \dfrac{x_1^{2m+1}}{\sinh^m x_1}$ and $\lambda_2 = \dfrac{x_2^{2m+1}}{\sinh^m x_2}$ such that x_1 and x_2 are the unique positive real roots of the equations $\tanh x = \dfrac{mx}{2m-1}$ and $\tanh x = \dfrac{mx}{2m+1}$ respectively.

Dynamics on $\hat{\mathrm{C}}$

The dynamics of functions from the one–parameter family H is indicated by describing the dynamics of f_λ ∈ H for z ∈ $\mathbb{C}$ in the present section. This includes the study of Julia set of f_λ in the extended complex plane $\hat{\mathrm{C}}$ for different values of λ ∈ R. If the singular values of transcendental function is bounded, then the Julia set is a closure of escaping points of function under iterations [11]; i.e., J(f) = I(f), where I(f) ={ z ∈ $\mathbb{C}$: $f^n(z) \to \infty$ as n→ ∞ and $f^n(z) \neq \infty$}

Proposition 5.1 let $f_\lambda \in$ H and $0 < \lambda < \lambda_1$. Then the Julia set J(f_λ) contains both real and imaginary axes.

Proof By Theorem 4.1(a), $f_\lambda^n(x) \to \infty$ for x $\in$ $[(-\infty,-r_2) \cup (-\alpha_1, 0) \cup (0, \alpha_1) \cup (r_2, \infty)] \backslash T_o$ and the orbits { $f_\lambda^n(x)$ } are chaotic for x $\in$ $[(-r_2,-r_1) \cup (-r_1,-\alpha_1) \cup (\alpha_1, r_1) \cup (r_1, r_2)] \backslash T_o$, it follows that $R \backslash T_o \subset J(f)$. In addition, since the poles and their preimages are contained in the Julia set, thus $R \subset J(f)$. Now, we have f_λ maps the real interval into iR. Then $R \cup iR \subset J(f)$. Hence the forward orbits of all singular values tend to ∞. Therefore, J(f) contains both real and imaginary axes.

Proposition 5.2 let $f_\lambda \in$ H and $\lambda = \lambda_1$. Then

1) The Fatou set F(f) contains a unique parabolic domain.

2) The Julia set contains the intervals $(-\infty,-x_3)$, $(-x_1, 0)$, $(0, x_1)$ and (x_3, ∞) and the Fatou set contains the intervals $(-x_3,-x_1)\backslash T_o$ and (x_1, x_3), where x_1 is indifferent fixed point, x_3 is a repelling fixed point of $f_\lambda(x)$.

Proof

1) Let $U_1 = \{z \in \mathbb{C} ;\ f_\lambda^n(x) \to x_1$ as $n \to \infty\}$. By Theorem 3.1(2), it follows that $f_\lambda(z)$ has an indifferent fixed point at $x = x_1$. Since, by Theorem 4.1(b), $f_\lambda^n(x) \to x_1$ as $n \to \infty$ for $x \in (x_1,x_3)\backslash T_o$ and $f_\lambda^n(x) \to \infty$ for $x \in (0,x_1)$, the indifferent fixed point x_1 lies on the boundary of U_1. Thus, U_1 is a parabolic domain in the Fatou set of $f_\lambda(z)$. Again, by Theorem 4.1(b), it follows that the forward orbits of all singular values either tend to x_1 or tend to ∞. Therefore, by Theorem 1.1, F(f) does not contain any parabolic domain other than U_1.

2) By Theorem 4.1(b), for $\lambda = \lambda_1$. $f_\lambda^n(x) \to x_1$ for $x \in [(-x_3,-x_1) \cup (x_1,x_3)]\backslash T_o$. Therefore, the intervals $(-x_3,-x_1)\backslash T_o$ and (x_1, x_3) are contained in the parabolic domain U_1. Since Fatou set contains parabolic domains, the intervals $(-x_3,-x_1)\backslash T_o$ and (x_1, x_3) belong to the Fatou set. Further, $f_\lambda^n(x) \to \infty$ for $x \in [(-\infty,-x_3) \cup (-x_1, 0) \cup (0, x_1) \cup (x_3, \infty)]\backslash T_o$. Thus, these intervals are contained in the Julia set of $f_\lambda(z)$. Since pole and preimages of the pole also belong to the Julia set.

Proposition 5.3 let $f_\lambda \in$ H and $\lambda_1 < \lambda < \lambda_2$. Then

1) The Fatou set F(f) does not contain any basin or parabolic domain except the basin of attraction of the real attracting fixed point a_λ of $f_\lambda(x)$.

2) The intervals $[(-r_\lambda, -\alpha_3) \cup (\alpha_3, r_\lambda)]\backslash T_o$ are contained in F(f) and the intervals $(-\infty,-r_\lambda)$, $(-\alpha_3, 0)$, $(0, \alpha_3)$, (r_λ, ∞) are contained in J(f).

Proof

1) By Theorem 3.1(3), $f_\lambda(z)$ has a real attracting fixed point a_λ. Let $A(a_\lambda) = \{z \in \mathbb{C} ;\ f_\lambda^n(z) \to a_\lambda$ as $n \to \infty\}$ be the basin of attraction of the attracting fixed point a_λ. For any point $z \in A(a_\lambda)$, the sequence of iterates { $f_\lambda^n(z)$ } tends to a_λ as $n \to \infty$ so that the sequence of iterates { $f_\lambda^n(z)$ } forms normal family at z. Consequently, $z \in F(f)$. Thus, $A(a_\lambda) \subset F(f)$. Further, by Theorem 4.1(c), it follows that the forward orbits of all singular values either tend to a_λ or tend to ∞. Therefore, by Theorem1.1, F(f) does not contain the basin of attractions other than $A(a_\lambda)$. That F(f) does not contain any parabolic domains follows similarly using Theorem 1.1.

2) By Theorem 4.1(c), $f_\lambda^n(x) \to \infty$ for $x \in [(-\infty,-r_\lambda) \cup (-\alpha_3, 0) \cup (0, \alpha_3) \cup (r_\lambda, \infty)]\backslash T_o$. Therefore, the intervals $[(-\infty,-r_\lambda) \cup (-\alpha_3, 0)]\backslash T_o$, $(0, \alpha_3)$ and (r_λ, ∞) belong to the Julia set of $f_\lambda(z)$. Since pole and preimages of the pole lie in the Julia set, it now follows that the Julia set contains the intervals $(-\infty,-r_\lambda)$, $(-\alpha_3, 0)$, $(0, \alpha_3)$ and (r_λ,∞).Again, by Theorem 4.1(c), $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $x \in [(-r_\lambda,-\alpha_3) \cup (\alpha_3,r_\lambda)]\backslash T_o$. Therefore, it follows that the intervals $(-r_\lambda,-\alpha_3)\backslash T_o$ and (α_3,r_λ) are contained in the basin of attraction $A(a_\lambda)$. Since the Fatou set contains basin of attractions, the intervals $(-r_\lambda,-\alpha_3)\backslash T_o$ and (α_3, r_λ) belong to the Fatou set of $f_\lambda(z)$.

Proposition 5. 4 let $f_\lambda \in$ H and $\lambda = \lambda_2$. Then

1) The Fatou set F(f) contains a unique parabolic domain.

2) The Julia set J(f) contains the intervals $(-\infty,-x_2)$, $(-\alpha_4, 0)$, $(0, \alpha_4)$ and (x_2, ∞) and the Fatou set F(f) contains the intervals $(-x_2, -\alpha_4)\backslash T_o$ and (α_4, x_2)

Proof The proof of proposition is analogous to that of proposition 5.2 for the case $\lambda = \lambda_1$ and is hence omitted.

Proposition 5.5 let $f_\lambda \in$ H and $\lambda > \lambda_2$. Then, the Julia set J(f) contains both real and imaginary axes.

Proof By Theorem 4.1(e), $f_\lambda^n(x) \rightarrow \infty$ for all x $\in$ R\T$_o$, it follows that R\T$_o$ $\subset$ J(f). Since $f_\lambda(x)$ maps imaginary axis on real axis and $f_\lambda^n(x) \rightarrow \infty$ for all x $\in$ R\T$_o$, it gives that iR\iT$_o$ $\subset$ J(f). Since the asymptotic value 0 is also a pole of $f_\lambda(z)$ $f_\lambda(z)$. 0 $\in$ J(f) and since preimages of pole are contained in Julia set, T$_o$ $\subset$ J(f). Therefore, J(f) contains both real and imaginary axes.

Proposition 5.6 For $\lambda > \lambda_2$, Fatou set cannot have any basin of attraction, parabolic domain.

Proof Since $f_\lambda^n(x) \rightarrow \infty$ for all x $\in$ R\T$_o$, the forward orbit of critical values on real axis tend to ∞. Further, the asymptotic value 0 is also a pole of $f_\lambda(z)$ so that orbit of 0 termintes. Therefore, Fatou set cannot have any basin of attraction, parabolic domain.

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