DOI: 10.24996/ijs.2025.66.12.26

Iraqi Journal of Science

ISSN: 0067-2904

Solving Hirota-Satsuma Coupled KdV equations of time-fractional order by using Sumudu Transform with Adomian Decomposition Method

Mohamed A. Khadr*, Abdulghafor M. Al-Rozbayani
Department of mathematics, College of Computer Sciences and Mathematics, University of Mosul, Mosul, Iraq

Received: 18/3/2024 Accepted: 17/10/2024 Published: 30/12/2025

Abstract
The primary goal of this research is to obtain an approximate solution to the Hirota-Satsuma coupled KdV equations with the lowest error rate, specifically when the time derivative is associated with a fraction and subject to specific initial conditions. In this work, two basic concepts are presented, the fractional Caputo derivative of order α>0 and the Sumudu Transform (ST), along with some inherent properties of this transformation. To address the problem at hand, a hybrid approach was adopted using numerical methods, combining the Sumudu Transform and the Adomian Decomposition Method (ADM). The Adomian Decomposition Method, as it is known, works to deal with complex nonlinear boundaries that are difficult to deal with directly. The article's methodology is carefully designed to prioritize clarity and simplicity and avoid unnecessary complexity. The results derived from this proposed method show a commendable level of accuracy and agree closely with exact solutions. The paper performs a comprehensive error analysis, ensuring the robustness and accuracy of the results.

Keywords: Adomain Decomposition Method, Caputo derivative, Discrete invers Sumudu, Fractional-time Hirota-Satsuma Coupled KdV equations, Sumudu Transform.

حل معادلات Hirota–Satsuma Coupled KdV المقترنة ذات الرتبة الزمنية الكسرية باستخدام طريقة تحويل سومودو مع تحليل ادوميان

محمد خضر*، عبد الغفور الروزبياني
قسم الرياضيات، كلية علوم الحاسوب و الرياضيات، جامعة الموصل، الموصل، العراق

الخلاصة
الهدف الأساسي من هذا البحث هو الحصول على حل تقريبي لمعادلات Hirota–Satsuma المقترنة KdV ذات الرتبة الزمنية الكسرية بأقل معدل خطأ، ويخضع لشروط أولية محددة. في هذا العمل تم عرض مفهومين أساسيين، مشتقة كابوتو الجزئية من الرتبة $\alpha > 0$ وتحويل السومودو (ST)، إلى جانب بعض الخصائص المتأصلة في هذا التحويل. ولمعالجة المشكلة المطروحة، تم اعتماد نهج هجين باستخدام الطرق العددية، والجمع بين تحويل سومودو وطريقة ادوميان التحليلية (ADM). تعمل طريقة ادوميان التحليلية كما هو معروف على التعامل مع الحدود غير الخطية المعقدة التي يصعب التعامل معها بشكل مباشر. تم تصميم

*Email: mohamed.csp121@student.uomosul.edu.iq

منهجية البحث بعناية لإعطاء الأولوية للوضوح والبساطة وتجنب التعقيد غير الضروري. تظهر النتائج المستمدة من هذه الطريقة المقترحة مستوى جديرًا بالثناء من الدقة وتتفق بشكل وثيق مع الحلول الدقيقة. يقوم البحث بإجراء تحليل شامل للأخطاء، مما يضمن قوة النتائج ودقتها.

1. Introduction

In 1981, Hirota and Satsuma presented the coupled Hirota-Satsuma KdV equations (HS-CKdVEs), commonly known as HS-CKdVEs. This equation expresses the interaction of two long waves with dissimilar dispersion relations. It is known to be partial non-linear differential equations are used over a large range to express many significant phenomena. The HS-CKdVEs problem forms a crucial basis in applied mathematics and physics and has several applications in physics and engineering. Fractional partial differential equations (FPDEs) have received the attention of many investigators because of their potential for real-world development subjects used in many areas of engineering and physics. Various approximate and exact procedures have been proposed for solving linear and nonlinear FPDEs [1], such as Laplace decomposition method [2], homogeneous perturbation. Method [3, 4], Laplace Homotopy perturbation method [5], Sumudu variable iteration method [6], Variance frequency transformation method [7], and other methods [8, 9, 10].

In this research, Sumudu Transform with Adomian Decomposition Method (SDM) are provided for solving HS-CKdVEs of time-fractional order, which is defined as follows [11, 12]:

$$D_t^\alpha u = \tfrac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x \qquad (1)$$
$$D_t^\alpha v = -v_{xxx} + 3uv_x, \qquad t > 0, 0 < \alpha \le 1$$
$$D_t^\alpha w = -w_{xxx} + 3uw_x.$$

Subject to the initial conditions:

$$u(x,0) = \frac{\beta - 2k^2}{3} + 2k^2 tanh^2(kx), \qquad (2)$$
$$v(x,0) = -\frac{4k^2 c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)\tanh(kx)}{3c_1},$$
$$w(x,0) = c_0 + c_1 \tanh(kx),$$

where $k, c_0, c_1 \neq 0, and\ \beta$ are arbitrary constants. The definition of Caputo's derivative is used because the initial conditions are of integer order. The circumstance of α=1 in Eq.1 was answered through Wu et al [1]. Here it should be noted that previously the system Eq.1 was also solved approximately in an approximate manner [12, 13, 14].

1.1 Caputo Derivative.

The Caputo fractional derivatives of order $\alpha > 0$ is clear as [15]

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial_t^\alpha} \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n} d\tau, n-1 < \alpha < n, & n \in N, \\ \dfrac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n, n \in N. \end{cases}$$

1.2 Sumudu Transform (ST) [16]

The Sumudu transform (ST) over the set,

$A = \left\{ u(t) | \exists M, T_1, T_2 > 0,\ |u(t)| < Me^{\frac{|t|}{Tj}},\ if\ t \in (-1)^j \times [0, \infty] \right\}$, is clear by,

$$G(f) = S[u(t)] = \int_0^\infty u(ft)e^{-t} dt = \frac{1}{f}\int_0^\infty u(t)e^{-\frac{t}{f}} dt,\ f \in (-T_1, T_2).$$

The ST satisfying the following properties [17]**:**
1. $S[1] = 1$;
2. $S\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right] = f^{\alpha},\ \alpha > 0$;
3. $S[e^{at}] = \frac{1}{1-af}$;
4. $S[\alpha g(x) \mp \beta h(x)] = \alpha S[g(x)] \mp \beta S[h(x)]$;
5. $S[D_t^{\alpha} u(x,t)] = \frac{S[u(x,t)]}{f^{\alpha}} - \sum_{k=0}^{n-1} f^{-\alpha+k} u^{(k)}(x,0), \quad n-1 < \alpha \leq n.$

where S is Sumudu transform operator.

Let the variable t inverse Sumudu transform to variable f. Therefore, discrete inverse Sumudu transform of $U(f) = \sum_{n=0}^{\infty} b_n f^n$, of the power series is define as [18]:
$S^{-1}[U(f)] = u(t) = \sum_{n=0}^{\infty} (\frac{1}{n!}) b_n t^n$.

1.3 Least Square Weighted Function (LSWF)

Suppose the exact solution $y(x,t)$ of nonlinear FPDEs defined on a closed square $D = [a,b] \times [0,T]$ as follows:

$$Ny(x,t) = f(x,t), \tag{3}$$

where N be a nonlinear differential operator, a, b are real constant and $T > 0$. Assume $Y(x,t)$ is an approximate solution of y. The following residual error $R(x,t)$ is evaluated by substituted Y in Eq.3:

$$R(x,t) = NY\,(x,t) \; - \; f(x,t) \tag{4}$$

The main objective of this definition is to minimize $R(x,t)$. The least squares weighted method is considered an appropriate numerical measure to determine the amount of error resulting from approximation and is defined as follows [19, 20]:

$$\text{LSWF} = \int_a^b \int_0^T |R(x,t)|^2 \, dt dx . \tag{5}$$

2. Methodology and Application

Now, a new approach represented by ST combined with the Adomian decomposition method is presented to find the solution of the system in Eq.1:
Using ST of system Eq.1

$$S[D_t^{\alpha} u] = \frac{1}{2} S[u_{xxx}] - 3\, S[\, u u_x] + 3S[(vw)_x\,], \tag{6}$$
$$S[D_t^{\alpha} v] = -S[v_{xxx}] + 3S[u \mathrm{v_x}],$$
$$S[D_t^{\alpha} w] = -S[w_{xxx}] + 3S[u w_x].$$

By using properties of the ST:

$$\left[\frac{S[u(x,t)]}{f^{\alpha}} - \sum_{k=0}^{m-1} f^{-\alpha+k} u^{(k)}(x,0)\right] = -\frac{1}{2} S[u_{xxx}] - 3S[u u_x] + 3S[(vw)_x], \tag{7}$$
$$\left[\frac{S[v(x,t)]}{f^{\alpha}} - \sum_{k=0}^{m-1} f^{-\alpha+k} v^{(k)}(x,0)\right] = -S[v_{xxx}] + 3S[u v_x],$$
$$\left[\frac{S[w(x,t)]}{f^{\alpha}} - \sum_{k=0}^{m-1} f^{-\alpha+k} w^{(k)}(x,0)\right] = -S[w_{xxx}] + 3S[u w_x].$$

So,

$$S[u(x,t)] = \sum_{k=0}^{m-1} f^k u^{(k)}(x,0) - \frac{1}{2} f^{\alpha} S\left[\frac{\partial^3 u(x,t)}{\partial x^3}\right] - 3 f^{\alpha} S\left[u \frac{\partial u(x,t)}{\partial x}\right] + 3 f^{\alpha} S\left[v \frac{\partial w(x,t)}{\partial x} + w \frac{\partial v(x,t)}{\partial x}\right], \tag{8}$$

$S[v(x,t)] = \sum_{k=0}^{m-1} f^k v^{(k)}(x,0) - f^\alpha S\left[\frac{\partial^3 v(x,t)}{\partial x^3}\right] + 3f^\alpha S\left[u\frac{\partial v(x,t)}{\partial x}\right],$

$S[w(x,t)] = \sum_{k=0}^{m-1} f^k w^{(k)}(x,0) - f^\alpha S\left[\frac{\partial^3 w(x,t)}{\partial x^3}\right] + 3f^\alpha S\left[u\frac{\partial w(x,t)}{\partial x}\right].$

Using Eq.2

$$S[u(x,t)] = \frac{\beta-2k^2}{3} + 2k^2 tanh^2(kx) - \frac{1}{2}f^\alpha S\left[\frac{\partial^3 u(x,t)}{\partial x^3}\right] - 3f^\alpha S\left[u\frac{\partial u(x,t)}{\partial x}\right] + 3f^\alpha S\left[v\frac{\partial w(x,t)}{\partial x} + w\frac{\partial v(x,t)}{\partial x}\right], \quad (9)$$

$$S[v(x,t)] = -\frac{4k^2 c_0(\beta+k^2)}{3c_1^2} + \frac{4k^2(\beta+k^2)\, tanh(kx)}{3c_1} - f^\alpha S\left[\frac{\partial^3 v(x,t)}{\partial x^3}\right] + 3f^\alpha S\left[u\frac{\partial v(x,t)}{\partial x}\right],$$

$$S[w(x,t)] = c_0 + c_1 tan\, h(kx) - f^\alpha S\left[\frac{\partial^3 w(x,t)}{\partial x^3}\right] + 3f^\alpha S\left[u\frac{\partial w(x,t)}{\partial x}\right].$$

Now, by using the Adomian decomposition method for right of system in Eq.9 it can be writing the solution of u, v and w as follows:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (10)$$

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t),$$

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t).$$

The nonlinear terms: as follows:

$u\frac{\partial u(x,t)}{\partial x} = \sum_{n=0}^{\infty} A_n,$

$v\frac{\partial w(x,t)}{\partial x} = \sum_{n=0}^{\infty} B_n,$

$w\frac{\partial v(x,t)}{\partial x} = \sum_{n=0}^{\infty} M_n,$

$u\frac{\partial v(x,t)}{\partial x} = \sum_{n=0}^{\infty} E_n,$

$u\frac{\partial w(x,t)}{\partial x} = \sum_{n=0}^{\infty} F_n.$

Where,

$$A_0 = u_0\frac{\partial u_0(x,t)}{\partial x}, \quad (11)$$

$$A_1 = u_1\frac{\partial u_0(x,t)}{\partial x} + u_0\frac{\partial u_1(x,t)}{\partial x},$$

$$A_2 = u_2\frac{\partial u_0(x,t)}{\partial x} + u_1\frac{\partial u_1(x,t)}{\partial x} + u_0\frac{\partial u_2(x,t)}{\partial x},$$

$$B_0 = v_0\frac{\partial w_0(x,t)}{\partial x},$$

$$B_1 = v_1\frac{\partial w_0(x,t)}{\partial x} + v_0\frac{\partial w_1(x,t)}{\partial x},$$

$$B_2 = v_2\frac{\partial w_0(x,t)}{\partial x} + v_1\frac{\partial w_1(x,t)}{\partial x} + v_0\frac{\partial w_2(x,t)}{\partial x},$$

$$M_0 = w_0\frac{\partial v_0(x,t)}{\partial x},$$

$$M_1 = w_1\frac{\partial v_0(x,t)}{\partial x} + w_0\frac{\partial v_1(x,t)}{\partial x},$$

$$M_2 = w_2\frac{\partial v_0(x,t)}{\partial x} + w_1\frac{\partial v(x,t)}{\partial x} + w_0\frac{\partial v_2(x,t)}{\partial x},$$

$$E_0 = u_0\frac{\partial v_0(x,t)}{\partial x},$$

$$E_1 = u_1\frac{\partial v_0(x,t)}{\partial x} + u_0\frac{\partial v_1(x,t)}{\partial x},$$

$$E_2 = u_2\frac{\partial v_0(x,t)}{\partial x} + u_1\frac{\partial v_1(x,t)}{\partial x} + u_0\frac{\partial v_2(x,t)}{\partial x},$$

$F_0 = u_0 \frac{\partial w_0(x,t)}{\partial x}$,
$F_1 = u_1 \frac{\partial w_0(x,t)}{\partial x} + u_0 \frac{\partial w_1(x,t)}{\partial x}$
$F_2 = u_2 \frac{\partial w_0(x,t)}{\partial x} + u_1 \frac{\partial w_1(x,t)}{\partial x} + u_0 \frac{\partial w_2(x,t)}{\partial x}$

Substituting Eq.10 and Eq.11, in Eq.9 so we get:

$$S[\sum_{n=0}^{\infty} u_n(x,t)] = \frac{\beta-2k^2}{3} + 2k^2 tanh^2(kx) + \frac{1}{2} f^\alpha \left[\frac{\partial^3 S[\sum_{n=0}^{\infty} u_n(x,t)]}{\partial x^3}\right] - 3f^\alpha S[\sum_{n=0}^{\infty} A_n] + 3f^\alpha S[\sum_{n=0}^{\infty} B_n] + 3f^\alpha S[\sum_{n=0}^{\infty} M_n], \quad (12)$$

$$S[\sum_{n=0}^{\infty} v_n(x,t)] = -\frac{4k^2 c_0(\beta+k^2)}{3c_1^2} + \frac{4k^2(\beta+k^2)\, tanh(kx)}{3c_1} - f^\alpha \left[\frac{\partial^3 S \sum_{n=0}^{\infty} v_n(x,t)}{\partial x^3}\right] + 3f^\alpha S[\sum_{n=0}^{\infty} E_n],$$

$$S\left[\sum_{n=0}^{\infty} w_n(x,t)\right] = c_0 + c_1 tan\,h(kx) - f^\alpha \left[\frac{\partial^3 S \sum_{n=0}^{\infty} w_n(x,t)}{\partial x^3}\right] + 3f^\alpha S\left[\sum_{n=0}^{\infty} F_n\right]$$

Take the inverse Sumudu transform of two side Eq.12 to obtain the following:

$$\sum_{n=0}^{\infty} u_n(x,t) = S^{-1}\left[\frac{\beta-2k^2}{3} + 2k^2 tanh^2(kx)\right] + S^{-1}\frac{1}{2} f^\alpha \left[\frac{\partial^3 S[\sum_{n=0}^{\infty} u_n(x,t)]}{\partial x^3}\right] - S^{-1}\left[3f^\alpha S[\sum_{n=0}^{\infty} A_n]\right] + S^{-1}\left[3f^\alpha S[\sum_{n=0}^{\infty} B_n]\right] + S^{-1}\left[3f^\alpha S[\sum_{n=0}^{\infty} M_n]\right], \quad (13)$$

$$\sum_{n=0}^{\infty} v_n(x,t) = S^{-1}\left[-\frac{4k^2 c_0(\beta+k^2)}{3c_1^2} + \frac{4k^2(\beta+k^2)\, tanh(kx)}{3c_1}\right] - f^\alpha S^{-1}\left[\frac{\partial^3 S \sum_{n=0}^{\infty} v_n(x,t)}{\partial x^3}\right] + 3S^{-1} f^\alpha S[\sum_{n=0}^{\infty} E_n],$$

$$\sum_{n=0}^{\infty} w_n(x,t) = S^{-1}[c_0 + c_1 tan\,h(kx)] - S^{-1} f^\alpha \left[\frac{\partial^3 S \sum_{n=0}^{\infty} w_n(x,t)}{\partial x^3}\right] + 3S^{-1} f^\alpha S[\sum_{n=0}^{\infty} F_n].$$

The recursive relation are given by

$$u_{n+1}(x,t) = S^{-1}\frac{1}{2} f^\alpha \left[\frac{\partial^3 S[u_n]}{\partial x^3}\right] - S^{-1}\left[3f^\alpha S[A_n]\right] + S^{-1}\left[3f^\alpha S[B_n]\right] + S^{-1}\left[3f^\alpha S[M_n]\right], \quad (14)$$

$$v_{n+1}(x,t) = -f^\alpha S^{-1}\left[\frac{\partial^3 S[v_n(x,t)]}{\partial x^3}\right] + 3S^{-1} f^\alpha S[E_n],$$

$$w_{n+1}(x,t) = -S^{-1} f^\alpha \left[\frac{\partial^3 S[w_n(x,t)]}{\partial x^3}\right] + 3S^{-1} f^\alpha S[\mathrm{F_n}], \quad (n = 0,1,2,\dots).$$

Where,

$$u_0 = \frac{\beta-2k^2}{3} + 2k^2 tanh^2(kx), \quad (15)$$

$$v_0 = -\frac{4k^2 c_0(\beta+k^2)}{3c_1^2} + \frac{4k^2(\beta+k^2)\, tanh(kx)}{3c_1},$$

$$w_0 = c_0 + c_1 tan\,h(kx).$$

Replacing Eq.15 into Eq.14, and applying the overhead equations, yields

$$u_1 = \frac{t^\alpha}{\Gamma(\alpha+1)}\Bigg(-16k^5(1 - tanh^2(kx))^2\, tanh(kx) + 8k^5 tanh^3(kx)(1 - tanh^2(kx)) - 12\left(-\frac{2k^2}{3} + \frac{\beta}{3} + 2k^2 tanh^2(kx)\right)k^3\, tanh(kx)\,(1 - tanh^2(kx)) + 3\left(-\frac{4k^2 c_0(k^2+\beta)}{3c_1^2} + \frac{4k^2(k^2+\beta)\, tanh(kx)}{3c_1}\right)k(1 - tanh^2(kx))c_1 + (c_0 + c_1 tanh\,(kx))\left(\frac{4k^3(k^2+\beta)(1-tanh^2(kx))}{c_1}\right)\Bigg), \quad (16)$$

$$v_1 = \frac{t^\alpha}{\Gamma(\alpha+1)}\Bigg(\frac{8k^5(k^2+\beta)(1-tanh^2(kx))^2}{3c_1} - \frac{16k^5(k^2+\beta) tanh^2(kx)(1-tanh^2(kx))}{3c_1} +$$

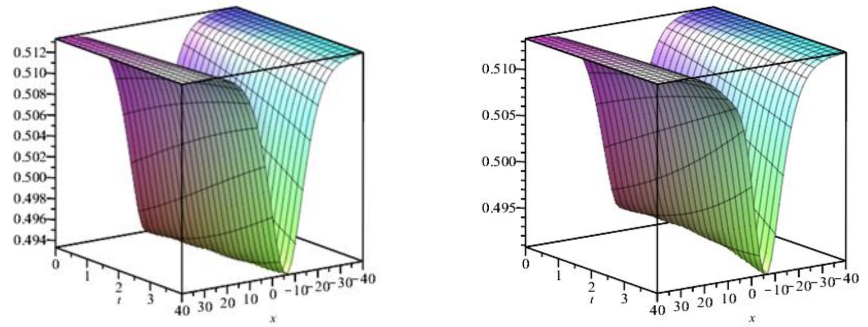
$$\left.\frac{4\left(-\frac{2k^2}{3}+\frac{\beta}{3}+2k^2tanh^2(kx)\right)k^3\left(k^2+\beta\right)\left(1-tanh^2(kx)\right)}{c_1}\right),$$

$$w_1=\frac{t^\alpha}{\Gamma(\alpha+1)}\Big(2c_1k^3(1-tanh^2(kx))^2-4c_1k^3tanh^2(kx)\left(1-tanh^2(kx)\right)+$$
$$3\left(-\frac{2k^2}{3}+\frac{\beta}{3}+2k^2tanh^2(kx)\right)c_1k\left(1-tanh^2(kx)\right)\Big).$$

It is possible to find u_2, v_2 *and* w_2 by repeating the same previous operations when $n=2$. So the approximate solution (SDM) would be:

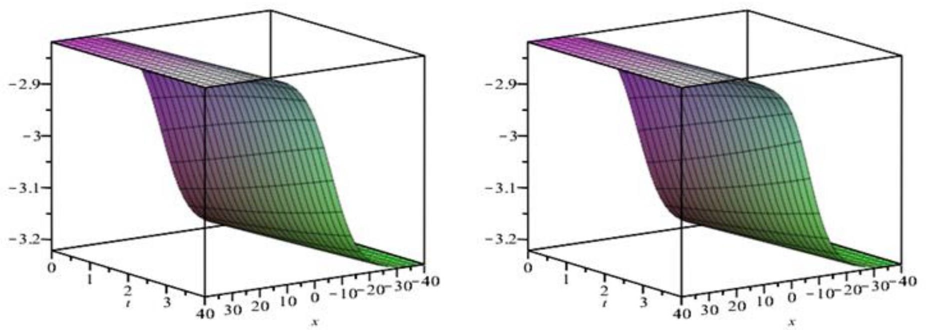
$$u_{SDM}=\sum_{i=0}^{2}u_i(x,t), v_{SDM}=\sum_{i=0}^{2}v_i(x,t), w_{SDM}=\sum_{i=0}^{2}w_i(x,t). \quad (17)$$

Comparison of the results of the proposed method with the exact solution will be shown in Figures (1-6) and Tables (1-3). The LSWF was studied when $\alpha=0.25$, $\alpha=0.5$ and $\alpha=0.75$ which represented in the Table 4.



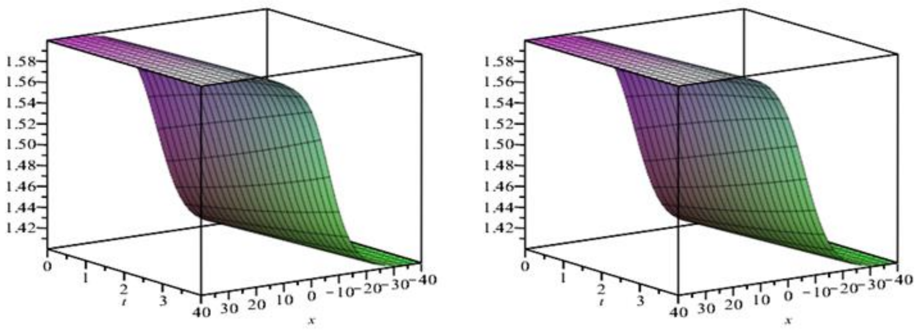
(a) **(b)**

Figure 1: The solution of $u(x,t)$, SDM consequence (a) and exact solution (b), while $k=0.1, \alpha=1, \beta=1.5, c_0=1.5, c_1=0.1, \quad t=0..4, x=-40..40$.



(a) **(b)**

Figure 2: The solution of $v(x,t)$, SDM consequence (a) and exact solution (b), while k = 0.1, α =1, β=1.5, c_0=1.5, $c_1=0.1$, $t=0..4, x=-40..40$.



(a) **(b)**

Figure 3: The solution of $w(x,t)$, SDM consequence (a) and exact solution (b), while k = 0.1, α =1, β=1.5, c_0=1.5, $c_1 = 0.1,\ t = 0..4, x = -40..40$ **.**

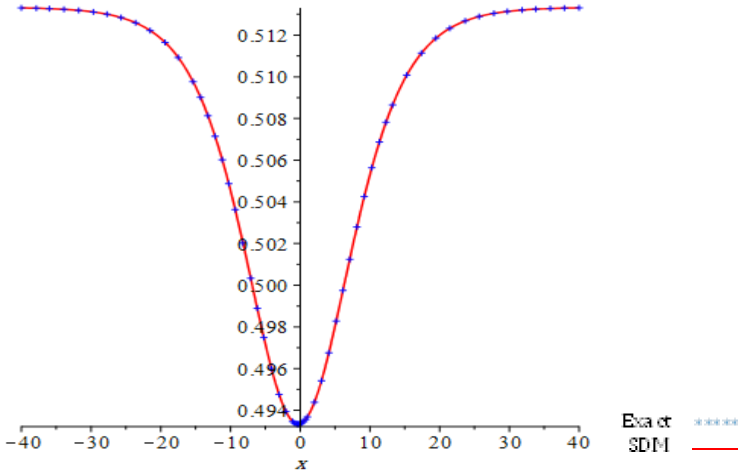


Figure 4: SDM consequence and exact solution of $u(x,t)$ at $k = 0.1,\ \beta = 1.5, c_0 = 1.5,$ $c_1 = 0.1,\ t = 0.2,\ x = -40..40$ when $\alpha = 1$.

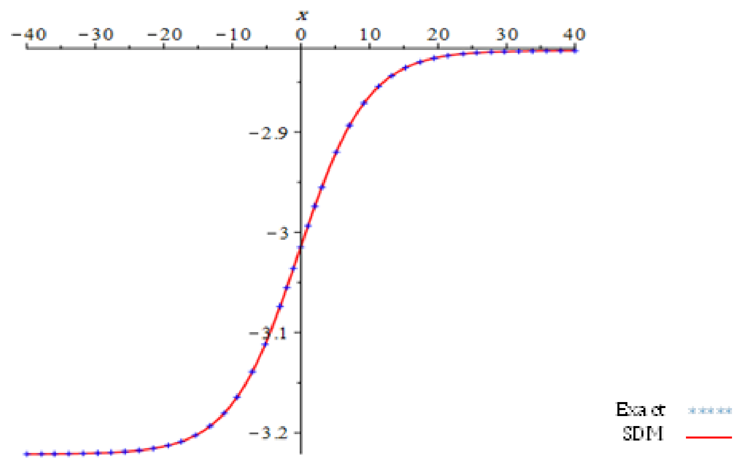


Figure 5: SDM consequence and exact solution of $v(x,t)$ at $k = 0.1,\ \beta = 1.5, c_0 = 1.5,$ $c_1 = 0.1,\ t = 0.2,\ x = -40..40$ when $\alpha = 1$.

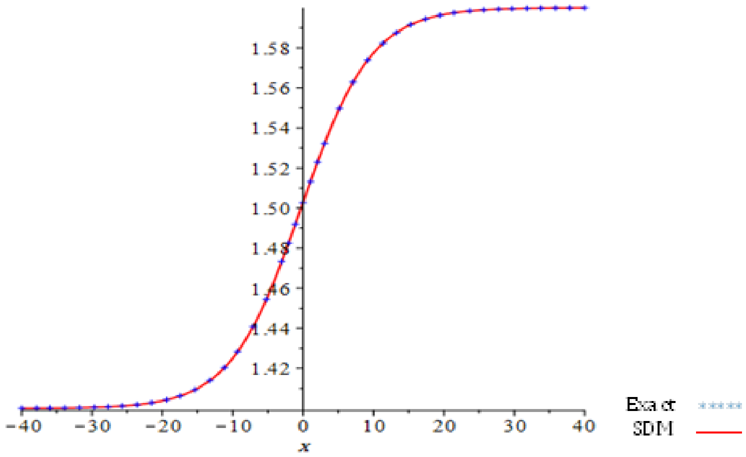


Figure 6: SDM consequence and exact solution of $w(x,t)$ at $k = 0.1,\ \beta = 1.5, c_0 = 1.5,\ c_1 = 0.1,\ t = 0.2,\ x = -40..40$ when $\alpha = 1$.

Table 1: The estimated solutions when $\alpha = 0.25, 0.5, 0.75, 1, k = 0.1, c_0 = 1.5, c_1 = 0.1, \beta = 1.5,\ t = 0.2$, and the exact solutions of u and the absolute error between u_e and u_a $for\ u(x,t)$.

	Approximate u				Exact u	Error $\lvert u_e - u_a \rvert$
x	52α = 0.	α = 0.5	57α = 0.	α = 1	α = 1	α = 1
0	0.493787	0.493513	0.493394	0.4933513333	0.4933513225	1.07900000×10^{-8}
0.2	0.493883	0.493582	0.493441	0.4933832896	0.4933832501	3.94902354×10^{-8}
0.4	0.493993	0.49366	0.493503	0.4934310820	0.4934310141	6.794121741×10^{-8}
0.6	0.494117	0.493763	0.493581	0.4934945584	0.4934944625	9.591820988×10^{-8}
0.8	0.494255	0.493876	0.493675	0.4935735177	0.4935733945	1.232609070×10^{-7}
1	0.494405	0.494004	0.493783	0.4936677110	0.4936675613	1.497662126×10^{-7}

Table 2: The estimated solutions when $\alpha = 0.25, 0.5, 0.75, 1, k = 0.1, c_0 = 1.5, c_1 = 0.1, \beta = 1.5,\ t = 0.2$, and the exact solutions of v and the absolute error between v_e and v_a $for\ v(x,t)$.

	Approximate v				Exact v	Error $\lvert v_e - v_a \rvert$
x	α = 0.25	α = 0.5	α = 0.75	α = 1	α = 1	α = 1
0	-2.997719	-3.0047603	-3.010173	-3.013960	-3.013961811	1.811348×10^{-6}
0.2	-2.993793	-3.0007765	-3.006163	-3.009940	-3.009941714	1.806279106×10^{-6}
0.4	-2.989888	-2.996808	-3.002164	-3.005928	-3.005929641	1.795457400×10^{-6}
0.6	-2.986006	-2.9928577	-2.998179	-3.001927	-3.001928766	1.778975627×10^{-6}
0.8	-2.982151	-2.988929	-2.994211	-2.997940	-2.997942228	1.75687427×10^{-6}
1	-2.978325	-2.985024	-2.990264	-2.993971	-2.993973120	1.72937758×10^{-6}

Table 3: The estimated solutions when $\alpha = 0.25, 0.5, 0.75, 1, k = 0.1, c_0 = 1.5, c_1 = 0.1, \beta = 1.5,\ t = 0.2$, and the exact solutions of w and the absolute error between w_e and w_a $for\ w(x,t)$.

	Approximate w				Exact w	Error $\lvert w_e - w_a \rvert$
x	α = 0.25	α = 0.5	α = 0.75	α = 1	α = 1	α = 1
0	1.511067	1.507569	1.504881111	1.503000000	1.502999100	8.99676×10^{-7}
0.2	1.513017	1.509548	1.506872840	1.504996734	1.504995837	8.97159654×10^{-7}
0.4	1.514956	1.5115519	1.508859093	1.506989479	1.506988589	8.91784571×10^{-7}
0.6	1.516884	1.513481	1.510838312	1.508976662	1.508975778	8.83586860×10^{-7}
0.8	1.518799	1.515433	1.512808966	1.510956719	1.510955847	8.72622254×10^{-7}
1	1.520700	1.512927	1.514769546	1.512928116	1.512927258	8.58962040×10^{-7}

To use LSWF, rewrite Eq. 1 as follows:

$$L1(u,v,w) = D_t^{\alpha} u - \frac{1}{2} u_{xxx} + 3 u u_x - 3(vw)_x = 0 \quad (18)$$

$$L2(u,v,w) = D_t^{\alpha} v + v_{xxx} - 3 u v_x = 0, \qquad t > 0, 0 < \alpha \le 1, \quad (19)$$

$$L3(u,v,w) = D_t^{\alpha} w + w_{xxx} - 3 u w_x = 0. \quad (20)$$

By using LSWF:

Ei $=\int_0^1 \int_{-40}^{40} [Li(u_n, v_n, w_n)]^2 dx dt\ , i = 1,2,3.$

Table 4: provided the LSWF quantitive when $t \in [0,1], x \in [-40,40],\quad c_0 = 1.5, c_1 = 0.1, \beta = 1.5$.

Quantities	LSWF		
	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
E1	4.5678×10^{-6}	4.5824×10^{-6}	6.7345×10^{-8}
E2	5.6696×10^{-4}	5.6808×10^{-4}	1.9931×10^{-6}
E3	1.3987×10^{-4}	1.4015×10^{-4}	4.9170×10^{-7}

3. Results and Discussion

In this, unit nine figures of SDM results and exact solutions for KdV solutions Eq.1 with initial conditions Eq.2 are presented. To demonstrate the accuracy of the SDM, numerical results are displayed and only a very small number of iterations are required to reach accurate solutions. There are no noticeable differences in the two solutions for each pair of plots. Tables 1, 2,3 and 4 show the numerical values by SDM when α = 0.25, 0.5, 0.75, 1.0 and the error value between precision and estimation when α = 1.0 for u, v, w respectively.

4. Conclusions

In this work, an effective method is presented to find approximate solutions to the Hirota-Satsuma coupled KdV equations for fractional time with initial conditions. These equations have wide applications in many real-world problems, especially in the physics and engineering sciences. In this regard, a powerful numerical method was proposed by using the Sumudu transform with Adomian decomposition method. Acceptable results were obtained in several tests. The method proved its efficiency for solving fractional order nonlinear equations.

Acknowledgments

We extend our thanks to the University of Mosul and the College of Computer Science and Mathematics for their continued support of the researchers.

References

[1] G. X. H. X. Z. S. Wu Y, "A generalized Hirota–Satsuma coupled Korteweg–de Vries equation and Miura transformations," *Physics Letters A,,* vol. 255, no. 1, pp. 259-264, 1999.

[2] H. K. Jassim, "The Approximate Solutions of Three-Dimensional Diffusion and Wave Equations within Local Fractional Derivative Operator," *Abstract and Applied Analysis,* vol. 2016, no. 1, p. 5, 2016.

[3] V. F. Oleksandr Boichuk, "Weakly perturbed linear boundary-value problem for system of fractional differential equations with Caputo derivative," *Results in Applied Mathematics,,* vol. 21, no. 1, p. 100424, 2024.

[4] J. D. K. a. R. S. Singh, "Numerical solution of time-and space-fractional coupled Burgers' equations via homotopy algorithm," *Alexandria Engineering Journal,,* vol. 55, no. 2, pp. 1753-1763, 2016.

[5] E. P. B. K. G. O. Kolade M. Owolabi, "Laplace transform-homotopy perturbation method for fractional time diffusive predator–prey models in ecology,," *Partial Differential Equations in Applied Mathematics,,* vol. 9, no. 1, p. 100607, 2024.

[6] S. A. Bhanotar, "Exploration of novel analytical solutions of boundary layer equation via the modified sumudu transform," *Examples and Counterexamples,* vol. 5, no. 1, p. 100140, 2024.

[7] G. Monzon, "Fractional variational iteration method for higher-order fractional differential equations," *Journal of Fractional Calculus and Applications,* vol. 15, no. 1, pp. 1-15, 2024.

[8] A.-K. K., "Numerical solution of time-fractional partial differential equations using Sumudu decomposition method," *Rom. J. Phys,* vol. 60, no. (1-2), pp. 99-110, 2015.

[9] M. S. Azzo SM, "Sumudu-Decomposition Method to Solve Generalized Hirota-Satsuma Coupled Kdv System," *Science Journal of University of Zakho,* vol. 10, no. (2), pp. 43-47, 2022.

[10] A. H. Mousa MG, "A new technique for solving fractional nonlinear equations by sumudu transform and adomian decomposition method," *Ibn Al-Haitham Journal for Pure and Applied sciences,* vol. 35, no. (3), pp. 128-134, 2022.

[11] H. K. a. H. K. Jassim, "Fractional Sumudu decomposition method for solving PDEs of fractional order," *Journal of Applied and Computational Mechanics,* vol. 7, no. 1, pp. 302-311, 2020.

[12] M. a. D. G. Shateri, "Solitary wave solutions for a time-fraction generalized Hirota-Satsuma coupled KdV equation by a new analytical technique," *Applied Mathematical Modelling,* vol. 33, no. 7, pp. 3107-3113, 2009.

[13] M. e. a. Merdan, "Solution of time-fractional generalized Hirota-Satsuma coupled KdV equation by generalised differential transformation method," *International Journal of Numerical Methods for Heat & Fluid Flow,* vol. 23, no. 5, pp. 927-940, 2013.

[14] R. O. A. a. A. Q. Saadeh, "Analytical solution of coupled hirota–satsuma and KdV equations," *Fractal and Fractional,* vol. 6, no. 12, p. 694, 2022.

[15] M. A. A. N. N. b. A. H. Muhammad Asim Khan, "Traveling wave solutions for space-time fractional Cahn Hilliard equation and space-time fractional symmetric regularized long-wave equation," *Alexandria Engineering Journal,* vol. 60, no. 1, pp. 1317-1324, 2021.

[16] R. M. Fawzi, "Solving Fractional Damped Burgers' Equation Approximately by Using The Sumudu Transform (ST) Method," *Baghdad Science Journal,* vol. 18, no. 1, p. 0803, 2021.

[17] A. M. A. S. M. a. A. A. M. Mahdy, "Implementation of the homotopy perturbation Sumudu transform method for solving Klein-Gordon equation.," *Applied Mathematics,* vol. 6, no. 3, pp. 617-628, 2015.

[18] F. B. M. a. A. A. K. Belgacem, "Sumudu transform fundamental properties investigations and applications," *International Journal of Stochastic Analysis,* vol. 2006, no. 1, pp. 1-23, 2006.

[19] A. T. A. Azzam SY Aladool, "Solving Linear and Nonlinear Fractional Differential Equations Using Bees Algorithm," *Iraqi Journal of Science,* vol. 1854, no. 1, pp. 1322-1330, 2023.

[20] A. S. A. Abdulrazzaq T Abed, "Applying particle swarm optimization based on Padé approximant to solve ordinary differential equation," *Numerical Algebra, Control & Optimization,* vol. 12, no. 2, p. 321, 2022.

[21] S. R. H.-G. S. J. M. M. A. K. Zhuo-Jia Fu, "A robust kernel-based solver for variable-order time fractional PDEs under 2D/3D irregular domains," *Applied Mathematics Letters,* vol. 94, no. 1, pp. 105-111, 2019.

[22] H. K. Jassim, "Analytical approximate solutions for local fractional wave equations," *Mathematical Methods in the Applied Sciences,* vol. 43, no. 2, pp. 939-947, 2020.

[23] S. A. E. M. H. A. A. &. H. W. Ahmed, "Numerical Solutions of Time-Fractional Whitham–Broer–Kaup Equations via Sumudu Decomposition Method," *Journal of Mathematics,* vol. 2023, no. 1, pp. 1-10, 2023.