STABILITY OF A PREY-PREDATOR MODEL WITH *SIS* EPIDEMIC DISEASE IN PREY

Dahlia Khaled Bahlool
Department of Mathematics, College of Science, University of Baghdad, Baghdad-Iraq.

Abstract

In this paper, an eco-epidemiolo-gical model consisting of prey-predator model with SIS disease is proposed and analyzed. The existence and the stability analysis of all possible equilibrium points are carried out. The persistent conditions of the proposed system are established. The global dynamics of the system is studied numerically.

أستقرارية نموذج الفريسة والمفترس مع المرض الوبائي (*SIS*) في الفريسة

داليا خالد بهلول
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق.

الخلاصة

في هذا البحث، تم اقتراح ودراسة نموذج بيئي وبائي يتكون من نظام الفريسة والمفترس مع وجود مرض من النوع (*SIS*) في الفريسة. شروط وجود واستقرارية النقاط الثابته بحثت. شروط الاصرار للنموذج المقترح وجدت. واخيرا الديناميكية الشاملة للنظام درست عدديا.

Introduction

Initially, models for ecological interactions and models for infectious diseases were developed separately. It has been observed that a strong interaction may arise between these factors. Indeed, the main aspects regarding population dynamics concern with the effects of infectious diseases in regulating natural populations, decreasing their population sizes, reducing their natural fluctuations, or causing destabilizations of equilibria into oscillations of the population states. The study of interacting species in which a disease spreads is known as eco-epidemiology. The study of eco-epidemio-logy has important ecological significance as it involves persistence-extinction thre-shold of each population in systems of two or more interacting species subjected to disease [1-4].

It is well known that, when the infective individual still infective and the susceptible individual still susceptible for all the time, the disease is called SI disease. However, when the infection does not lead to immunity, so that infective becomes susceptible again after recovery, the disease is called an SIS disease. Finally, when infective has permanent immunity after recovery, the disease is called an SIR disease. SIS epidemic model is one of the most basic and most important models in describing the spread of many diseases and hence it attached many author's attention, see [5] and the references therein.

Recently, numbers of ecological models involving SI and SIR epidemic disease have been considered, see [6], [7] and the references therein. In this paper an eco-epidemiological model consisting of Beddington-DeAnglis prey-predator model involving SIS epidemic disease has been proposed and analyzed.

Model Formulation

In this section, a system consists of a prey with density $N(t)$, which may has disease, interacting with predator with density $y(t)$ is proposed depending on the following assumptions.

A1: In the absence of disease and predation, the prey population density grows according to a logistic curve with carrying capacity $K(K>0)$ and an intrinsic birth rate constant $r(r>0)$. Hence the evolution of prey population can be described as:

$$\frac{dN}{dt}=rN(1-\frac{N}{K}) \qquad (1)$$

A2: In the presence of disease we assume that the total prey population N is composed of two population classes: one is the class of susceptible prey denoted by S, and the other is the class of the infected prey, denoted by I. Therefore, at any time *t*, the total density of prey population is

$$N(t)=S(t)+I(t) \qquad (2)$$

A3: It is assumed that only susceptible prey S is capable of reproducing with logistic law Eq. (1), the infected prey I is removed before having the possibility of reproducing. However, the infected prey population I still contributes with S to population growth toward the carrying capacity.

A4: A susceptible prey S becomes infected at a rate proportional to SI and an infected prey can recover and becomes susceptible again at a rate γI. Moreover, the infected prey population faces the natural death due to the effect of disease at a rate μI. Therefore the evolution equations for the susceptible prey S and infected prey I, according to Eq.(1), A3 and A4 are

$$\left.\begin{aligned}\frac{dN}{dt}&=rS(1-\frac{N}{K})-\lambda SI+\gamma I\\ \frac{dI}{dt}&=\lambda SI-\gamma I-\mu I\end{aligned}\right\} \qquad (3)$$

where $N=S+I$ and λ,γ,μ are the positive parameters, which stand for infection rate constant, recover rate constant and natural death rate constant respectively.

A5: In the presence of predator both the prey populations (susceptible and infected) will be consumed, according to Beddington-DeAnglis type of functional response, due to the effect of predation. However, the predator population will be reproduced depending on the availability of food (i.e. prey populations). Finally, it is assumed that the predator population decaying due to either the natural death with constant rate β ($\beta>0$) or the amount of infected prey consumed with proportionality constant rate θ ($\theta>0$).

Therefore, by using the above assumptions, the dynamic of prey-predator model with SIS disease in the prey can be represented in the following set of equations.

$$\frac{dS}{dt}=rS\left(1-\frac{S+I}{K}\right)-\lambda SI+\gamma I-q_1Y \qquad (4a)$$

$$\frac{dI}{dt}=\lambda SI-\gamma I-\mu I-q_2Y \qquad (4b)$$

$$\frac{dY}{dt}=(e_1q_1+e_2q_2)Y-\beta Y-\theta q_2Y \qquad (4c)$$

where $q_1=\frac{a_1S}{bY+S+\alpha I+c}$ and $q_2=\frac{a_2I}{bY+S+\alpha I+c}$, while the parameters a_1,a_2,b,α,c,e_1 and e_2 are positive constants represent respectively the attack rate of susceptible prey, attack rate of infected prey, the predator encounter rate, the predator's preference rate, the half saturation constant, the predator's conversion rate from S and I respectively. Since the density of any species can't be negative, therefore we will solve system (4) with the following initial condition $S(0)\geq 0$, $I(0)\geq 0$ and $Y(0)\geq 0$.

It is easy to verify that all the functions on the right hand side of Eq. (4)(a-c) are continuous and have continuous partial derivatives with respect to dependent variables S, I and Y. Accordingly they are Lipichitzian functions and hence system (4) has a unique solution for each non-negative initial condition. Further the boundedness of the system is shown in the following theorem.

Theorem 1. All the solutions of system (4) which initiate in the R_+^3 are uniformly bounded.

Proof: Since we have $N(t)=S(t)+I(t)\ \forall t\geq 0$ and $\frac{dN}{dt}=rN(1-\frac{N}{K})$, hence we obtain:

$$N(t)=\frac{N_0K}{N_0+(K-N_0)e^{-rt}}$$

Where $N(0)=N_0=S_0+I_0$. Hence it is easy to verify that $N(t)\leq M_1$, $\forall t\geq 0$ where $M_1=\max\{N_0,K\}$. Consequently, $S(t)<M_1$, $\forall t\geq 0$.

Assume that, $W=S+I+Y$ then we obtain

$$\frac{dW}{dt}=\frac{dS}{dt}+\frac{dI}{dt}+\frac{dY}{dt}\leq\gamma-\delta W$$

Where $\gamma=(1+r)M_1$ and $\delta=\min\{1,\mu,\beta\}$ then we have

$$\frac{dW}{dt}+\delta W\leq\gamma$$

Now by solving the above linear differential inequality [8], we get that

$$W(t) \le \frac{\gamma}{\delta} + \left(w_0 - \frac{\delta}{\gamma}\right)e^{-\delta t}$$

here $W(0) = w_0$. Hence $W(t) \le M_2,\ \forall t \ge 0$ where $M_2 = \max\{w_0, \frac{\delta}{\gamma}\}$.

Hence, all the solution of system (4) are uniformly bounded and the proof is complete.

The stability analysis

In this section, we study the existence of equilibrium points of system (4) and their local stability. It is observed that, system (4) has at most five nonnegative equilibrium points.

The equilibrium points $E_0 = (0,0,0)$ and $E_1 = (K,0,0)$ are always exist.

The disease-free equilibrium point $E_2 = (\overline{S}, 0, \overline{Y})$, where

$$\overline{S} = \frac{1}{2rbe_1a_1}\Big[a_1K(rbe_1 - e_1a_1 + \beta) + \sqrt{(a_1K)^2(-rbe_1 + e_1a_1 - \beta)^2 + 4rbe_1{a_1}^2\beta cK}\Big] \quad \text{..........(5a)}$$

$$\overline{Y} = \frac{r(\overline{S}+c)(K-\overline{S})}{a_1K - rb(K-\overline{S})} \quad (5b)$$

exists in the $Int.R_+^2$ of $SY-$ plane provided that the following two conditions hold

$$\frac{K(rb-a_1)}{rb} < \overline{S} < K \quad (6a)$$

$$\beta < e_1a_1 \quad (6b)$$

The predator-free equilibrium point $E_3 = (\widetilde{S}, \widetilde{I}, 0)$, where

$$\widetilde{S} = \frac{\gamma+\mu}{\lambda} \quad (7a)$$

$$\widetilde{I} = \frac{r(K-\widetilde{S})\widetilde{S}}{K\mu + r\widetilde{S}} \quad (7b)$$

exists in the $Int.R_+^2$ of $SI-$plane provided that

$$\widetilde{S} < K \quad (8)$$

Finally, the positive equilibrium point $E_4 = (S^*, I^*, Y^*)$, which is known as endemic point, exists in the $Int.R_+^3$ if and only if there is a positive solution to the following set of algebraic equations

$$f_1(S,I,Y) = r\left(1 - \frac{S+I}{K}\right) - \lambda I + \gamma\frac{I}{S} - q_1\frac{Y}{S} = 0$$

$$f_2(S,I,Y) = \lambda S - \gamma - \mu - q_2\frac{Y}{I} = 0$$

$$f_3(S,I,Y) = e_1q_1 + e_2q_2 - \beta - \theta q_2 = 0$$

Straightforward computations give that

$$I^* = \frac{be_1a_1S^*[\lambda S^* - \gamma - \mu] - a_2[(e_1a_1 - \beta)S^* - c\beta]}{a_2[(e_2-\theta)a_2 - \alpha\beta] - ba_2(e_2-\theta)[\lambda S^* - \gamma - \mu]} \quad (10a)$$

$$Y^* = \frac{[(e_2-\theta)ca_2 + ((e_2-\theta)a_2 - e_1a_1\alpha)S^*][\lambda S^* - \gamma - \mu]}{a_2[(e_2-\theta)a_2 - \alpha\beta] - ba_2(e_2-\theta)[\lambda S^* - \gamma - \mu]} \quad \text{....... (10b)}$$

while, S^* is the positive root for the equation

$$f_1(S, I^*, Y^*) = 0 \quad (10c)$$

Obviously, E_4 exists uniquely in the $Int.R_+^3$ provided that there is a positive root for Eq. (10c) that is satisfy the following set of conditions:

$$\frac{a_2[(e_1a_1-\beta)S^* - c\beta]}{e_1a_1S^*[\lambda S^* - \gamma - \mu]} < b < \frac{(e_2-\theta)a_2 - \alpha\beta}{(e_2-\theta)[\lambda S^* - \gamma - \mu]} \quad (11a)$$

$$S^* > \max.\left\{\frac{\gamma+\mu}{\lambda}, \frac{\beta c}{e_1a_1 - \beta}\right\} \quad (11b)$$

$$\text{with } 0 < \alpha < \min.\left\{\frac{(e_2-\theta)a_2}{e_1a_1}, \frac{(e_2-\theta)a_2}{\beta}\right\} \quad (11c)$$

Consequently, if $e_1a_1 > \beta$, the condition (11c) is replaced by

$$0 < \alpha < \frac{(e_2-\theta)a_2}{e_1a_1} \quad (11d)$$

Now, the local stability analysis near the above equilibrium points of system (4) can be summarized as follows:

The Jacobian matrix of system (4) at the equilibrium point E_0 is

$$J(E_0) = \begin{bmatrix} r & 0 & 0 \\ 0 & -(r+\mu) & 0 \\ 0 & 0 & -\beta \end{bmatrix}$$

Accordingly, the eigenvalues of $J(E_0)$ are:

$$\lambda_{0S} = r > 0, \lambda_{0I} = -(\lambda + \mu) < 0, \quad \lambda_{0Y} = -\beta < 0 \quad (12)$$

Where λ_{iu} represents the eigenvalue of $J(E_i)$; $i = 0,1,2,3,4$ that describe the dynamics in the u -direction. Therefore, E_0 is a saddle point with locally stable manifold in the $IY-$plane and with locally unstable manifold in the $S-$direction.

The Jacobian matrix of system (4) at the equilibrium point E_1 is given by

$$J(E_1) = \begin{bmatrix} -r & -r + \gamma - \lambda K & -\frac{a_1K}{c+K} \\ 0 & \lambda K - \gamma - \mu & 0 \\ 0 & 0 & \frac{e_1a_1K}{K+c} - \beta \end{bmatrix}$$

Therefore, the eigenvalues of $J(E_1)$ can be written as

$$\lambda_{1S} = -r < 0, \lambda_{1I} = -(\gamma + \mu) + \lambda K, \quad \lambda_{1Y} = \frac{e_1a_1K}{K+c} - \beta \quad (13a)$$

Clearly, if the following two conditions hold

$$\frac{e_1 a_1 K}{K+c} < \beta \qquad (13b)$$

$$\lambda < \frac{\gamma+\mu}{K} \qquad (13c)$$

then E_1 is the locally asymptotically stable. While, if at least one of conditions (13b) and (13c) violate then E_1 is a saddle point.

Further, the local stability analyses near the equilibrium points E_2 and E_3 of system (4) are discussed in the following two theorems respectively.

Theorem 2. Assume that, the disease-free equilibrium point $E_2 = (\overline{S}, 0, \overline{Y})$ of system (4) exists in the $Int.R_+^2$ of $SY-$plane, so that the following condition holds

$$\frac{a_1 \overline{Y}}{N_1^2} < \frac{r}{K} < \frac{e_1 a_1 b \overline{Y}}{N_1^2} \qquad (14a)$$

Then E_2 is locally asymptotically stable in the $Int.R_+^2$ of $SY-$plane. In addition to condition (14a), if the following condition holds

$$\lambda \overline{S} < (\gamma + \mu) + \frac{a_2 \overline{Y}}{N_1} \qquad (14b)$$

Then E_2 is locally asymptotically stable in R_+^3. Finally, if at least one of conditions (14a), (14b) violates then E_2 is a saddle point.

Proof. According to system (4), the Jacobian matrix of system (4) at the equilibrium point E_2 can be written as: $J(E_2) = (b_{ij})_{3\times3}$; where

$b_{11} = -\frac{r\overline{S}}{K} + \frac{a_1 \overline{S}\overline{Y}}{N_1^2}$, $\quad b_{12} = \frac{-r\overline{S}}{K} + \frac{a_1 \alpha \overline{S}\overline{Y}}{N_1^2} + \gamma - \lambda \overline{S}$,

$b_{13} = -\frac{a_1 \overline{S}^2 + a_1 c \overline{S}}{N_1^2}$, $b_{21} = b_{23} = 0$,

$b_{22} = \lambda \overline{S} - (\gamma + \mu) - \frac{a_2 \overline{Y}}{N_1}$,

$b_{31} = \frac{e_1 a_1 b \overline{Y}^2 + e_1 a_1 c \overline{Y}}{N_1^2}$, $b_{32} = \frac{N_1 (e_2 - \theta) a_2 \overline{Y} - e_1 a_1 \alpha \overline{S}\overline{Y}}{N_1^2}$,

$b_{33} = -\frac{e_1 a_1 b \overline{S}\overline{Y}}{N_1^2}$ with $N_1 = b\overline{Y} + \overline{S} + c$.

Consequently, it is easy to verify that, the eigenvalues of $J(E_2)$ satisfy the following relations:

$$\lambda_{2S} + \lambda_{2Y} = -\frac{r\overline{S}}{K} + \frac{a_1 \overline{S}\overline{Y}}{N_1^2} - \frac{e_1 a_1 b \overline{S}\overline{Y}}{N_1^2} \qquad (15a)$$

$$\lambda_{2S} . \lambda_{2Y} = \frac{e_1 a_1 b \overline{S}\overline{Y}}{N_1^3} (rb\overline{S} N_1 + a_1 c K) > 0 \qquad (15b)$$

$$\lambda_{2I} = \lambda \overline{S} - (\gamma + \mu) - \frac{a_2 \overline{Y}}{N_1} \qquad (15c)$$

Clearly, according to Eq. (15a), if condition (14a) holds, then both the eigenvalues λ_{2S} and λ_{2Y}, which describe the dynamics in the $S-$and $Y-$direction respectively are negative. Hence E_2 is locally asymptotically stable in the $Int.R_+^2$ of $SY-$plane.

Further more, depending on the Eq. (15c) if in addition to condition (14a) condition (14b) holds too, than all the eigenvalues of $J(E_2)$ are negative and hence E_2 is locally asymptotically stable in R_+^3.

Finally, if at least one of conditions (14a), (14b) violate then $J(E_2)$ has at most two positive eigenvalues while the third one still negative and hence E_2 is a saddle point. ■

Theorem 3. Assume that, the predator-free equilibrium point $E_3 = (\widetilde{S}, \widetilde{I}, 0)$ of system (4) exists in the $Int.R_+^2$ of SI-plane, so that the following condition holds

$$\gamma < \frac{\widetilde{S}(K\mu + r\widetilde{S})}{K(K - \widetilde{S})} \qquad (16a)$$

Then E_3 is locally asymptotically stable in the $Int.R_+^2$ of SI-plane. In addition to condition (16a), if the following condition holds

$$\frac{e_1 a_1 \widetilde{S} + (e_2 - \theta) a_2 \widetilde{I}}{N_2} < \beta \qquad (16b)$$

where $N_2 = \widetilde{S} + \alpha \widetilde{I} + c$. Then E_3 is locally asymptotically stable in the R_+^3. Finally, if at least one of the conditions (16a), (16b) violates then E_3 is a saddle point.

Proof. Straightforward computations show that, the eigenvalues of the Jacobian matrix at E_3, i.e. $J(E_3)$, satisfy the following relations:

$$\lambda_{3S} + \lambda_{3I} = -\frac{r\widetilde{S}}{K} + \frac{\gamma \widetilde{I}}{\widetilde{S}} \qquad (17a)$$

$$\lambda_{3S} . \lambda_{3I} = \frac{\lambda r}{K} (K - \widetilde{S}) \widetilde{S} > 0 \qquad (17b)$$

Which is positive under the existence condition (8).

$$\lambda_{3Y} = \frac{e_1 a_1 \widetilde{S} + e_2 a_2 \widetilde{I} - \theta a_2 \widetilde{I}}{N_2} - \beta \qquad (17c)$$

Then by using similar arguments as those in the proof of theorem, the proof follows directly. ■

Finally, in order to study the dynamical behavior near the positive equilibrium point $E_4 = (S^*, I^*, Y^*)$ of system (4) in the $Int.R_+^3$, the Jacobain matrix $J(E_4)$ is computed as follows:

$$J(E_4) = (a_{ij})_{3\times3} \qquad (18)$$

Where $a_{11}=-\frac{rS^*}{K}+\frac{a_1S^*Y^*}{N_3^2}-\frac{\gamma I^*}{S^*}$,

$a_{12}=-\frac{rS^*}{K}+\frac{\alpha a_1S^*Y^*}{N_3^2}+\gamma-\lambda S^*$,

$a_{13}=-\frac{a_1(S^*+\alpha I^*+c)S^*}{N_3^2}<0$,

$a_{21}=\frac{a_2I^*Y^*}{N_3^2}+\lambda I^*>0$, $a_{22}=\frac{\alpha a_2I^*Y^*}{N_3^2}>0$,

$a_{23}=-\frac{a_2(S^*+\alpha I^*+c)I^*}{N_3^2}<0$,

$a_{31}=\frac{(bY^*+c)e_1a_1Y^*+[e_1a_1\alpha-(e_2-\theta)a_2]I^*Y^*}{N_3^2}$,

$a_{32}=\frac{(bY^*+c)(e_2-\theta)a_2Y^*+[(e_2-\theta)a_2-e_1a_1\alpha]S^*Y^*}{N_3^2}$,

$a_{33}=-\frac{b[e_1a_1S^*+(e_2-\theta)a_2I^*]Y^*}{N_3^2}<0$ with

$N_3=bY^*+S^*+\alpha I^*+c$.

Consequently, the characteristic equation of $J(E_4)$, can be written as

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0 \qquad (19)$$

where the coefficients are given by:

$$A_1=-(a_{11}+a_{22}+a_{33})$$
$$A_2=a_{11}(a_{22}-a_{33})-(a_{12}a_{21}+a_{13}a_{31})+a_{22}a_{33}-a_{23}a_{32}$$
$$A_3=a_{11}(a_{23}a_{32}-a_{22}a_{33})+a_{21}(a_{21}a_{33}-a_{23}a_{31})+a_{13}(a_{22}a_{31}-a_{21}a_{32})$$

According to the above, the local stability conditions of the positive equilibrium point E_4 of system (4) can be derived easily as shown in the following theorem.

Theorem 4. Suppose that the positive equilibrium point $E_4=(S^*,I^*,Y^*)$ of system (4) exists in the $Int.R_+^3$. Let the following conditions hold:

$$\left(a_1S^*+a_2\alpha I^*\right)Y^*<\left(\frac{rS^*}{K}+\frac{\gamma I^*}{S^*}\right)N_3^2 \qquad (20a)$$

$$\frac{a_2I^*}{be_1}[\alpha-b(e_2-\theta)]<a_1S^*Y^*<\left(\frac{r}{K}+\lambda\right)\frac{S^*N_3^2}{\alpha}-\frac{\gamma N_3^2}{\alpha} \qquad (20b)$$

$$\frac{(bY^*+c)e_1a_1}{(e_2-\theta)-e_1a_1\alpha}<I^* \qquad (20c)$$

$$B_8>B_9 \qquad (20d)$$

where B_8 and B_9 are given below. Then E_4 is locally asymptotically stable point in the $Int.R_+^3$.

Proof. According to Routh-Hurwitz criterion, all the eigenvalues of $J(E_4)$ have negative real parts, and hence the proof follows, if and only if $A_1>0$, $A_3>0$ and $\Delta=A_1A_2-A_3>0$.

Therefore, by substituting the value of coefficients a_{ij} of $J(E_4)$ and then simplifying the resulting terms, it is easy to verify that A_1, A_3 and Δ can be rewritten as follows:

$$A_1=\frac{1}{N_3^2}[B_1+B_2]$$
$$A_3=\frac{I^*Y^*}{N_3^5}[B_1B_4+B_3(B_5+B_6)+a_1S^*(S^*+\alpha I^*+c)(B_6+B_7)]$$
$$\Delta=\frac{1}{N_3^6}[B_8-B_8]$$

Where

$B_1=\left(\frac{rS^*}{K}+\frac{\gamma I^*}{S^*}\right)N_3^2-a_1S^*Y^*$,

$B_2=Y^*[e_1a_1bS^*-a_2(\alpha-(e_2-\theta)b)I^*]$,

$B_3=\left(\frac{rS^*}{K}+\lambda S^*\right)N_3^2-[a_1\alpha S^*Y^*+\gamma N_3^2]$,

$B_4=[(S^*+c)(e_2-\theta)a_2-e_1a_1\alpha S^*]a_2$,

$B_5=[(e_2-\theta)a_2I^*-e_1a_1(\alpha I^*+c)]a_2$

$B_6=\lambda N_3^2[(bY^*+S^*+c)(e_2-\theta)a_2-e_1a_1\alpha S^*]$,

$B_7=[(e_2-\theta)a_2-e_1a_1\alpha]a_2Y^*$,

$$B_8=B_1^2B_2+B_1B_2^2+I^*Y^*B_2B_4+I^*B_3(a_2Y^*+\lambda N_3^2)(B_1-a_2\alpha I^*Y^*)$$

$$B_9=Y^*(S^*+\alpha I^*+c)\left[\left(\frac{B_5}{a_2}-be_1a_1Y^*\right)[a_1S^*B_1+a_1S^*(B_2+a_2\alpha I^*)+a_2I^*B_3]+\frac{a_1S^*I^*}{\lambda N_3^2}B_6\left(a_2Y^*+\lambda N_3^2\right)\right]$$

Note that, B_1 and B_2 are positive provided that conditions (20a) and (20b) are satisfied respectively . Hence we obtain $A_1>0$.

Also, we have $B_3>0$ provided that condition (20b) holds, while B_4, B_6 and B_7 are positive under the existence condition of E_4. Further, $B_5>0$ under condition (20c). Therefore we obtain $A_3>0$.

Finally, since $B_1-a_2\alpha I^*Y^*>0$ due to condition (20a) and $\frac{B_5}{a_2}-be_1a_1Y^*>0$ due to condition (20c). Hence B_8 and B_9 are positive too.

In addition, since we have $B_8 > B_9$ due to condition (20d). Then $\Delta = A_1 A_2 - A_3 > 0$ and all the eigenvalues of $J(E_4)$ have negative real parts. Thus E_4 is locally asymptotically stable in the $Int.R_+^3$. ■

Now, the persistent of system (4) is studied. A system is persistent if there exist a compact region Ω subset of the interior of the state space such that all solutions with positive initial conditions are attracted to Ω [9]. This is equivalent to that, the boundary of the positive cone in the state space where the solutions exist, act as repellers.

Before go further to establish the persistent conditions of system (4) as shown in the next theorem the following two needed lemmas are proved.

Lemma 5. Assume that the disease-free equilibrium point $E_2 = (\overline{S}, 0, \overline{Y})$ is locally asymptotically stable in the $Int.R_+^2$ of SY-plane, then it is a globally asymptotically stable in the $Int.R_+^2$ of SY-plane.

Proof. Obviously at any point in the $Int.R_+^2$ of SY-plane system (4) reduces the following disease-free 2D subsystem

$$\frac{dS}{dt} = rS\left(1 - \frac{S}{K}\right) - \frac{a_1 SY}{bY+S+c} = g_1(S,Y) \qquad (21a)$$

$$\frac{dY}{dt} = \frac{e_1 a_1 SY}{bY+S+c} - \beta Y = g_2(S,Y) \qquad (21b)$$

Now, define $D_1 = \frac{1}{SY}$. Clearly D_1 is a continuously differentiable function in $Int.R_+^2$ of SY-plane. Further, we have that

$$\nabla_1 = \frac{\partial(D_1 g_1)}{\partial S} + \frac{\partial(D_1 g_2)}{\partial Y} = -\frac{r}{KY} - \frac{a_1(e_1 b - 1)}{(bY+S+c)^2}$$

Since $e_1 b > 1$, due to the local stability condition (14a). Thus $\nabla_1 (< 0)$ is not identically zero and does not change sign in $Int.R_+^2$ of SY-plane. Then, according to Dulac criterion [10], system (21) does not have limit cycle in $Int.R_+^2$ of SY-plane.

Now, since E_2 is the only equilibrium point in $Int.R_+^2$ of SY-plane, which is locally asymptotically stable. Hence, according to Poincare-Bendixson theorem E_2 is globally asymptotically stable in $Int.R_+^2$ of SY-plane. ■

Lemma 6. Assume that the predator-free equilibrium point $E_3 = (\widetilde{S}, \widetilde{I}, 0)$ is locally asymptotically stable in the $Int.R_+^2$ of SI-plane, then it is a globally asymptotically stable in the $Int.R_+^2$ of SI-plane.

Proof. Similar to the proof of lemma (5) with the Dulac function $D_2 = \frac{1}{SI}$.

Theorem 7. Assume that the planer equilibrium points $E_2 = (\overline{S}, 0, \overline{Y})$ and $E_3 = (\widetilde{S}, \widetilde{I}, 0)$ exist and they are globally asymptotically stable in the $Int.R_+^2$ of SY-plane and SI-plane respectively. In addition, let the following two conditions hold

$$\lambda_{2I} = \lambda \overline{S} - (\gamma + \mu) - \frac{a_2 \overline{Y}}{N_1} > 0 \qquad (22a)$$

$$\lambda_{3Y} = \frac{e_1 a_1 \widetilde{S} + e_2 a_2 \widetilde{I} - \theta a_2 \widetilde{I}}{N_2} - \beta > 0 \qquad (22b)$$

Then system (4) is persistent.

Proof. According to theorem (1) all solutions of system (4) are uniformly bounded in R_+^3 and the trajectories of system (4) belong to the region:

$$\Omega = \{(S,I,Y) \in R_+^3 : S(t) + I(t) \le M_1 \text{ with } S(t) + I(t) + Y(t) \le M_2\}$$

where M_1 and M_2 are given in theorem (1).

Also we have that; the equilibrium point $E_0 = (0,0,0)$ exists and is unstable in the S-direction. The axial equilibrium point $E_1 = (K,0,0)$ exists and is unstable in I-direction due to existence condition (8) of predator-free equilibrium point E_3. While E_2 and E_3 are globally asymptotically stable in the $Int.R_+^2$ of SY-plane and SI-plane respectively. Further, from the conditions (22)(a-b) the equilibrium points E_2 and E_3 are unstable in the positive directions I and Y orthogonal to SY-plane and SI-plane respectively. Consequently, all the boundary equilibrium points of system (4) in R_+^3 act as a repellers. Hence all the solutions of system (4) with positive initial conditions are attracted to the interior of Ω. Thus system (4) is persistent. ■

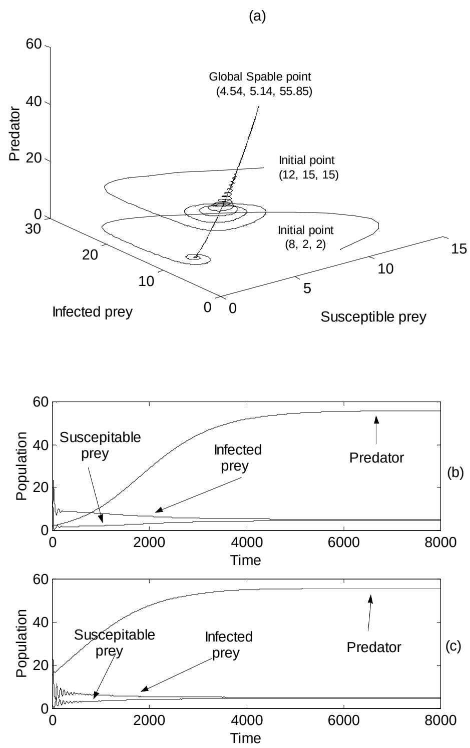
Numerical simulation

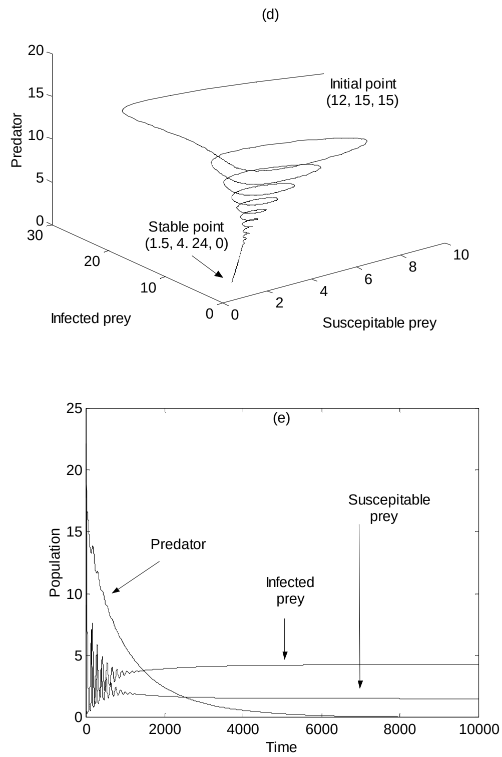
In this section the global dynamics of system (4) in the $Int.R_+^3$ is investigated numerically. In order to verify the results we have made throughout this paper and understand the effects of varying the parameters, system (4)

is solved numerically for different sets of parameters and different sets of initial conditions and then number of attracting sets along with their time series are drawn.
For the following set of parameters:

$$r = 3.5, K = 100, \lambda = 0.4, \gamma = 0.1, a_1 = 1, \\ b = 0.2, \alpha = 1, c = 25, \mu = 0.5, a_2 = 1, \quad (23) \\ e_1 = 0.5, e_2 = 0.5, \beta = 0.1, \theta = 0.05.$$

The trajectory of system (4) approaches asymptotically to global stable point in $Int.R_+^3$ as shown in Fig. 1(a-c). However for the parameters set (23) with $r = 1.5$ the trajectory of system (4) approaches asymptotically to (1.5, 4.24, 0) in $Int.R_+^2$ of SI-plane see Fig. 1(d-e). In addition to the above, it is observed that, for $r \le 1.93$ the trajectory of system (4) approaches to predator-free equilibrium point in $Int.R_+^2$ of SI-plane, while it has globally asymptotically stable point in $Int.R_+^3$ for $r > 1.93$. Hence the system still persistent for all values of $r > 1.93$.





Figuer 1: The attractor sets of system (4) with their time series for parameters set (23). (a) Global stable point in $Int.R_+^3$ for $r = 3.5$. (b) Time series of the attractor in (a) starting at (8, 2, 2). (c) Time series of the attractor in (a) starting at (12, 15, 15). (d) Stable point in $Int.R_+^2$ of SI - plane for $r = 1.5$. (e) Time series of the attractor in (d) starting at (12, 15, 15).

For the parameters set (23) with the infection rate $\lambda \le 0.16$, system (4) losses the persistent and the trajectory approaches asymptotically to disease-free equilibrium point in $Int.R_+^2$ of SY-plane, see for example Fig. 2(a-b) when $\lambda = 0.1$. However, for $0.16 < \lambda \le 0.65$ system (4) persists and the trajectory approaches asymptotically to globally stable point in $Int.R_+^3$ as shown in Fig. 1(a-c). Finally, increasing the value of infected rate further, i.e. $\lambda > 0.65$, system (4) losses the persistent again and the trajectory approaches asymptotically to predator-free equilibrium point in $Int.R_+^2$ of SI-plane see for example Fig.2(c-d) where $\lambda = 0.75$.
For the set of parameters (23) with the recover rate $\gamma \le 0.07$ system (4) has periodic attractor in $Int.R_+^3$ see for example Fig. 3(a-b) when $\gamma = 0.05$, while the trajectory of system (4)

approaches asymptotically to globally stable point in $Int.R_+^3$ for γ in the range (0.07, 4.92] as shown in Fig. 1(a-c). Finally increasing the recover rate further $\gamma > 4.92$ system (4) losses its persistent and the trajectory approaches to globally stable point in $Int.R_+^2$ of SY-plane, see for example Fig. 3(c-d) when $\gamma = 5$.

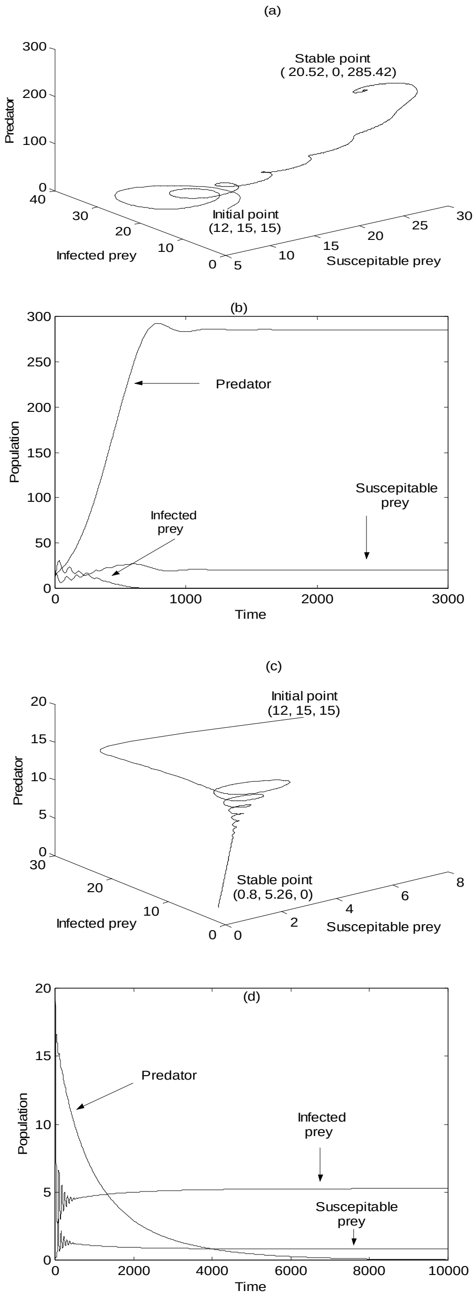


Figure 2: The attractor sets of system (4) with their time series for parameters set (23). (a) Stable point in $Int.R_+^2$ of SY-plane when $\lambda = 0.1$. (b) Time series of the attractor in (a) starting at (12, 15, 15). (c) Stable point in $Int.R_+^2$ of SI-plane when $\lambda = 0.75$. (d) Time series of the attractor in (c) starting at (12, 15, 15).

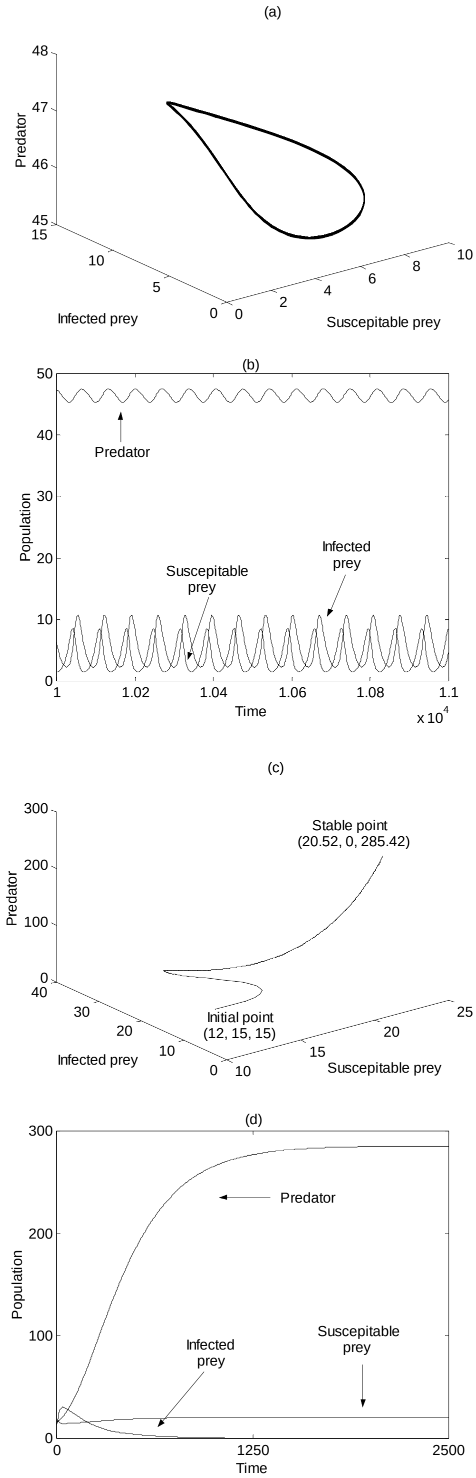
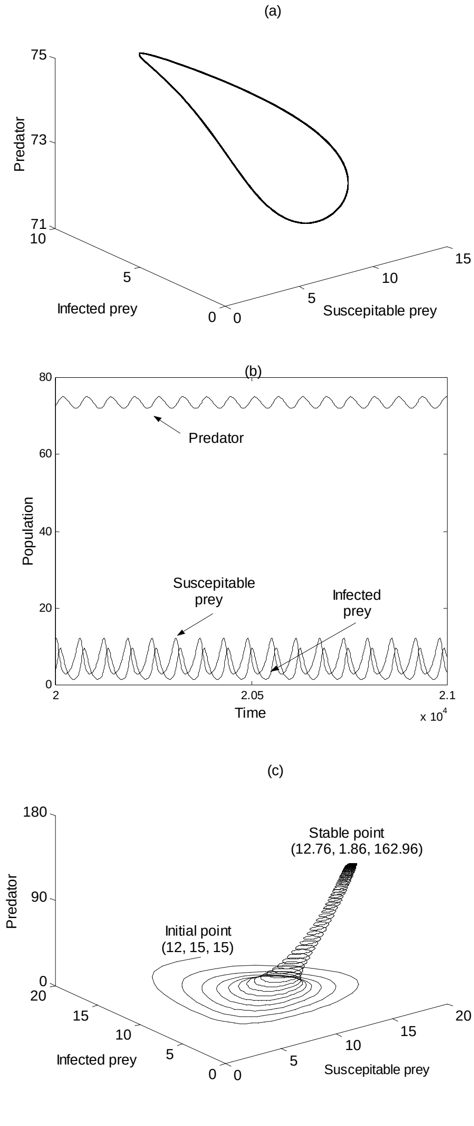


Figure 3: The attractor sets of system (4) with their time series for parameters set (23). (a) Periodic attractor in $Int.R_+^3$ when $\gamma = 0.05$. (b) Time series of the attractor in (a). (c) Stable point in $Int.R_+^2$ of SY-plane when $\gamma = 5$. (d) Time series of the attractor in (c) starting at (12, 15, 15).

For the parameters set (23) with the death rate of infected prey in the ranges $0 < \mu < 0.75$, $0.75 \le \mu \le 2.55$ and $2.55 < \mu \le 5$ system (4) still persistent and the trajectory approaches asymptotically to globally stable point in $Int.R_+^3$, periodic attractor in $Int.R_+^3$ and globally stable point in $Int.R_+^3$ as shown in Fig. 1(a-c), Fig. 4(a-b) and Fig. 4(c-d) respectively. Finally it is observed that increasing the death rate further $\mu > 5$, system (4) losses the persistent and the trajectory approaches asymptotically to stable point in the $Int.R_+^2$ of SY-plane.



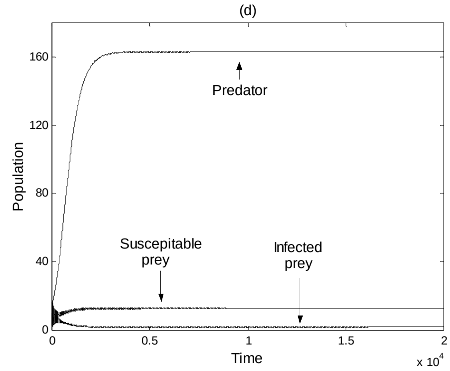
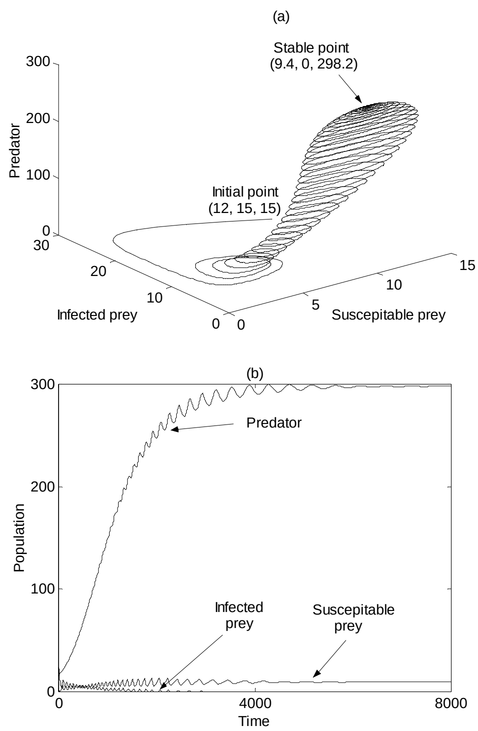


Figure 4: The attractor sets of system (4) with their time series for parameters set (23). (a) Periodic attractor in $Int.R_+^3$ when $\mu = 1$. (b) Time series of the attractor in (a). (c) Stable point in $Int.R_+^3$ when $\mu = 2.75$. (d) Time series of the attractor in (c) starting at (12, 15, 15).

Now, for the parameters set (23) with the natural death rate of predator species in the ranges $0 < \beta \le 0.053$, $0.053 < \beta \le 0.096$, $0.096 < \beta < 0.14$ and $\beta \ge 0.14$ the trajectory of system (4) approaches asymptotically to globally stable point in $Int.R_+^2$ of SY-plane, periodic attractor in $Int.R_+^3$, globally stable point in $Int.R_+^3$ and globally stable point in $Int.R_+^2$ of SI-plane respectively as shown in Fig. 5(a-f).



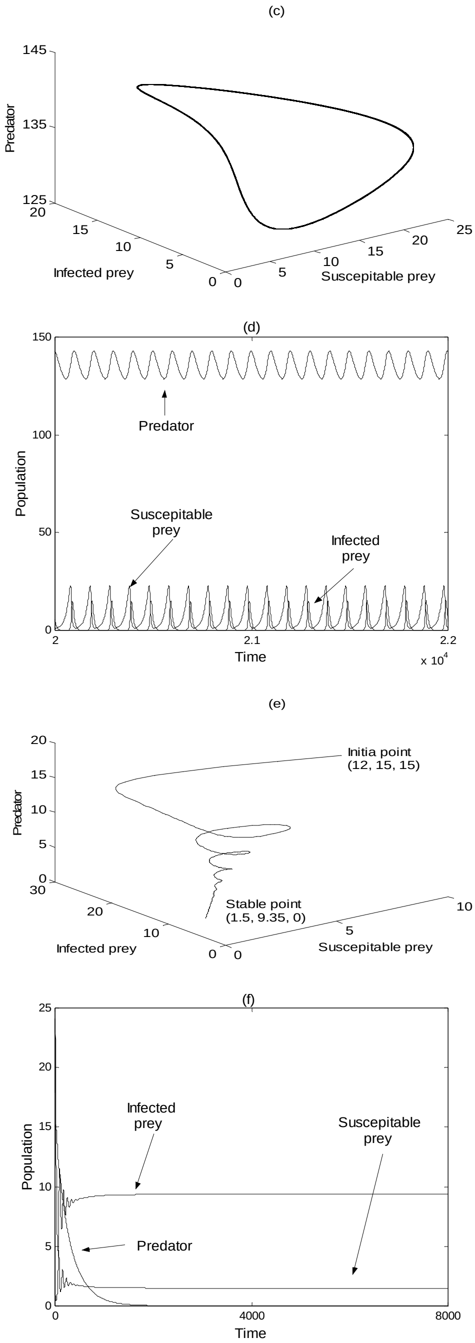


Figure 5: The attractor sets of system (4) with their time series for parameters set (23). (a) Stable point in $Int.R_+^3$ when $\beta = 0.05$. (b) Time series of the attractor in (a) starting at (12, 15, 15). (c) Periodic attractor in $Int.R_+^3$ when $\beta = 0.07$. (d) Time series of the attractor in (c). (e) Stable point in $Int.R_+^2$ of SI -plane when $\beta = 0.2$. (f) Time series of the attractor in (e).

Conclusion and Discussion

An eco-epidemiological system consisting of Biddington-DeAngils prey-predator interacting involving SIS epidemic disease in prey has been proposed and analyzed. Analytically, it is observed that the system has at most five nonnegative equilibrium points in R_+^3. The trajectory of system (4) approaches asymptotically to locally stable positive point (endemic point) under certain conditions. Further more, system (4) persists under conditions (22)(a-b). However, numerically it is observed that, system (4) has two type of attractors in $Int.R_+^3$: approaches to globally stable point or to globally stable limit cycle. Moreover the system is very sensitive to varying in parameters set as shown in Figurers. (1-5).

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