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On Rad_R *-lifting module

Mustafa Mohammed Jasim *, Wasan Khalid
Mathematics Department, College of Science, University of Baghdad, Iraq

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Abstract

The goal of this study is to introduce a new concept on a lifting module. Let M be a unitary left R-module, and let R be any ring with one. M is called Rad_R *-lifting for short (R^*-lifting) for any $A \le M$, there are some submodule B of A that is $M = B \oplus B_1$, $B_1 \le M$ and $A \cap B_1 \ll_{R^*} B_1$.We prove some characterization of this class of modules. We show that every R^*-hollow is R^*-lifting also. The quotient of R^*-lifting module is R^*-lifting under some conditions. Some results about this concept are given. The relation between this concept and other modules related with it are presented. In addition, FI-R^*-lifting was introduced as fallows: if for any submodule fully invariant $A \le M$, there is some submodule D of M that is $M = D \oplus D_1, D_1 \le M$ and $A \cap D_1 \ll_{R^*} D_1$.it was proved that for any direct summand of R^*-supplement is R^*lifting module.

Keywords: Lifting, R^*-coessentail submodule, R^*-Lifting module, fully R^*-lifting module.

حول مقاس الرفع من نمط Rad_R *−

مصطفى محمد جاسم*, وسن خالد
قسم الرياضيات, كلية العلوم, جامعة بغداد, بغداد, العراق

الخلاصة

الهدف من هذا البحث هو تقديم فكره جديده عن المقاس الرفع .نعطي مفهوم جديد للمقاس M على حلقة. لتكن مقاس الوحدانيه من اليسار R. اي حلقه مع عنصر المحايد. نطلق على M مقاس الرفع من نمط R^* باختصار (R^*−lifting) لكل مقاس جزئي $A \le M$ يوجد مقاس جزئي B في A بحيث $M = B \oplus B_1$, $B_1 \le M$ و $A \cap B_1 \ll_{R^*} B_1$. نثبت بعض الخواص المهمه لهذه الفئه من المقاسات نبين كل مقاس مجوف من نمط R^* هو مقاس رفع من نمط R^* . حاصل قسمه المقاس الرفع من نمط R^* تحت بعض الشروط والنتائج لهذا المفهوم يعطى. نقدم العلاقه بين فكره المقاس الرفع من نمط R^* والمقاسات الاخرى وبالاضافه نقدم ($FI-R^*$−lifting) مقاس الرفع ثابت كليا لكل مقاس جزئي ثابت كليا $A \le M$ يوجد مقاس جزئي D في M بحيث $M = D \oplus D_1, D_1 \le M$ و $A \cap D_1 \ll_{R^*} D_1$ نستنتج لكل جمع مباشر للمقاس المكمل هو مقاس رفع من نمط R^* وكل مقاس رفع هو ايضا مقاس رفع نمط R^* ثابت كليا.

*Email: mustafa.md27182@gmail.com

1. Introduction

Let M be a unitary RR-module, and R be any ring with identity. A submodule V of M is referred to as small in M if whenever $M = V + U$ for certain $U \leq M$ (Indicated by $V \ll M$) then $U = M$, [1] [2], [3]. A module M is hollow if $\forall A < M$ is small in M, [4], [5]. For module M is called lifting for any submodule A of M, there are some submodule B of M that is $M = B \oplus B_1$ $B_1 \leq M$ and $A \cap B_1 \ll B_1$, [1], [6], [7]. For every R-endomorphism f of M, if $f(A)$ contained in A and $A \leq M$, then f is called fully invariant, [8], [9]. The cosingular submodule of M was released by Oscan in the following manner:$Z^*(M)$={$m \in M; Rm \ll I(M)$},where $I(M)$ is the injective hull of M, see [10], [11].Where $Z^*(M)$ is called the cosingular submodule if $Z^*(M) = M$, M is called cosingular [12]. Several authors present generalizations of lifting module see [13], it was introduced a submodule V of M is called Rad_R $*$-small in M for short R^*-small submodule if whenever $M = V + U$ and $Rad_R * \left(\frac{M}{U}\right) = \frac{M}{U}$ (briefly $R^*\left(\frac{M}{U}\right) = \frac{M}{U}$) then $U = M$, notationally $V \ll_{R^*} M$, [14]. M is referred to as R^*-hollow module if it is non zero and every proper submodule of M is R^*-small [14]. We present a new generalization of lifting is called R^*-Lifting module**.** We begin by demonstrating some general features of R^*-lifting modules. It was proved that for an indecomposable module M, M is R^*-lifting iff M is R^*-hollow. In the same way, it will be concerned with Fully invariant R^*-lifting module (shortly FI-R^*-lifting) where M is called FI-R^*-lifting if for any submodule fully invariant $\leq M$, there is some submodule D of M that is $M = D \oplus D_1, D_1 \leq M$ and $A \cap D_1 \ll_{R^*} D_1$. A characterization of R^*-lifting.and (FI-R^*-lifting) are given and the main properties of these concepts are proven.

2. R^*-Lifting module

Recall that a submodul**e** A in M is called R^*-small submodule if whenever $M = A + B$ and $Rad_R * \left(\frac{M}{B}\right) = \frac{M}{B}$ (briefly $R^*\left(\frac{M}{B}\right) = \frac{M}{B}$) then $B = M$ notationally $A \ll_{R^*} M$, [14].

2.1 Definition: A Module M is called Rad_R $*$-lifting for short (R^*-lifting module) if each submodule A of M, there is submodule B of A such that $M = B \oplus B_1$, $B_1 \leq M$ and $A \cap B_1 \ll_{R^*} B_1$.

2.2 Remarks and examples

1. It seems that every lifting module is R^*-lifting module; the opposite is not true. For instance, Z as Z- module.
2. Let M be R-module and $R^*(M) = M$, then M is lifting iff M is R^*-lifting- module

Proof. $\Rightarrow$: Clear by (1)
$\Leftarrow$: Let $A \leq M$, since M is R^*-lifting module. Consequently $B \leq A$, there is $M = B \oplus B_1$, $B_1 \leq M$ and $A \cap B_1 \ll_{R^*} M$ .Hence $(A \cap B_1) + A_1 = M$ .To prove $R^*\left(\frac{M}{A_1}\right) = \frac{M}{A_1}$ and $A_1 = M$ **.** Let $\nu: M \to \frac{M}{A_1}$ is the natural projection, then $\nu(R^*(M)) \leq R^*(\frac{M}{A_1})$.By [3], hence $\nu(M) = \frac{M}{A_1} \leq R^*\left(\frac{M}{A_1}\right)$, thus $R^*\left(\frac{M}{A_1}\right) = \frac{M}{A_1}$ but $(A \cap B_1) + A_1 = M$ and $A \cap B_1 \ll_{R^*} M$, hence $A_1 = M$.

3. Every semi-simple module is R^*-lifting.
4. Z_6 as Z-module is R^*-lifting**.**
5. Q as Z-module is not R^*-lifting. It is known that Q does not lifting so by (2) Q as Z-module is not R^*-lifting.
6. Every R^*-hollow is R^*-lifting.

$Proof$. Suppose A contained in M. If $A \lneq M$, then $\ll_{R^*} M$, thus there is $\{0\} \leq A$, $\{0\}$ is a direct summand of M and $M \cap A = A \ll_{R^*} M$, hence M is R^*-lifting.

2.3 Remark

Let M be an R-module. Then M is R^*-lifting iff every submodule A of M, there exist a submodule B of M such that $M = B \oplus B_1$, $B_1 \leq M$ and $A \cap B_1 \ll_{R^*} M$.

$Proof$. See proposition (2.6) [14].

Take note of this:$B \leq A \leq M$, for module M.When $\frac{A}{B} \ll_{R^*} \frac{M}{B}$, then A is referred to as the Rad_R $*$-cossentail submodule of B in M (for short, R^*-coessential submodule), [14].

The following give characterization of R^*-lifting module

2.4 Theorem

Consider the R -module M. The following statements are comparable:

1. M is R^*- lifting module
2. For each submodule A of M there is $D \leq^{\oplus} M$,such that $A = D \oplus S$ and $S \ll_{R^*} M$.
3. For each submodule A of M there is $L \leq^{\oplus} M$ such that $L \leq A$ and $L \subseteq_{R^*ce} A \in M$.

$Proof$. (1) $\Rightarrow$ (2) : Let $A \leq M$ and since M is R^*-lifting, then there exist $B \leq M$ that is $M = B \oplus B_1$, $B_1 \leq M$ and $A \cap B_1 \ll_{R^*} M$ by remark (2.3). Now $A = A \cap M = A \cap (B \oplus B_1)$ $= B \oplus (A \cap B_1)$ (modular law) [3]. Take $D = B$, and $S = (A \cap B_1)$. Hence $D \leq^{\oplus} M$, $A = D \oplus S$ and $S \ll_{R^*} M$.

(2) $\Rightarrow$ (3): Let $A \leq M$. By (2) $= L \oplus S$,where $L \leq^{\oplus} M$ and $S \ll_{R^*} M$. We are done if we can show that $\frac{A}{L} \ll_{R^*} \frac{M}{L}$.Since $\frac{M}{L} = \frac{A}{L} + \frac{K}{L}$ and $R^*\left(\frac{\frac{M}{L}}{\frac{K}{L}}\right) = \frac{M}{K}$. Since $\frac{M}{L} = \frac{L+S}{L} + \frac{K}{L}$, hence $M = L + S + K = S + K$, but $S \ll_{R^*} M$, so $M = K$. Thus $\frac{M}{A} = \frac{K}{A}$, therefore $L \subseteq_{R^*ce} A$.

"(3)" $\Rightarrow$ "(1)" Let A be submodule of M, there is $U \leq^{\oplus} M$ such that $U \leq A$, $M = U \oplus U_1$ and $\frac{A}{U} \ll_{R^*} \frac{M}{U}$. By (3) We want demonstrate that $A \cap U_1 \ll_{R^*} U_1$. let $U_1 = (A \cap U_1) + N$ and $R^*\left(\frac{U_1}{N}\right) = \frac{U_1}{N}$,$N \leq U_1$ .Since $M = U + U_1 = U + (A \cap U_1) + N$ .Hence the quotient module $\frac{M}{U} = \frac{U+(A\cap U_1)+N}{U} = \frac{D+(A\cap U_1)}{U} + \frac{U+N}{U}$.Since $U \leq U + (A \cap U_1) \leq A$ and $U \subseteq_{R^*} A$ in M then $U \subseteq_{R^*Ce} U + (A \cap U_1)$ by [1]. Observe that (by second Isomorphism theorem) $\frac{M}{N+U} = \frac{U+U_1}{U+N} = \frac{U_1+(U+N)}{U+N} \cong \frac{U_1}{U_1 \cap (U+N)} = \frac{U_1}{(U_1 \cap U)+N} = \frac{U_1}{N}$ and $R^*\left(\frac{U_1}{N}\right) = \frac{U_1}{N}$, subsequently $R^*\left(\frac{M}{U+N}\right) = \frac{M}{U+N}$, by [14]. But $\frac{U+(A\cap U_1)}{U} \ll_{R^*} \frac{M}{U}$,so $\frac{M}{U} = \frac{U}{N}$ Thus $M = U + N$. However, ($N \leq U_1$ and $U \cap U_1 = 0$) , then $U \cap N = 0$ .Thus $M = U \oplus N$, that is $N = U_1$therefore M is R^*-lifting

2.5 Proposition

Let M be an indecomposable module M. Then M is R^*-lifting iff M is R^*-hollow.

$proof$. $\Rightarrow$ Let M be an R^*-lifting indecomposable module and $A < M$ .Since M is R^*-lifting there is $V \leq A$ such that $V \oplus V_1 = M$ and $A \cap V_1 \ll_{R^*} V_1$, but M is indecomposable, on the other hand $V = 0$ or $V = M$. If $V = M$,then $M = A$,which is contradiction. Consequently $V = 0$ and hence $M = V_1$, implies that $A \cap V_1 = A \ll_{R^*} V_1$.Thus $A \ll_{R^*} M$.

$\Leftarrow$ by Remarks and examples (2.2(6))

2.6 Proposition

Any direct summand of R^*-lifting module is R^*-lifting

$Proof.$ Let M be R^*-lifting module. Suppose that $M = M_1 \oplus M_2$, to prove M_1 is R^*-lifting. Let A be a submodule of M_1, so $A \leq M$ by theorem (2.4),$M = A \oplus L$, then $A \leq^{\oplus} M$, since $M_1 = M_1 \cap M = M_1 \cap (A \oplus L) = A \oplus (M_1 \cap L)$ by (modular law) [3] .Hence $A \leq^{\oplus} M_1$,and $L \ll_{R^*} M$, but $A \leq^{\oplus} M$,$L \leq M_1$ which is a direct summand of M, then (by (2.7), [14]) $L \ll_{R^*} M_1$ by theorem (2.4).Thus M_1 is R^*-lifting. Similarly, M_2 is R^*-lifting.

Recall that A and B are submodules of M, which could be R-module. A is referred to as Rad_R $*$-supplement modules of B in M (for short R^* -supplement) if we have $M = A + B$ and $A \cap B \ll_{R^*} A$, [1], [15], [16]. M is a R^*-supplemented module if any submodule of M contains has R^*-supplement submodule in M.

2.7 Theorem

Let M be a module in R. The following are comparable:

1. M is R^*-lifting module
2. For every $B \leq M$ possesses R^*-supplement A_1 in M, $A_1 \leq M$ that is $A_1 \cap B$ is direct summand of B.

$Proof.$ (1) $\Rightarrow$ (2): Suppose M is R^*-lifting and B is submodule of M by Theorem (2.4) there is $A \leq^{\oplus} M$ such that $M = A \oplus A_1, A \leq B$ and $A_1 \cap B \ll_{R^*} A_1$. Now $B = B \cap M = B \cap (A \oplus A_1) = A \oplus (B \cap A_1)$, (modular law) [3]. Since $A \leq B$,then $M = B + A_1$and $B \cap A_1 \ll_{R^*} A_1$ .Hence A_1 is R^*-supplement of B in $M$and $B \cap A_1 \leq^{\oplus} B$.

(2) $\Rightarrow$ (1): let $B \leq M$ by (2) B has R^*-supplement A in Msuch that $B \cap A \leq^{\oplus} B$.then $M = A + B$,$A \cap B \ll_{R^*} A$, so $B = (B \cap A) \oplus X$, but $M = A + B$ so $M = (A \cap B) \oplus X + A = A + X$ and $A \cap B \cap X = A \cap X = \{0\}$,then $M = A \oplus X$ and $A \cap B \ll_{R^*} A$, thus $M$is R^*-lifting.
The following proposition gives another characterization of R^*-lifting.

2.8 Proposition

Let M be an R-module. Then M is R^*-lifting module if and only if for any submodule B of M there is an idempotent $f \in End(M)$ such as $f(M) \leq B$ and $(1 - f)(B) \ll_{R^*} (1 - f)(M)$.

$Proof.$ $\Rightarrow$ Suppose that M is R^*-module and let $B \leq M$. By Theorem (2.7) B has R^*-supplement of A in M, such that $M = A + B$ and $A \cap B \ll_{R^*} A$, then $B \cap A \leq^{\oplus} B$. Consequently $B = (B \cap A) \oplus X, X \leq B$. Since $M = A + B = (B \cap A) + X + A$, hence $M = X + A$ and $B \cap A \cap X = A \cap X = \{0\}$, Therefore $M = A \oplus X$.Consider the projection map $f: A \oplus X \to X$. $f(M) \leq^{\oplus} M$, say that. Let $v \in M$.Subsequently $v = v + f(v) - f(v) = f(v) + \big(v - f(v)\big)$.Therefore $f(v) + (1 - f)(v)$,then $M = f(M) + (1 - f)(M)$. Hence $(1 - f)(B) = B \cap (1 - f)(M) = B \cap A \ll_{R^*} A = (1 - f)(M)$ See [14].

$\Leftarrow$ Let $B \leq M$ .By the theory there is an idempotent $f \in End(M)$ that is $f(M) \leq B$ and $(1 - f)(B) \ll_{R^*} (1 - f)(M)$.Now $M = f(M) + (1 - f)(M)$ and $B \cap (1 - f)(M) = (1 - f)(B) \ll_{R^*} (1 - f)(M)$.Thus M is R^*-lifting.

Next, we give some various conditions under which the quotient of R^*-lifting module is R^*-lifting.

A module M is said to be distributive if each of its submodule's A, B, C in M such that $(A + B) \cap C = A \cap C + B \cap C$, [17].

Not that a submodule N of M is called fully invariant for each $f \in And(M)$, $f(N) \subseteq N$, likewise a module M is said to be a duo module if for each of its submodules is fully invariant, [9], [8].

2.9 Proposition

If a module M is R^*-lifting, $B \leq M$, then the factor module is R^*-lifting module in the each of the following cases.

1. For every $D \leq^{\oplus} M$,Then $\frac{D+B}{B} \leq^{\oplus} \frac{M}{B}$.
2. M is distributive module.
3. B is fully invariant submodule $of\ M$.

$Proof.$

1. Suppose that M is R^*-lifting module. Let $\frac{X}{B} \leq \frac{M}{B}$,then there is $D \leq X$ such as $M = D \oplus D_1$,$D_1 \leq M$ and $D \subseteq_{R^*ce} X$ .By case (1) $\frac{D+B}{B} \leq^{\oplus} \frac{M}{B}$.Then $\frac{D+B}{B} \subseteq_{R^*ce} \frac{X}{B}$ in $\frac{M}{B}$ by [1].Therefore $\frac{M}{B}$ is R^*-lifting.
2. Suppose that M is distributive module, then $D \leq^{\oplus} M, M = D \oplus D_1, D_1 \leq M$. Hence the quotient module can be $\frac{M}{B} = \frac{D+B}{B} + \frac{D_1+B}{B}$ and $\frac{D+B}{B} \cap \frac{D_1+B}{B}$ .Then $\frac{(D_1 \cap B)}{B} + \frac{(D \cap B)+B}{B} = \frac{B}{B}$. Hence $\frac{D+B}{B} \leq^{\oplus} \frac{M}{B}$, so by (1) M is R^*-lifting.
3. Take $\frac{X}{B} \leq \frac{M}{B}$. Since M is R^*-lifting, there exist a submodule of D of X such that $D \subseteq_{R^*ce} X$ in M and $M = D \oplus D_1$, $D_1 \leq M$,by lemma [6]. Hence the quotient module $\frac{M}{B} = \frac{D+B}{B} \oplus \frac{D_1+B}{B}$. Let f: $(m + D) \to (m + D) + B$ such as $f(m + D) = (m + D) + B, \forall m \in M$ is epimorphism. Since $D \subseteq_{R^*ce} X$ in M,then $\frac{X}{D} \ll_{R^*} \frac{M}{D}$ and $f\left(\frac{X}{D}\right) = \frac{X}{B+D} \ll_{R^*} \frac{M}{B+D}$.Then $B + D \subseteq_{R^*ce} X$, hence $\frac{X}{D+B} \subseteq_{R^*ce} \frac{X}{B}$ in $\frac{M}{B}$ (by [1]).So $\frac{M}{B}$ is R^*-lifting.

2.10 Lemma

Let $M = A + B$ be R^*-lifting module if $R^*\left(\frac{M}{A}\right) = \frac{M}{A}$, there is $V \leq^{\oplus} M$ such that $M = A + V$ and $V \subseteq_{R^*ce} B$ in M.

$Proof.$ Assume $M = A + B$ be R^*-lifting and $R^*\left(\frac{M}{A}\right) = \frac{M}{A}$, $V \subseteq_{R^*ce} B$ such as $V \leq^{\oplus} M$.Hence the quotient module $\frac{M}{V} = \frac{A+V}{V} + \frac{B}{V}$ and $R^*\left(\frac{M}{A}\right) = \frac{M}{A}$, implies $R^*\left(\frac{M}{A+V}\right) = \frac{M}{A+V}$ by corollary((2.4) [14]). But $\frac{B}{V} \ll_{R^*} \frac{M}{V}$. Hence $\frac{M}{V} = \frac{A+V}{V}$,thus $M = A + V$.

Recall that submodule B of R-module A has ample R^*-supplement in A if for every $N \subseteq A$ with $+B = A$, there is R^*-supplement N_1 of B and $N_1 \subset N$ $i.e.$ $A = N_1 + B$ and $N_1 \cap B \ll_{R^*} N_1$, [1], [16].

2.11 Proposition

Let M be an amply R^*-supplemented module such that every R^*-supplement of M is direct summand, then M is R^*-lifting.

$Proof.$ Suppose M is amply R^*-supplemented module and let A be a submodule of M, then A has amply R^*-supplemented B hence $M = A + B$ and $A \cap B \ll_{R^*} B$,since M is amply R^*-supplemented and $M = A + B$ then A contains a R^*-supplement X of B by our assumption X is direct summand of M, $M = X \oplus Y. Y \leq M$.Now, $A = A \cap M = A \cap (X + Y) = X + (A \cap Y)$ by modular law .Since X is R^*-supplement of B in M, then $M = X + B$, hence $A = A \cap M = A \cap (X + B) = X + (A \cap B)$.Now Consider the projection map $P: M \to Y, P(A) = P\big(X + (A \cap Y)\big) = P(A \cap B)$,hence $P(A \cap B) = A \cap Y$.Since $A \cap B \ll_{R^*} M$,then by proposition ((2.6) [14]) $P(A \cap B) = A \cap Y \ll_{R^*} Y$.Thus M is R^*-liftting.

Recall that M is R-module and A be a sub module of M, we say that A is R^*-coclosed submodule of M indicated by ($A \subseteq_{R*CC} M$) if whenever $X \subseteq_{R*Ce} A$ in M for some X of A, we have $X = A$, [1].

2.12 Proposition

Let M be an R^*-lifting module. Then every R^*-coclosed contained in M is summand of M.

$Proof$. Let M be an R^*-lifting, so that there exists $D \leq A$ such that $M = D \oplus D`$ and $A \cap D` \ll_{R^*} M$ .Since M is R^*-coclosed. Thus $\cap D` \ll_{R^*} A$ . Now $A = A \cap M = A \cap (D \oplus D`) = D \oplus (A \cap D`)$ to show $R^*\left(\frac{A}{D}\right) = \frac{A}{D}$. let $f: A \to \frac{A}{D}$ be the projection homomorphism $f(A \cap D`) \cong \frac{(A \cap D`)+D}{D}$ Implies that , $\frac{A}{D} \cong \frac{(A \cap D`)+D}{D} \ll_{R^*} \frac{A}{D}$ and $\frac{(A \cap D`)+D}{D} \leq R^*\left(\frac{A}{D}\right)$, but $A \cap D` \ll_{R^*} A$,then $R^*\left(\frac{A}{D}\right) = \frac{A}{D}$ .Thus $A = D$.Therefore $A \leq^{\oplus} M$.

2.13 Proposition

Let $M = M_1 \oplus M_2$ be due module such that M_1 and M_2 are R^*-lifting modules. Then M is R^*-lifting module.

$Proof$. Let $M = M_1 \oplus M_2$ be duo module and $\leq M$, A is fully invariant submodule of M. Consequently $A \cap M = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2)$, by lemma ((5.4) [6]). Since M_1 and M_2 are R^*-lifting module. Then $A \cap M_1 = A_1 \oplus A_2$ and $A \cap M_2 = A_3 \oplus A_4$, A_1 and A_3 is direct decomposition of M_1 and M_2.Respectively A_2,A_4 are R^*-small submodule of M_1 and M_2,respectively by theorem (2.4), $A_1 \oplus A_3 \leq^{\oplus} M$ and $A_2 \oplus A_4 \ll_{R^*} M_1$.Thus M is R^*-lifting.

Take note that if M_1 and M_2 are R-module then M_1 is M_2-projective if any submodule A of M_2 and every $f: M_1 \to \frac{M_2}{A}$ is homomorphism, there is $g: M_1 \to M_2$ is homomorphism to the extent that $\pi \circ g = f$ where $\pi: M_2 \to \frac{M_2}{A}$ is the natural epimorphism . M_1 and M_2 are said relatively projective if M_1 is M_2-projective and M_2 is M_1-projective see [18].

2.14 Proposition

Let $M = M_1 \oplus M_2$ where M_1 is R^*-lifting module and let M_2 is M_1-projective. Then the following statement are identical

1. M is R^*-lifting module.
2. For every $A \leq M$ such as $M \neq A + M_1$, there is $D \leq^{\oplus} M$ that is $D \subseteq_{R^*ce} A$ in M.

$Proof$. $(1) \Rightarrow (2)$: It is clear.

$(2) \Rightarrow (1)$: Let A be a submodule of M and let $M = A + M_1$Since M_2 is M_1 projective ,then there exist a submodule $A_1 \leq A$ such that $M = A_1 \oplus M_1$,(by lemma(4.47) [18]) .But, M_1 is R^*-lifting and $\frac{M}{A_1} = \frac{A_1+M_1}{A_1} \cong \frac{M_1}{A_1 \cap M_1} = M_1$(by the second isomorphism theorem) ,therefore $\frac{M}{A_1}$ is R^*-lifting ,so there exist summand $\frac{D}{A_1}$ of $\frac{M}{A_1}$ such that $\frac{D}{A_1} \subseteq_{R^*ce} \frac{A}{A_1}$ in $\frac{M}{A_1}$. Hence $D \subseteq_{R^*ce} A$ in M.Now $D = D \cap M = A \cap (A_1 \oplus M_1) = A_1 \oplus (D \cap M_1)$,by modular law but $\frac{D}{A_1}$ is direct summand of M_1,so $\frac{A_1 \oplus (D \cap M_1)}{A_1} \leq^{\oplus} \frac{A_1 \oplus M_1}{A_1}$ .Furthermore,$\frac{A_1 \oplus (D \cap M_1)}{A_1} \cong \frac{D \cap M_1}{A_1 \cap (D \cap M_1)} = D \cap M_1$(by the second isomorohim theorem). Hence $D \cap M_1 \leq^{\oplus} M_1$.Let $M_1 = (D \cap M_1) \oplus Y$,for some submodule Y of M. Consequently $M = A_1 \oplus M_1 = A_1 \oplus (D \cap M_1) \oplus Y = D \oplus Y$ .Therefore M is R^*-lifting.

We conclude this section with the following proposition.

2.15 Proposition

Let M_1 and M_2 be R^*-lifting modules such that M_i is M_j-projective $(i,j = 1,2)$ .Then $M = M_1 \oplus M_2$ .Then M is R^*-lifting.

Proof. To show that M is R^*-lifting assume that M_1 and M_2 are R^*-lifting module and $A \leq M$ .Consider the submodule $M_1 \cap (A + M_2) \leq M_1$, since M_1 is R^*-lifting, so there is decomposition $M_1 = A_1 \oplus B_1$ such that $A_1 \leq M_1 \cap (A + M_2)$ and $[M_1 \cap (A + M_2)] \cap B_1 = B_1 \cap (A + M_2) \ll_{R^*} B_1$. Therefore $M = M_1 \oplus M_2 = A_1 \oplus B_1 \oplus M_2 = M_1 \cap (A + M_2) + B_1 + M_2 = A + (M_2 \oplus B_1)$.Since $M_2 \cap (A + B_1) \leq M_2$ and M_2 is R^*-lifting .Subsequently $M_2 = A_2 \oplus B_2$,such that $A_2 \leq M_2 \cap (A + B_1) \ll_{R^*} B_2$ .Because we have $M = A + (B_1 \oplus M_2) = A + B_1 + A_2 + B_2 = A + (B_1 \oplus B_2)$, so $M = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2)$.Since M_i is M_j-projective .Then M_1 is M_j-projective and M_2 is M_j projective $(j = 1,2)$ and hence A_1 is B_j-projective and A_2 is B_j-projective.So (by proposition 2.1.6 [19]) A_1 is $B_1 \oplus B_2$-projective and A_2 is $B_1 \oplus B_2$-projective .So (by proposition(2.1.6) [19]) $A_1 \oplus A_2$ is $B_1 \oplus B_2$-projective .Then there exist $X \leq A$ such that $M = X \oplus (B_1 \oplus B_2)$ by lemma (4.47) [18]) .Since $B_1 \cap (A + M_2) \ll_{R^*} B_1$ and $B_2 \cap (A + B_1) \ll_{R^*} B_2$,then $B_1 \cap (A + M_2) \oplus B_2 \cap (A + M_2) \ll_R B_1 \oplus B_2$.subsequently$A \cap (B_1 \oplus B_2) \leq B_1 \cap (A + M_2) \oplus B_2 \cap (A + B_1) \ll_{R^*} B_1 \oplus B_2$,(by proposition (3.6) [14]). As a result , $A \cap (B_1 \oplus B_2) \ll_{R^*} B_1 \oplus B_2$.

3. Fully invariant R^*-lifting modules

3.1 Definition

A module M is referred to as fully invariant R^*-lifting module (shortly FI-R^*-lifting) if for any fully invariant submodule A of M, there is submodule D of M such that $M = D \oplus D_1$, $D_1 \leq M$ and $A \cap D_1 \ll_{R^*} D_1$.

3.2 Remarks and example

1. It is evident that every R^*-lifting module is FI-R^*-lifting.
2. Z_6 as Z-module is FI-R^*-lifting.
3. Q as Z-module is not FI-R^*-lifting.

3.3Proposition

let M be an R-module, then M is FI-R^*-lifting if and only if for any fully invariant submodule A of M, there is $D \leq^{\oplus} M$ that is $M = D \oplus D_1, D_1 \leq M, A \cap D_1 \ll_{R^*} M$.

Proof . It is evident by proposition (2.6 (3) [14]).

3.4 Proposition

Let M be an R-module. The statements below are identical.

1. M is FI-R^*-lifting module
2. For each fully invariant submodule A of M can be written as $A = D \oplus S$, $D \leq^{\oplus} M$ and $S \ll_{R^*} M$,
3. For each fully invariant submodule A of M can be written as $A = D + S$, $D \leq^{\oplus} M$ and $S \ll_{R^*} M$.
4. For each fully invariant $A \leq M$ there exist $D \leq^{\oplus} M$ such that $D \leq A$ and $D \subseteq_{R^*ce} A$ in M.

Proof. (1) ⇒ (2): Suppose that M is FI-R^*-lifting module and $A \leq M$, A is fully invariant, then there is $D \leq^{\oplus} M$ such that $= D \oplus D_1$,$D_1 \leq M$ and $A \cap D_1 \ll_{R^*} M$ (proposition3.3). Now $A = M \cap A = D \oplus (A \cap D_1)$, take $S = (A \cap D_1)$,so we get the result.

(2) ⇒ (3): It is evident.

(3) ⇒ (4): Assume $A \leq M$, A fully invariant submodule, by (3) $A = D \oplus S$, where D is direct summand of M and $S \ll_{R^*} M$. So $M = D \oplus D_1$, $D \leq M$, then D_1 is R^*-supplement of $D + S = A$ in M by [1]. To show $D \subseteq_{R^*} A$ in M, suppose $\phi: D_1 \to \frac{M}{D}$ be function specified by $\phi(x) = x + D$, for each $x \in D_1$. Evidently ϕ is isomorphism. Since $A \cap D_1 \ll_{R^*} D_1$, then $\phi(A \cap D_1) = \frac{A}{D} \ll_{R^*} \frac{M}{D}$. Thus M is R^*-lifting.

(4) ⇒ (1): Suppose $A \leq M$, A fully invariant submodule, by (4) there is $D \leq^{\oplus} M$ such that $D \leq A$ and $\frac{A}{D} \ll_{R^*} \frac{M}{D}$ .Let $D_1 = (A \cap D_1) + B$ where $B \leq D_1$ and $R^*\left(\frac{D_1}{B}\right) = \frac{D_1}{B}$. To demonstrate $D_1 = B$. Since $M = D + D_1 = D + (A \cap D_1) + B$. Now $D \leq D + (A \cap D_1) \leq A$ and $D \subseteq_{R^*ce} A$ in M. Hence the quotient module $\frac{M}{D} = \frac{B+(A\cap D_1)}{D} + \frac{B+D}{D}$. Then $D \subseteq_{R^*} D + (A \cap D_1)$ in , (by [1]).Since $\frac{M}{D+B} = \frac{D+D_1}{D+B} = \frac{D_1+(D+B)}{D+B} \cong \frac{D_1}{D_1\cap(D+B)} = \frac{D_1}{B}$. (Isomorphism theorem and modular law) and $R^*\left(\frac{D_1}{B}\right) = \frac{D_1}{B}$. Therefore $R^*\left(\frac{M}{D+B}\right) = \frac{M}{D+B}$ and $\frac{B+(A\cap D_1)}{D} \ll_{R^*} \frac{M}{D}$ .Hence $\frac{M}{D} = \frac{D+B}{D}$, implies that $M = B + D$. Since $B \leq D_1$ and $D \cap D_1 = 0$, then $B \cap D = 0$.Hence $M = D \oplus B$,therefore $D_1 = B$.Thus M is FI-R^*-module.

The following theorem can be demonstrated using the same logic as the proof of theorem (2.8).

3.5 Theorem

Assume M is R-module. The statements below are identical

1. M is FI-R^*-lifting module
2. Each fully invariant submodule $B \leq M$ has R^*-supplement A in M, such that $A \cap B \leq^{\oplus} A, A \leq M$.

3.6 Theorem

Let M be an R-module. Then M is FI-R^*-lifting module iff $B \leq M$, A is fully invariant, there is an idempotent $f \in And\ (M)$ that is $f(M) \leq B$ and $(1-f)(B) \ll_{R^*} (1-f)(M)$.

Proof. By the same argument of proposition (2.9).

3.7 Proposition

Let M be an R-module. The following statement are identical.

1. M is FI-R^*-lifting
2. M has a direct summand R^*-supplement for any fully invariant submodule of M
3. For any fully invariant submodule A of M, there is a R^*supplement $D \leq M$ and a direct summand R^*supplement $D_1$of D such that $D \subseteq_{R^*ce} A$ in M and each homomorphism $f: M \to \frac{M}{D\cap D_1}$ can be lifted to an endomporphism $\alpha: M \to M$ such that $\alpha(m) + (D \cap D_1) = f(m)\ \forall m \in M$.

Proof. (1) ⇒ (2): Suppose M is R^*-lifting and let $A \leq M$, A is fully invariant, then there is $D \leq M$, $D \leq A$,$M = D \oplus D_1$,$D_1 \leq M$ and $A \cap D_1 \ll_{R^*} D_1$.Clearly that D_1 is R^*-supplement of M .Conversely. By our assumption, there is $B \leq^{\oplus} M$ such that $M = B \oplus B_1$ and B is R^*-supplement of A in M. It suffices to demonstrate that $B_1 \leq A$ .Consider the projection map. $\rho: M \to B_1$, since A fully invariant submodule of M, $\rho(A) = (A + B) \cap B_1 = M \cap B_1 = B_1 \leq A$.Thus M is FI-R^*-lifting.

(2) ⇒ (3): Let $A \leq M$, A fully invariant submodule Since M is FI-R^*-lifting, there exist a decomposition $M = D \oplus D_1, D \leq A$ and $D \subseteq_{R^*ce} A$ in M then by [1] D is R^*-coclosed

submodule of A in M and it is evident D_1 is summand R^*-supplement of D in M .Since $D \cap D_1 = 0$,then the result is obtained.
(3) $\Rightarrow$ (1): Let $A \leq M$, A fully invariant, by (3) there is $D \leq A$ such that $D \subseteq_{R^*ce} A$ in M and R^*-supplement D_1 of D in M that is $D_1$is a direct summand of M and $D \subseteq_{R^*ce} A$ in M and each homomorphism $f: M \to \frac{M}{D \cap D_1}$ can be lifted to an endomorphism $\alpha: M \to M$ such that $\alpha(m) + (D \cap D_1) = f(m)$ $\forall m \in M$.it follow from lemma ((2.2) [20]).That is D direct summand of M. Thus A is FI-R^*-lifting.

3.8 Lemma

1. Any sum or intersection of fully invariant submodules is again fully invariant submodule of .
2. If $A \leq B \leq M$ such that A fully invariant submodule of B and B is fully invariant submodule of M,then A is fully invariant submodule of M
3. If $M = \oplus_{i=1}^{n} M_i$ and A is fully invariant submodule of M, then $A = \oplus_{i=1}^{n} (A \cap M_i)$ and $A \cap M_i$is fully invariant submodule of M_i, [21]

3.9 Proposition

Let M be FI-R^*-lifting and A is direct summand fully invariant submodule of M, then A is FI-R^*-lifting.

Proof. Let $M = A \oplus B$ be FI-R^*-lifting ,$B \leq M$, and A fully invariant submodule of M, to show that A is FI-R^*-lifting let X fully invariant submodule of A, (lemma (3.8)) hence $X = D \oplus S$,where D is direct of $M, B \leq X \leq M$ and $S \ll_{R^*} M$ implies that D is a direct summand of A (by proposition 2.7 [14]) $S \ll_{R^*} M$. Thus A FI-R^*-lifting.

The following theorem demonstrates that a finite direct summand of the FI-R^*-lifting module is FI-R^*-lifting

3.10 Theorem

If $M_i, i = 1,2, \ldots, n$ are FI-R^*-lifting then the direct decomposition $M = \oplus_{i=1}^{n} M_i$ is FI-R^*-lifting.

Proof. Suppose A is fully invariant submodule of M, then $A = \oplus_{i=1}^{n} (A \cap M_i)$ and $A \cap M_i \leq M_i$, $A \cap M_i$ is fully invariant of $M_i, \forall\, i = 1,2,3, \ldots\ldots, n$ by lemma (3.8). Then $\cap M_i = B_i \oplus S_i$, where B_i is a direct summand of M_i and $S_i \ll_{R^*} M_i$, $\forall\, i = 1,2, \ldots, n$ .Let $B = \oplus_{i=1}^{n} B_i$ and $S = \oplus_{i=1}^{n} S_i$. Since each B_i is FI-R^*-lifting, then B_i is direct summand of M and $S \ll_{R^*} M$.Then M is FI-R^*-lifting.

We end this section by showing that the connection between FI-R^*-lifting and R^*-supplement with the quotient submodule

3.11 Proposition

Let $M = M_1 \oplus M_2$ .Then M is FI-R^*-lifting iff for each fully invariant submodule $\frac{A}{M_1} \leq \frac{M}{M_1}$,there is $B \leq^{\oplus} M$ such that $B \leq M_2, M = B + A$ and $A \cap B \ll_{R^*} B$.

Proof . $\Rightarrow$: Suppose that M_2 is FI-R^*-lifting and for each fully invariant submodule $\frac{A}{M_1} \leq \frac{M}{M_1}$, by lemma (3.8), $A \cap M_2 \leq M_2$,$A \cap M_2$ is fully invariant submodule of M_2. Since M_2 is FI-R^*-lifting, there is $B \leq^{\oplus} M$ such that $B \leq A \cap M$ and $M_2 = B \oplus B_1$, hence $M_2 = (A \cap M_2) + B_1$ and $A \cap B_1 \ll_{R^*} B_1$.Thus $M = B_1 + A$.

⇐: To demonstrate that M_2 is $FI\text{-}R^*$-lifting. Suppose A is fully invariant submodule of M_2. Then $\frac{A \oplus M_1}{M_1} \leq \frac{M}{M_1}$, $\frac{A \oplus M_1}{M_1}$ is fully invariant submodule of $\frac{M}{M_1}$, by [6]. By our theory there is a direct summand B of M such that $B \leq M_2, M = B + A + M_1$ and $B \cap (A + M_1) \ll_{R^*} B$,but $B \cap A \leq B \cap (A + M_1) \ll_{R^*} B$,so $B \cap A \ll_{R^*} B$ .Now $M_2 = M_2 \cap M = M_2 \cap (B + A + M_1) = B + A$.Subsequently B is R^*-supplement of A in M_2.Therefore M_2 is $FI\text{-}R^*$-lifting by proposition (3.7).

4. Conclusion

This document introduces a new generalization of lifting module called R^*-lifting module, we illustrate the connection of R^*-lifting module with another concept, for instance the indecomposable module M is R^*-lifting iff M is R^* hollow and for duo module where $M = M_1 \oplus M_2$ be due module that is M_1 and M_2 are R^*-lifting modules. Then M is R^*-lifting module. Finally the concept of fully invariant R^*-lifting module are given for each fully submodule. For instance, the direct summand of M where M be $FI\text{-}R^*$-lifting and A is fully invariant submodule direct summand of M, then A is $FI\text{-}R^*$-lifting and $M = \bigoplus_{i=1}^{n} M_i$ be a direct decomposition of $FI\text{-}R^*$-lifting. Then M is $FI\text{-}R^*$-lifting.

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