DOI: 10.24996/ijs.2026.67.1.28

ISSN: 0067-2904

On algebraic characteristics of nonequivalent degree three arcs in $PG(2,17)$

Zainab Shehab Hamed [1] , J.W.Hirschfeld[2]

[1] Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq.

[2] Department of Mathematics, School of Mathematical and Physical Sciences, Sussex University, Brighton, UK.

Received: 2/6/2024 Accepted: 29/12/2024 Published: 30/1/2026

Abstract

In this paper, the algebraic characteristics of nonequivalent arcs of degree three in $PG(2,17)$ are discussed. An approach to construct these sets is introduced. The approach focuses on obtaining a large size of complete degree three arc in $PG(2,17)$. This work starts by fixing a set, $\mathcal{X}=\{P_1,P_2,P_3\}$ that have three points lie on the same line $Ł_x$ of the projective space of order seventeen, $PG(2,17)$. This set is a (3;3)-Arc. Then, the process is continued to establish the sets of (ƙ;3)-Arcs by adding the points of the projective plane, $PG(2,17)$ that satisfied the condition $\mathcal{X} \cap lines\ Ł_i \setminus Ł_x = 3$, where $Ł_i \in PG(2,17)$. So, the sets of (4;3)-Arcs, (5;3)-Arcs, (6;3)-Arcs, (7;3)-Arcs, (8;3)-Arcs, (9;3)-Arcs, (10;3)-Arcs, (11;3)-Arcs, (12;3)-Arcs, (13;3)-arcs, (14;3)-Arcs, (15;3)-Arcs, (16;3)-Arcs, (17;3)-Arcs, (18;3)-Arcs, (19;3)-Arcs, (20;3)-Arcs, (21;3)-Arcs, (22;3)-Arcs, (23;3)-Arcs, (24;3)-Arcs, (25;3)-Arcs, (26;3)-Arcs, (27;3)-Arcs, and (28;3)-Arcs are obtained. So that this approach gives the number of (ƙ;3)-Arcs in each construction for ƙ = 4,5,6,7,8,…28, and then the number of nonequivalent (ƙ;3)-Arcs for ƙ = 4,5,6,7,8,…,28 is given as well. This number is established according to the number of nonequivalent secant distributions of degree three arcs, (ƙ;3)-Arcs. Thus, the spectrum of nonequivalent arcs in each process is 2, 6, 16, 32, 49, 71, 97, 122, 149, 170, 192, 205, 220, 230, 233, 234, 229, 218, 190, 160, 101, 34, 4, 3, 1, respectively. Also, the associated stabilizer group for each constructed nonequivalent arc is computed. In addition, the action of each stabilizer group on the corresponding nonequivalent arc is discussed. As a result of these actions, there are different sizes of orbits. These sizes are one, two, three, four, and six. The largest size of degree three arc established in this process is ƙ = 28 .

Keywords: (ƙ; 3)-Arc, Complete arc, Nonequivalent secants, $PG(2,17)$, Group.

حول الخصائص الجبرية لاقواس الدرجة الثالثة الغير متكافئة في (PG(2,17

زينب شهاب حامد [1]، هرشفيلد وليم جيمس [2]

[1] قسم الرياضيات , كلية العلوم, الجامعة المستنصرية, بغداد, العراق

[2] قسم الرياضيات, مدرسة الرياضيات والعلوم الفيزيائية, برايتون, المملكة المتحدة

الخلاصة

في هذا البحث، الخصائص الجبرية للاقواس الغير متكافئة من الدرجة الثالثة تم مناقشتها حيث تم

*Email: zainab.sh91@yahoo.com

تقديم طريقة لتصنيف مجموعات أقواس الدرجة الثالثة غير المكافئة في المستوى الإسقاطي من الرتبة السابعة عشرة . يركز النهج على الحصول على حجم كبير من الدرجة الثالثة التامة لهذا الفضاء. يبدا هذا التصنيف بتثبيت مجموعة من النقاط التي تقع على نفس الخط .هذه المجموعة هي قوس− (3;3) . بعد ذلك، تستمر العملية في إنشاء مجموعات الأقواس الاسقاطية الثلاثية الدرجة عن طريق اضافة النقاط المنتمية الى الفضاء الاسقاطي

Ł ه. PG(2,17)∈ هذه الخطوط $X \cap Ł_i \backslash Ł_x = 3$ حيث من الدر جة السابعة عشر والتي تحقق

لذا المجموعات التالية تم الحصول عليها تباعا. هذه المجاميع هي اقواس الدرجة الثالثة لاحجام مختلفة وهي (5;3)−Arcs, (6;3)−Arcs, (7;3)−Arcs, (8;3)−Arcs, (9;3)−Arcs, (10;3)−Arcs,(11;3)−Arcs, (12;3)−Arcs, (13;3)−Arcs, (14;3)−Arcs, (15;3)−Arcs, (16;3)−Arcs,(17;3)−Arcs , (18;3)−Arcs,(19;3)−Arcs, (20;3)−Arcs, (21;3)−Arcs, (22;3)−Arcs, (23;3)−Arcs, (24;3)−Arcs, (25;3)−Arcs, (26;3)−Arcs, (27;3)−Arcs, (28;3)−Arcs, for ƙ= 4,5,6,7,8,...,28 .

ايضا يعطي هذا النهج عدد الاقواس الثلاثية الغير متكافئ . يتم إنشاء هذا العدد الغير متكافئ وفقًا لعدد التوزيعات القاطعة غير المكافئة لأقواس الدرجة الثالثة وبالتالي فإن نطاق ألاقواس من الدرجة الثالثة والغير متكافئة في كل عملية على التوالي هو كالاتي 2, 6, 16, 32, 49, 71, 97, 122, 149, 170, 192, 205, 220, 230, 233, 234, 229, 218, 190, 160, 101, 34,4,3,1 كما يتم حساب مجموعة التثبيت المرتبطة لكل قوس غير مكافئ تم إنشاؤه. بالإضافة إلى ذلك، تمت مناقشة عمل كل مجموعة استقرار على القوس غير المكافئ المقابل. ونتيجة لهذه الإجراءات، هناك أحجام مختلفة من المدارات هذه الأحجام هي واحد ، اثنان، ثلاثة، أربعة، وستة. أكبر حجم لاقواس الدرجة الثالثة الذي تم ايجاده هو : ƙ=28

1. Introduction

Finite field theory goes back to the 17th and 18th centuries. Many mathematicians contributed to its development. For instance, Pierre de Fermat (1601-1665) and Leonard Euler (1707-1783) when they presented the structural theory of particular finite fields. In contrast, the theory of finite fields began in (1811-1832) when the French scientist Everest gave the necessary and sufficient conditions for a polynomial to have an algebraic solution. Nowadays, finite field theory has become very important in applied mathematics, engineering science, and computer science due to its many applications in these fields [1], [2]. The history of projective geometry goes back to the Renaissance times in Italy through the advanced techniques used by painters and artists at that time, as they represented reality in scenes. These techniques were a method of interpreting ideas that were slowly formed, which later became a new branch of geometry called projective geometry. In the nineteenth century, projective geometry became an important branch of mathematics and one of the most important major achievements. For instance, non-Euclidean mathematics. For over fifty years, projective geometry has been set in a new direction by its communications. Describing classical geometric structure in terms of properties may seem at first glance, in-depth, but it has done a lot for finite geometry [3], [4]. Classical projective geometry structures are very suitable for modern communications. In particular, projective geometry is applied to theories of coding and cryptography [5], [6]. Projective geometry works with the properties of fixed shapes in projections. This appears clearly in the most important theory in projective geometry, which is Babus' theory. This theory gave great attention to points and lines and the relationship of incidence between points and lines. This incident structure is the study of geometry structures that satisfy certain geometric axioms inspired by the properties of occurrence of points, lines, planes, etc [7], [8], and [9].The point P in $PG\,(2,q)$ represents in three coordinates of the forms $\{P_o, P_1, 1\}, \{P_0, 1, 0\}$, and $\{1,0,0\}$, the number of these forms is

$q^2, q,$ and 1 respectively. In addition, each line Ł in *PG* (2,q) satisfies the equation $\alpha X_1 + \beta X_2 + \gamma X_3 = 0$, when α,β,γ ∈ F_q not all zero. So, the structure of the projective plane is obtained by the incidence relation of points and lines and it is satisfied if the following statements hold.

1- For any two distinct points, there is one line containing them.
2- For any two lines, there is only one point of intersection between them.
3- For any four points, there is a quadrangle containing them.

So that the above structure pays more attention to studying one of the most fundamental and important problems in projective geometry which is the problem of establishing a large complete arc. In this problem, the subjects of arcs and complete arcs have seen considerable improvements in *PG* (2,q). These sets of arcs of *PG* (2,q) are constructed over the Galois field F_q that is the field of a finite number of elements, where every element in this field satisfies the form $y^q - y = 0$.
and when q is a prime number, then $F_p = \{0,1,\dots,p-1\}$. The number of points and lines in any geometric plane, *PG* (2,q), is $q^2 + q + 1$ and each line has $q + 1$ points and each point lies on $q + 1$ lines [10-13]. Thus, the projective plane, *PG* (2,17) has 307 points and lines, where each line has 18 points. Also, there are 18 lines intersecting with a point. In addition, an arc, ƙ in the projective space is defined to be the set of zero dimensional subspaces which not contain three of these spaces on the same line. This set is named complete when ƙ does not belongs to another ƙ +1 set of arc [14], [15], and [16]. Also, one dimensional subspace *Ł* is j-secant of ƙ when the condition | *Ł* ∩ ƙ | = j holds. This study our aims to establish the sets of (ƙ;3)-Arcs in *PG* $(2, q)$ for q=17, to establish the nonequivalent sets of (ƙ;3)-Arcs in *PG* (2,17), and to discuss the algebraic characteristics of nonequivalent (ƙ;3)-Arcs throughout calculate the stabilizer groups of the number of nonequivalent (ƙ;3)-Arcs, and then to introduce the stabilizer properties of these nonequivalent arcs. Some of these stabilizers are cyclic groups, where this group is the group in which each element is a power of fixed element that satisfies e = r^n, [17].

2. Materials and methods

This section presents some materials for this work [18]. Also, the method used to construct the nonequivalent degree three arcs with the corresponding secants is the method given in [19]. The programming language used to implement the results in Section 3 is Gap [20]

Corollary 2.1: An arc of degree d has an upper bound as $m_d(2,q) \le (d-1)q + d$.
Theorem 2.2: Let *K* be a maximal (ƙ; d)-arc in *PG* (2,q), then one of the bellow statements satisfies:
1. $d = q + 1$ and $K = PG\ (2, q)$.
2. $d = q$ and $K = AG(2,q) = PG\ (2,q) \setminus$ *Ł* for some line *Ł*.
3. $2 \le d < q, d|q$, and the dual of the external lines of *K* construct a ($\frac{(q+1-d)q}{d}$, $\frac{q}{d}$)-arc is maximal.
Corollary 2.3: A (ƙ; d)-arc is a maximal if and only if each line in $PG(2,q)$ is either an d-secant or an external line.

Corollary 2.4: If 2 < d < q and d does not divide q, then $m_d(2,q) \le (d-1)q + d - 2$.
In addition, Barlotti [21] in 1956 indicated that $m_d(2,q) \le \begin{Bmatrix} (d-1)q + d\ for\ (d,q) = 1 \\ (d-1)q + d - 2\ for\ d > 2 \end{Bmatrix}$,
but Lunelli and Sce [22] in 1964 improved the bounds as below:

$$m_d(2,q) \leq \begin{cases} (d-1)q+d-3 & for\ 4 \leq d \leq q \\ (d-1)q+d-4 & for\ 9 \leq d \leq q \end{cases}.$$

3. Results and discussion

3.1 (4;3)-Arcs

The construction process of (4;3)-Arcs illustrated that there are 289 sets of (4; 3)-Arcs, A_1. While the sets of nonequivalent arcs are two each has a stabilizer group of the form $Z_{16} \times S_3$.

Table 1: Nonequivalent secant distributions of (4; 3)-Arcs.

$A_1^{i=1,2}$	$\{\theta_3, \theta_2, \theta_1, \theta_0\}$
A_1^1	{1, 3, 62, 241}
A_1^2	{1, 3, 63, 240}

Remark 3.1.1:

The stabilizer groups of the nonequivalent (4; 3)-Arcs divide the associated arcs into two orbits as follows:

i- Orbit (A_1^1) = {18, 304, 196}, {158}.

ii- Orbit (A_1^2) = {18, 304, 196}, {24}.

3.2 (5;3)-Arcs

The construction sets of (5;3)-Arcs, in this process, showed that there are 575 sets of (5; 3)-Arcs, A_2. While there are six nonequivalent sets of (5; 3)-Arcs. Also, the stabilizer groups of these arcs are Z_2 and D_8.

Table 2: Nonequivalent secant distributions of (5; 3)-Arcs.

$A_2^{i=1,..,6}$	$\{\theta_3, \theta_2, \theta_1, \theta_0\}$
A_2^1	{1,7,73,226}
A_2^2	{1, 6, 73, 227}
A_2^3	{1, 6, 74, 226}
A_2^4	{1, 7, 72, 227}
A_2^5	{2, 4, 75, 226}
A_2^6	{2, 4, 76, 225}

Remark 3.2.1:

The stabilizer groups of the nonequivalent (5; 3)-Arcs split the associated arcs into a number of orbits as follows:

i- The (ƙ; 3)-Arcs A_2^1, A_2^2, A_2^3, A_2^4, each has a group Z_2 that divides the associated arc into two orbits of sizes 2 and 1.

ii- The (ƙ; 3)-Arcs A_2^5, A_2^6, each has a dihedral group of order eight. This group partitions the associated arcs in two orbits of sizes 4 and 1.

3.3 (6;3)-Arcs

The constructed sets of (6; 3)-Arcs, A_3, are 1685. Among these arcs, there are sixteen nonequivalent (6; 3)-Arcs. The sixteen arcs have four types of stabilizer groups which are I, Z_2, Z_3, and S_4.

Table 3: Nonequivalent secant distributions of (6; 3)-Arcs.

$A_3^{i=1,..,16}$	$\{\theta_3, \theta_2, \theta_1, \theta_0\}$
A_3^1	{1, 9, 85, 212}
A_3^2	{1, 10, 84, 212}
A_3^3	{1, 11, 81, 214}
A_3^4	{1, 11, 82, 213}
A_3^5	{1, 12, 80, 214}
A_3^6	{1, 12, 81, 213}
A_3^7	{2, 8, 84, 213}
A_3^8	{2, 8, 85, 212}
A_3^9	{2, 9, 83, 213}
A_3^{10}	{2, 9, 84, 212}
A_3^{11}	{3, 5, 87, 212}
A_3^{12}	{3, 5, 88, 211}
A_3^{13}	{3, 6, 86, 212}
A_3^{14}	{3, 6, 87, 211}
A_3^{15}	{4, 3, 89, 211}
A_3^{16}	{4, 3, 90, 210}

Remark 3.3.1:

The stabilizers of the nonequivalent (6; 3)-Arcs divide the associated arcs into orbits as follows:

i- The (ƙ; 3)-Arcs $A_3^1, A_3^3, A_3^4, A_3^5, A_3^6, A_3^7, A_3^8, A_3^9, A_3^{10}$ have the identity group that partitions the corresponding arc in a number of orbits of size 1.

ii- The (ƙ; 3)-Arc A_3^2 has the cyclic group of order two. This group splits the associated arc into five orbits of sizes 1, 1, 2, 1, 1.

iii- The (ƙ; 3)-Arcs $A_{3,}^{11}$ $A_{3,}^{12}$ $A_{3,}^{13}$ $A_{3,}^{14}$ each has the cyclic group of order three. This group divides the corresponding arc in two orbits of sizes 3, 3.

v- The (ƙ; 3)-Arcs $A_{3,}^{15}A_{3,}^{16}$ each has a group S_4. This group permutes the corresponding arc in one orbit of size 6.

3.4 (7;3)-Arcs

In this process, there are 4279 (7; 3)-Arcs, A_4 and there are thirty two nonequivalent (7; 3)-Arcs. In addition, the 32 (7; 3)-Arcs have four types of stabilizer groups, which are I, Z_2, Z_4, and S_4.

Table 4: Nonequivalent secant distributions of $(7;3)$-Arcs.

$A_4^{i=1,..,32}$	$\{\theta_3,\theta_2,\theta_1,\theta_0\}$
A_4^1	{1, 18, 87, 201}
A_4^2	{1, 13, 95, 198}
A_4^3	{1, 15, 91, 200}
A_4^4	{1, 16, 90, 200}
A_4^5	{1, 17, 87, 202}
A_4^6	{1, 17, 88, 201}
A_4^7	{1, 18, 86, 202}
A_4^8	{2, 12, 94, 199}
A_4^9	{2, 12, 95, 198}
A_4^{10}	{2, 13, 93, 199}
A_4^{11}	{2, 14, 90, 201}
A_4^{12}	{2, 14, 91, 200}
A_4^{13}	{2, 15, 89, 201}
A_4^{14}	{2, 15, 90, 200}
A_4^{15}	{3, 9, 97, 198}
A_4^{16}	{3, 10, 96, 198}
A_4^{17}	{3, 11, 93, 200}
A_4^{18}	{3, 11, 94, 199}
A_4^{19}	{3, 12, 92, 200}
A_4^{20}	{3, 12, 93, 199}
A_4^{21}	{4, 6, 100, 197}
A_4^{22}	{4, 7, 99, 197}
A_4^{23}	{ 4, 8, 96, 199}
A_4^{24}	{4, 8, 97, 198}
A_4^{25}	{4, 9, 95, 199}
A_4^{26}	{4, 9, 96, 198}
A_4^{27}	{5, 5, 99, 198}
A_4^{28}	{5, 5, 100, 197}
A_4^{29}	{5, 6, 98, 198}
A_4^{30}	{5, 6, 99, 197}
A_4^{31}	{6, 3, 101, 197}
A_4^{32}	{6, 3, 102, 196}

Remark 3.4.1:

The stabiliser groups of the nonequivalent $(7;3)$-Arcs split these arcs into a number of orbits as follows:

i-The $(\text{k};3)$-Arcs $A_4^1,\dots,A_4^{20},A_4^{23},\dots,A_4^{26}$ have the identity group that partitions the corresponding arc in a number of orbits of size 1.

ii- The $(\text{k};3)$-Arcs $A_4^{21},A_4^{22},A_4^{27},A_4^{28}$, A_4^{30}have the cyclic group of order two. This group divides the associated arc into orbits of sizes 1 and 2.

iii-The $(\text{k};3)$-Arcs A_4^{29} has the cyclic group of order four. The group Z_4divides the corresponding arc into three orbits of sizes 4, 2, 1.

v- The $(\text{k};3)$-Arcs A_4^{31} , A_4^{32} each has a group S_4. This group divides the corresponding arc into two orbits of size 3 and 4.

3.5 (8;3)-Arcs

The sets of $(8;3)$-Arcs, A_5 are 8135 where there are 49 nonequivalent $(8;3)$-Arcs. Also, there are three types of stabilizer groups, which are $I,\ Z_2,\ Z_4$.

Table 5: Nonequivalent secant distributions of $(8;3)$-Arcs.

$A_5^{i=1,..,49}$	$\{\theta_3, \theta_2, \theta_1, \theta_0\}$
A_5^1	{2, 22, 94, 189}
A_5^2	{1, 20, 99, 187}
A_5^3	{1, 22, 95, 189}
A_5^4	{1, 23, 94, 189}
A_5^5	{1, 24, 91, 191}
A_5^6	{1, 24, 92, 190}
A_5^7	{1, 25, 90, 191}
A_5^8	{1, 25, 91, 190}
A_5^9	{2, 15, 106, 184}
A_5^{10}	{2, 17, 102, 186}
A_5^{11}	{2, 19, 98, 188}
A_5^{12}	{2, 19, 99, 187}
A_5^{13}	{2, 20, 97, 188}
A_5^{14}	{2, 21, 94, 190}
A_5^{15}	{2, 21, 95, 189}
A_5^{16}	{2, 22, 93, 190}
A_5^{17}	{3, 14, 105, 185}
A_5^{18}	{3, 16, 101, 187}
A_5^{19}	{3, 16, 102, 186}
A_5^{20}	{3, 17, 100, 187}
A_5^{21}	{3, 18, 97, 189}
A_5^{22}	{3, 18, 98, 188}
A_5^{23}	{3, 19, 96, 189}
A_5^{24}	{3, 19, 97, 188}
A_5^{25}	{4, 11, 108, 184}
A_5^{26}	{4, 13, 104, 186}
A_5^{27}	{4, 13, 105, 185}
A_5^{28}	{4, 14, 103, 186}
A_5^{29}	{4, 15, 100, 188}
A_5^{30}	{4, 15, 101, 187}
A_5^{31}	{4, 16, 99, 188}
A_5^{32}	{4, 16, 100, 187}
A_5^{33}	{5, 10, 107, 185}
A_5^{34}	{5, 10, 108, 184}
A_5^{35}	{5, 11, 106, 185}
A_5^{36}	{5, 12, 103, 187}
A_5^{37}	{5, 12, 104, 186}
A_5^{38}	{5, 13, 102, 187}
A_5^{39}	{5, 13, 103, 186}
A_5^{40}	{6, 7, 110, 184}

A_5^{41}	{6, 8, 109, 184}
A_5^{42}	{6, 9, 106, 186}
A_5^{43}	{6, 9, 107, 185}
A_5^{44}	{6, 10, 105, 186}
A_5^{45}	{6, 10, 106, 185}
A_5^{46}	{7, 6, 109, 185}
A_5^{47}	{7, 6, 110, 184}
A_5^{48}	{7, 7, 108, 185}
A_5^{49}	{7, 7, 109, 184}

Remark 3.5.1:

The stabilizer groups of the nonequivalent $(8;3)$-Arcs divide the associated arcs into a number of orbits as below:

i- The $(\text{ḱ};3)$-Arcs $A_5^1, \dots, A_5^{39}, A_5^{41}, \dots, A_5^{45}$ have the identity group that splits the associated arc in a number of orbits of size 1.

ii- The $(\text{ḱ};3)$-Arcs A_5^{40}, A_5^{46}, A_5^{47}, A_5^{49}have the cyclic group of order two that divide the associated arc into orbits of sizes 1 and 2.

iii- The $(\text{ḱ};3)$-Arcs A_4^{48} has the cyclic group of order four. This group partitions the associated arc in four orbits of sizes 4, 2, 1, 1.

3.6 (9;3)-Arcs

The constructed sets of $(9;3)$-Arcs, A_6 are 11921. Here, The number of nonequivalent $(9;3)$-Arcs. is 71. These 71 arcs have the stabilizer groups I, S_3, $Z_2 \times Z_2$ **.**

Table 6: Nonequivalent secant distributions of $(9;3)$-Arcs.

$A_6^{i=1,..,71}$	$\{\theta_3, \theta_2, \theta_1, \theta_0\}$
A_6^1	{3, 27, 99, 178}
A_6^2	{1, 28, 101, 177}
A_6^3	{1, 30, 97, 179}
A_6^4	{1, 31, 96, 179}
A_6^5	{1, 32, 93, 181}
A_6^6	{1, 32, 94, 180}
A_6^7	{1, 33, 92, 181}
A_6^8	{1, 33, 93, 180}
A_6^9	{ 2, 23, 108, 174}
A_6^{10}	{2, 25, 104, 176}
A_6^{11}	{2, 27, 100, 178}
A_6^{12}	{2, 27, 101, 177}
A_6^{13}	{2, 28, 99, 178}
A_6^{14}	{2, 29, 96, 180}
A_6^{15}	{2, 29, 97, 179}
A_6^{16}	{2, 30, 95, 180}
A_6^{17}	{2, 30, 96, 179}
A_6^{18}	{3, 20, 111, 173}
A_6^{19}	{3, 22, 107, 175}
A_6^{20}	{3, 24, 103, 177}
A_6^{21}	{3, 24, 104, 176}
A_6^{22}	{3, 25, 102, 177}
A_6^{23}	{3, 26, 99, 179}

A_6^{24}	{3, 26, 100, 178}
A_6^{25}	{3, 27, 98, 179}
A_6^{26}	{4, 17, 114, 172}
A_6^{27}	{4, 19, 110, 174}
A_6^{28}	{4, 21, 106, 176}
A_6^{29}	{4, 21, 107, 175}
A_6^{30}	{4, 22, 105, 176}
A_6^{31}	{4, 23, 102, 178}
A_6^{32}	{4, 23, 103, 177}
A_6^{33}	{4, 24, 101, 178}
A_6^{34}	{4, 24, 102, 177}
A_6^{35}	{5, 14, 117, 171}
A_6^{36}	{5, 16, 113, 173}
A_6^{37}	{5, 18, 109, 175}
A_6^{38}	{5, 18, 110, 174}
A_6^{39}	{5, 19, 108, 175}
A_6^{40}	{5, 20, 105, 177}
A_6^{41}	{5, 20, 106, 176}
A_6^{42}	{5, 21, 104, 177}
A_6^{43}	{5, 21, 105, 176}
A_6^{44}	{6, 13, 116, 172}
A_6^{45}	{6, 15, 112, 174}
A_6^{46}	{6, 15, 113, 173}
A_6^{47}	{6, 16, 111, 174}
A_6^{48}	{6, 17, 108, 176}
A_6^{49}	{6, 17, 109, 175}
A_6^{50}	{6, 18, 107, 176}
A_6^{51}	{6, 18, 108, 175}
A_6^{52}	{7, 10, 119, 171}
A_6^{53}	{7, 12, 115, 173}
A_6^{54}	{7, 12, 116, 172}
A_6^{55}	{7, 13, 114, 173}
A_6^{56}	{7, 14, 111, 175}
A_6^{57}	{7, 14, 112, 174}
A_6^{58}	{7, 15, 110, 175}
A_6^{59}	{7, 15, 111, 174}
A_6^{60}	{8, 9, 118, 172}
A_6^{61}	{8, 9, 119, 171}
A_6^{62}	{8, 10, 117, 172}
A_6^{63}	{8, 11, 114, 174}
A_6^{64}	{8, 11, 115, 173}
A_6^{65}	{8, 12, 113, 174}
A_6^{66}	{8, 12, 114, 173}
A_6^{67}	{ 9, 7, 120, 171}
A_6^{68}	{ 9, 8, 117, 173}
A_6^{69}	{ 9, 8, 118, 172}
A_6^{70}	{ 9, 9, 116, 173}
A_6^{71}	{9, 9, 117, 172}

Remark 3.6.1:

The stabilizer groups of the nonequivalent (9; 3)-Arcs split the associated arcs into a number of orbits as follows:

i- The (ƙ; 3)-Arcs $A_6^1, \dots, A_6^{66}$, A_6^{68} , A_6^{70}, A_6^{71} have the identity group. This group partitions the associated arc in orbits of size 1.

ii- The (ƙ; 3)-Arc A_6^{67} has the group S_3 that split this arc into two orbits of sizes 6 and 3.

iii- The (ƙ; 3)-Arcs A_6^{69} has the cyclic group $Z_2 \times Z_2$. This group partitions the associated arc into four orbits of sizes 1, 2, 4, 2.

Note 3.6.2:

Table 7 introduces the information of the sets of (ƙ; 3)-Arcs and the nonequivalent sets of (ƙ; 3)-Arcs,$\boldsymbol{A}_m^{i=1,..,n}$, m=7,…,25 with their groups.

Table 7: Number of (ƙ; 3)-Arcs for ƙ = 10,…,28.

(ƙ; 3)-Arcs	Number of (ƙ; 3)-Arcs	$\boldsymbol{A}_m^{i=1,..,n}$, m=7,…,25	Stabilizers of nonequivalent (ƙ; 3)-Arcs
(10; 3)-Arcs	16169	$\boldsymbol{A}_7^{i=1,..,97}$	I = 93, Z_2= 4
(11; 3)-Arcs	20268	$\boldsymbol{A}_8^{i=1,..,122}$	I = 120, Z_2= 2
(12; 3)-Arcs	23440	$\boldsymbol{A}_9^{i=1,..,149}$	I = 145, Z_2= 2, S_3= 2
(13; 3)-Arcs	25675	$\boldsymbol{A}_{10}^{i=1,..,170}$	I = 169, Z_2= 1
(14; 3)-Arcs	25649	$\boldsymbol{A}_{11}^{i=1,..,192}$	I = 191, Z_2= 1
(15; 3)-Arcs	24892	$\boldsymbol{A}_{12}^{i=1,..,205}$	I = 205
(16; 3)-Arcs	22055	$\boldsymbol{A}_{13}^{i=1,..,220}$	I = 220
(17; 3)-Arcs	18842	$\boldsymbol{A}_{14}^{i=1,..,230}$	I = 230
(18; 3)-Arcs	14604	$\boldsymbol{A}_{15}^{i=1,..,233}$	I = 233
(19; 3)-Arcs	10640	$\boldsymbol{A}_{16}^{i=1,..,234}$	I = 234
(20; 3)-Arcs	7076	$\boldsymbol{A}_{17}^{i=1,..,229}$	I = 229
(21; 3)-Arcs	4447	$\boldsymbol{A}_{18}^{i=1,..,218}$	I = 218
(22; 3)-Arcs	2419	$\boldsymbol{A}_{19}^{i=1,..,190}$	I = 190
(23; 3)-Arcs	1143	$\boldsymbol{A}_{20}^{i=1,..,160}$	I = 160
(24; 3)-Arcs	401	$\boldsymbol{A}_{21}^{i=1,..,101}$	I = 101
(25; 3)-Arcs	76	$\boldsymbol{A}_{22}^{i=1,..,34}$	I = 34
(26; 3)-Arcs	7	$\boldsymbol{A}_{23}^{i=1,..,4}$	I = 4
(27; 3)-Arcs	5	$\boldsymbol{A}_{24}^{i=1,..,3}$	I = 3
(28; 3)-Arcs	1	$\boldsymbol{A}_{25}^{i=1}$	Z_3

The previous tables show that the major result that has been achieved in this study is a large complete (ƙ;3)-Arc. This complete arc is of size ƙ=28. The coordinates of complete (28;3)-Arc are given below.

ƙ= { 5, 8, 1 },{ 2, 0, 1 },{ 0, 6, 1 },{ 5, 7, 1 },{ 9, 1, 0 },{ 0, 9, 1 },{ 9, 8, 1 },{ 16, 10, 1 },{ 2, 1, 1 },{ 16, 5, 1 },{ 5, 5, 1 },{ 9, 0, 1 },{ 12, 16, 1 },{ 3, 0, 1 },{ 1, 2, 1 },{ 7, 1, 1 },{ 1, 7, 1 },{ 12, 5, 1 },{ 1, 1, 1 },{ 7, 16, 1 },{ 15, 16, 1 },{ 2, 1, 0 },{ 0, 2, 1 },{ 9, 9, 1 },{ 7, 10, 1 },{ 15, 7, 1 },{ 16, 2, 1 },{ 6, 1, 0 }.

This complete (ƙ;3)-Arc has the group Z_3 that partitions this arc in ten orbits of sizes 3,3,3,3,3,3,3,3,3,1, respectively. Also, the associated secant distribution of complete (28;3)-Arc is $\theta_{3=}108$, $\theta_{2=}54$, $\theta_{1=}72$, $\theta_{0=}73$.

Table 8: Data of Nonequivelant (ƙ; 3)-Arcs for for ƙ = 4,…,28.

(ƙ;3)-Arc	D_i	$s_g: n$
(4:3)-Arcs	2	$Z_{16} \times S_3$:2
(5:3)-Arcs	6	Z_2:4, D_8 :2
(6;3)-Arcs	16	I : 9, Z_2:1, Z_3:4, S_4 :2
(7;3)-Arcs	32	I :24 , Z_2:5, Z_4:1, S_4 :2
(8;3)-Arcs	49	I : 44, Z_2: 4, Z_4:1
(9;3)-Arcs	71	I : 69, S_3 :1, $Z_2 \times Z_2$:1
(10; 3)-Arcs	97	I : 93, Z_2: 4
(11; 3)-Arcs	122	I :120, Z_2: 2
(12; 3)-Arcs	149	I : 145, Z_2: 2, S_3:2
(13; 3)-Arcs	170	I : 169, Z_2: 1
(14; 3)-Arcs	192	I : 191, Z_2: 1
(15; 3)-Arcs	205	I :205
(16; 3)-Arcs	220	I :220
(17; 3)-Arcs	230	I :230
(18; 3)-Arcs	233	I :233
(19; 3)-Arcs	234	I : 234
(20; 3)-Arcs	229	I : 229
(21; 3)-Arcs	218	I : 218
(22; 3)-Arcs	190	I : 190
(23; 3)-Arcs	160	I : 160
(24; 3)-Arcs	101	I : 101
(25; 3)-Arcs	34	I : 34
(26; 3)-Arcs	4	I =: 4
(27; 3)-Arcs	3	I : 3
(28; 3)-Arcs	1	Z_3

The previous calculation constructed different sizes of (ƙ;3)-Arc for ƙ = 4, … ,28. Therefore, the bellow theorem is stated.

Theorem 3.6.3:
There is at least a complete (28, 3)-Arc in PG (2,17).

4. Conclusions

This study introduces a certain calculation method to compute the sets of (ƙ; 3)-Arcs in $PG(2,17)$ throughout establishing a maximum size of arc, and then establish the sets of nonequivalent (ƙ; 3)-Arcs in $PG(2,17)$. Then, discuss the most essential algebraic characteristics of each nonequivalent arc in terms of the secant distributions, the stabilizer groups, and the orbits. In this paper, the method used to classify these arcs is relies on the number of nonequivalent secants of arcs. Table 8 summarized all related details of nonequivalent (ƙ; 3)-Arcs in $PG(2, 17)$, where the symbol D_i indicates the number of nonequivalent (ƙ; 3)-Arcs, the symbol $s_g: n$ indicates the form of stabilizer. Here, n is the number of group.

5. Acknowledgment

The author would like to thank University of Mustansiriyah (https://uomustansiriyah. edu.iq) Baghdad-Iraq, and University of Sussex, Brighton-UK for their support in the present work.

6. Disclosure and conflict of interest

The authors confirm that there are no conflicts of interest.

References

[1] R. Lidl and H. Niederreiter, Introduction to finite fields and their applications, Cambridge University Press, 1994.

[2] A. A. Albert and R. Sandler, An introduction to finite projective planes, California: Courier Corporation, 2015.

[3] G. Kiss and T. Szonyi, Finite geometries, New York: Chapman and Hall/CRC, 2019.

[4] A. Barlotti, Some topics in finite geometrical structures, Carolina: University of Carolina, 1965.

[5] E. B. Al-Zangana, "Projective MDS Codes Over GF(27)," *Baghdad Science Journal,* vol. 18, no. 2 (Suppl.), pp. 1125-1132, 2021.

[6] E. B. Al-Zangana and E. A. Shehab, "Certain Types of Linear Codes Over the Finite Field of order Twenty-Five," *Iraqi Journal of Science,* vol. 62, no. 11, pp. 4019-4031, 2021.

[7] A. H. ALI, Classification of arcs in the Galois plane of order thirteen, Ph.D. Thesis, Brighton: University of Sussex, 1993.

[8] A. L. Yasin, Cubic Arcs in The Projective Plane of Order Eight, Ph.D. Thesis, Brighton: University of Sussex, 1986.

[9] A. J. S. Al-Rikabi, Construction of Some Types of Sets in Three Dimensional Projective Space over Finite Field of Order Eight, Ph.D. Thesis, Baghdad: Mustansiriyah University, 2022.

[10] E. B. Al-Zangana and M. Ibrahim, "Classification of subsets in finite projective line over Galois field of order twenty-seven," *Journal of Physics: Conference Series, IOP Publishing,* vol. 1818, no. 1, p. 012087, 2021.

[11] M. D. Boeck and G. V. d. Voorde, "Elation KM-arcs," *Combinatorica ,* vol. 39, no. 3, pp. 501-544, 2019.

[12] J. Thas, Arcs, Caps, Generalisations: Results and Problems. In50 years of Combinatorics, Graph Theory, and Computing, Chapman and Hall/CRC, 2019.

[13] R. N. Daskalov, "New Large (n;r)-Arcs in PG(2,q)," *Iranian Journal of Mathematical Sciences and Informatics,* vol. 17, no. 1, pp. 125-133, 2022.

[14] E. B. Al-Zangana and J. Radhi, "Construction of complete (k; r)-arcs from orbits in PG (3, 11)," *Al-Mustansiriyah Journal of Science,* vol. 33, no. 3, pp. 48-53, 2022.

[15] D. Bartoli, A. Davydov, G. Faina, S. Marcugini and F. Pambianco, "On sizes of complete arcs in PG (2, q)," *Discrete Mathematics,* vol. 312, no. 3, pp. 680-698, 2012.

[16] D. Bartoli, A. Davydov, G. Faina, S. Marcugini and F. Pambianco, "New upper bounds on the smallest size of a complete arc in a finite Desarguesian projective plane," *Journal of Geometry,* vol. 104, no. 1, pp. 11-43, 2013.

[17] M. Hall, The theory of groups, New York: Courier Dover Publications, 2018.

[18] M. Braun, "New lower bounds on the size of (n;r)-arcs in PG(2,q)," *Journal of Combinatorial Designs,* vol. 27, no. 11, pp. 682-687, 2019.

[19] Z. S. Hamed, "New sizes of complete (k, 4)-arcs in PG (2, 17)," *Baghdad Science Journal,* vol. 20, no. 2, pp. 0338-0338, 2023.

[20] T. G. Group, "GAP -- Groups Algorithms and Programming Version 4.11.0," 2021. [Online]. Available: https://www.gap-system.org..

[21] A. Barlotti, "Sui fk; ng-archi di un piano lineare finito," *Bollettino dell'Unione Matematica*

Italiana, vol. 11, no. 4, pp. 553-556, 1956.
[22] L. Lunelli and M. Sce, "Considerazioni arithmetiche e risultati sperimentali sui {K; n} q-archi," *Lombardo Accademia Science,* vol. 98, no. 15, pp. 3-52, 1964.