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Operational Matrix for Solving Variable Order Generalized Caputo Fractional Derivatives

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Abstract

In this study, new generalized derivative and integral operators are introduced, stemming from the newly developed new generalized Caputo variable order fractional derivatives (NGCVFDs). Utilizing these operators, a numerical method is devised to address variable order fractional differential equations (VOFDEs). The solutions of VOFDEs are approximated using shifted Legendre polynomials (SLPs) as basis vectors, and the derivative operational matrix of SLPs is extended to a generalized derivative operational matrix in the context of NGCVFDs. The efficiency of the numerical method is assessed through various test examples. Additionally, the outcomes of the proposed method are compared with existing methodologies in the literature. The variable-order fractional differential operators of the generalized Caputo is categorized into three types in this paper:

(i) Different values in ρ and Fractional variable order parameters, (ii) Different values in fractional parameter whilst *ρ* parameters and Fractional variables orders are constant, and (iii) Different values in Fractional variables orders parameters controlled fraction and *ρ* parameters. The example of numerical methods show theoretical interpretation and prove effectiveness of suggested technique.

Keywords: Shifted Legendre polynomials ,Fractional orders differential equations, Generalized Caputo-type derivative ,Operational matrix.

المصفوفة التشغيلية لحل المعادلات التفاضلية الكسرية ذات الرتب المتغيرة مع نوع كابوتو المعممة

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الخلاصة

في هذه الدراسة، تم تقديم عوامل مشتقة وتكاملية معممة جديدة، تنبع من مشتقات كابوتو الكسرية ذات الترتيب المتغير المعمم الجديدة (NGCVFDs) التي تم تطويرها حديثًا. باستخدام هذه العوامل، تم تصميم طريقة عددية لمعالجة المعادلات التفاضلية الكسرية ذات الترتيب المتغير يتم تقريب حلول VOFDEs باستخدام متعددات حدود لا جندر المتحولة (SLPs) كمتجهات أساسية، مع توسيع المصفوفة التشغيلية المشتقة لـ SLPs إلى مصفوفة تشغيلية مشتقة معممة في سياق NGCVFDs. يتم تقييم كفاءة الطريقة العددية من خلال أمثلة الاختبار المختلفة. بالإضافة إلى ذلك، تتم مقارنة نتائج الطريقة المقترحة مع المنهجيات الموجودة في الأدبيات. يتم تصنيف العامل التفاضلي الكسري ذو الترتيب المتغير لـ قيم متفاوتة من المعلمات في هذه الورقة :أ) قيم متفاوتة في ρ والمعلمات الكسرية ذات ترتيب المتغير ب) قيم متفاوتة في

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المعلمات الكسرية ذات ترتيب المتغير بينما تكون معلمات ρ ثابته ج) قيم متفاوتة في المعلمات التي تتحكم في المعلمات الكسرية وρ . تم إعطاء أمثلة عددية لتوضيح التحليل النظري والتحقق من كفاءة الطريقة المقترحة.

1. Introduction

Fractional calculus (FC) is a field that studies derivatives of random order. Initially ,FC was considered only theoretical mathematical concepts with few practical implementations. However, in following decades, FC has made significant progress in both applied and abstract areas of mathematics [1], [2]. Various fractional operator are documented in scientific literature, among which Caputo differential operators (CDO) have important uses. This is often used to resolve initial value problem of fractional orders [3], [4].The implementation of mathematical models, especially Caputo-type differential equations, to many real world phenomenon has been extensively studied. For example, in [5], researchers used convection a differential equation (CFDE) as mean to study behavior of the Casson fluid flows. In [6], scientists used CFDE to study in detail how electricity works in circuits. The popular integrated-operator that is fractionally based is primarily affected by the value of the parameters ρ and α in a fractionally computed value. The generalized-Caputo type fractionally derived property [7] possesses the same properties as the Caputo property. Additionally, it provides a practical method for creating and managing mathematical model with fractional computations. These versions contain the old-style Caputo, Atangna Baleanu and Caputo- Fabrezio a fraction derivatives [8]. excepting parameters for fractional order, the ρ parameters have a similar effect on graphs made from acquired information. The specific settings of parameters can be altered to create variety of graph. The most of Functional Differential Equations (FDEs) have a significant impact on analysis. As a result, there is a pressing need for a numerical resolution approaches.

Several a numerical approaches for solving a functional differential equation are documented in literature, but method of operational matrixes, when combine with collocations technique and Tau technique, is frequently employed to the resolve both ordinary and partial FDEs [9], [10]. The foundation of this approach is to convert FDEs into algebraic a system of equation that computers can easily solve. Lastly, this method determines the solution's representation via the orthogonal basis of polynomials. Atangana's work in [11] dedicated three a new variants of fractional number: the Mittag-Leffler ,exponential and power laws. Many of the Mathematician developing this new a field, they used it to derive various fractions from various problems in life, include the real-world issues and systems of the FDEs [12],diffusion equations, physiology, economy, electrochemical process, physics, fluids and physiology [13].

Inspired by the aforementioned research, we propose new operators for integration and differentiation, a new general form of Caputo's derivatives for fractional variable orders, and we develop numerical technique use recently discover derivatives, matrix of shifted Legendre polynomial, in context of the new general forms of Caputo's derivative for a solution to Fractional Functioned Differential Equation (FFDE). Additionally, we evaluate practicality of our develop numerical methods by comparing result obtained from the suggested method with these establish in the academic literature. Our analysis showed that numerical result obtained through our traditional technique were in agreement with these obtained through other methods.

The variable-order generalized Caputo fractional derivative is an extension of the classical Caputo fractional derivative, allowing the order of differentiation to vary as a function of space or time. This derivative is particularly useful in modeling complex systems with dynamic behaviors that cannot be adequately described using constant-order derivatives. It has applications in fields such as viscoelasticity, anomalous diffusion, and control theory [14], [15].

2. Preliminaries

Let us clarify the generalize Caputo-type derivatives of the fraction and provide example that demonstrate the concept. Our subsequent project is to investigate its attributes.

Definition 2.1 [16]. Variables order fractional-differential equations: Definition of orders (t) for function $u(t) \in C^m[0, b]$ given by:

$$D^{(t)}\ u(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_{0+}^{t}(t-\tau)^{-\alpha(t)}u'(\tau)d\tau + \frac{u(0^+)-u(0^-)}{\Gamma(1-\alpha(t))}t^{-\alpha(t)},$$

in the case of $0 < \alpha(t) < 1$, we get:

$$D^{(t)}\ u(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_{0+}^{t}(t-\tau)^{-\alpha(t)}u'(\tau)\mathrm{d}\tau$$

Definition 2.2 [17]. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous function . The variable-order generalized Caputo fractional derivatives of order α(x) of the function f(x) is given by:

$$^{c}D_{X}^{\alpha(x)}f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha(x))}\int_0^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha(x)+1-m}}d(\tau), & m-1<\alpha<m \\ \frac{d^m}{dx^m}f(x), & \alpha = m \end{cases}$$

Definition 2.3 [18] **.** The new generalized Caputo-type fractional derivative of variable order $\alpha > 0$ is defined as:

$$^{c}D_{a+}{}^{\alpha(t),\rho}v(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{n-\alpha-1}\left\{(s^{1-\rho}\frac{d}{ds})^n v(s)\right\}ds\,, t>a$$

where $\rho > 0\,, a \geq 0\,, n-1 < \alpha < n, n = \lceil\alpha\rceil$, $v(t) \in C^n[a,b]$.

Furthermore, in new generalize Caputo-fractional derivatives[15], we obtain that:

$^{c}D^{\alpha(t),\rho}C = 0$, where C is fixed.

Moreover, if $m-1<\alpha<m, k>m-1$ and $k \notin \mathbb{N}$

$$^{c}D^{\alpha(x),\rho}(t^\rho - a^\rho)^k = \begin{cases}\rho^\alpha \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}(t^\rho-a^\rho)^{k-\gamma}, & k\in\mathbb{N}_0 \text{ and } k\geq\lceil\alpha\rceil \text{ or } k\in\mathbb{N} \text{ and } k>\lfloor\alpha\rfloor \\ 0 & k\in\mathbb{N}_0 \text{ and } k<\lceil\alpha\rceil\end{cases},$$

Theorem.2.4 [18] **.** Generalized Caputo-type fractional derivative of variable order:

let $m-1<\alpha\leq m, a\geq 0,\ \rho>0$ and $f\in C^m[a,b]$. then , for $a<t\leq b$,

$$(D_{a+}^{\alpha(t),\rho}f)(t) = D_{a+}^{\alpha(t),\rho}\{f(t) - \sum_{n=0}^{m-1}\frac{1}{\rho^n n!}(t^\rho - a^\rho)[(x^{1-\rho}\frac{d}{dx})^n f(x)]|_{x=a}\}.$$

Proof.

$$D_{a+}^{\alpha(t),\rho}\{f(t) - \sum_{n=0}^{m-1}\frac{1}{\rho^n n!}(t^\rho - a^\rho)[(x^{1-\rho}\frac{d}{dx})^n f(x)]|_{x=a}\}$$

$$=\frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}(t^{1-\rho}\frac{d}{dt})^m\int_a^t s^{\rho-1}(t^\rho-s^\rho)^{m-\alpha-1}\{f(s) - \sum_{n=0}^{m-1}\frac{1}{\rho^n n!}\left(s^\rho - a^\rho\right)^n[x^{1-\rho}\frac{d}{dx})^n f(x)\Big]|_{x=a}\}ds$$

$$= \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}(t^{1-\rho}\frac{d}{dt})^m\int_a^t \frac{(t^\rho-s^\rho)^{m-\alpha}}{m-\alpha}\{f'(s) - s^{\rho-1}\sum_{n=0}^{m-1}\frac{1}{\rho^{n-1}(n-1)!}(s^\rho - a^\rho)^{n-1}[\,(x^{1-\rho}\frac{d}{dx})^n f(x)]|_{x=a}\}ds$$

$$= \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}(t^{1-\rho}\frac{d}{dt})^{m-1}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{m-\alpha-1}\{ (s^{1-\rho}\frac{d}{ds})f(s) - \sum_{n=1}^{m-1}\frac{1}{\rho^{n-1}(n-1)!}(s^\rho - a^\rho)^{n-1}[(x^{1-\rho}\frac{d}{dx})^n f(x)]|_{x=a}\}ds.$$

$$D_{a+}^{\alpha(t),\rho}\{f(t) - \sum_{n=0}^{m-1}\frac{1}{\rho^n n!}(t^\rho - a^\rho)^n[(x^{\rho-1}\frac{d}{dx})^n f(x)]|_{x=a}\}$$

$$= \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{m-\alpha-1}\{(s^{1-\rho}\frac{d}{ds})^m f(s)\}ds$$

Theorem.2.5 [18]**.** Generalized Caputo fractional derivatives of variable orders let $m-1<\alpha\le m, a\ge 0, \rho>0$ *and* $f\in C^m[a,b]$. *then , for* $a<t\le b$,

$$I_{a+}^{\alpha(t),\rho}\, D_{a+}^{\alpha(t),\rho} f(t) = f(t) - \sum_{n=0}^{m-1}\frac{1}{\rho^n n!}(t^\rho - a^\rho)^n[(x^{1-\rho}\frac{d}{dx})^n f(x)]|_{x=a}$$

Proof: $I_{a+}^{\alpha(t),\rho}\, D_{a+}^{\alpha(t),\rho} f(t) = \frac{\rho^{\alpha-1}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - a^\rho) D_{a+}^{\alpha(t),\rho} f(s)\, ds.$

$$= \frac{\rho^{2-m}}{\Gamma(\alpha)\Gamma(m-\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}\int_a^s z^{\rho-1}(s^\rho - z^\rho)^{m-\alpha-1}(z^{1-\rho}\frac{d}{dz})^m f(z)\, ds\, dz$$

$$= \frac{\rho^{2-m}}{\Gamma(\alpha)\Gamma(m-\alpha)}\int_a^t z^{\rho-1}(z^{1-\rho}\frac{d}{dz})^m f(z)\int_z^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}(s^\rho - z^\rho)^{m-\alpha-1}\, ds\, dz.$$

Making the substitution $s^\rho = z^\rho + (t^\rho - z^\rho)r$,

$$\int_z^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}(s^\rho - z^\rho)^{m-\alpha-1}\, ds = \frac{1}{\rho}(t^\rho - z^\rho)^{m-1}\int_0^1 (1-r)^{\alpha-1} r^{m-\alpha-1}\, dr$$

$$= \frac{1}{\rho}(t^\rho - z^\rho)^{m-1}\frac{\Gamma(\alpha)\Gamma(m-\alpha)}{\Gamma(m)}$$

Then, we obtain

$$I_{a+}^{\alpha(t),\rho}\, D_{a+}^{\alpha(t),\rho} f(t) = \frac{\rho^{1-m}}{(m-1)!}\int_a^t z^{\rho-1}(t^\rho - z^\rho)^{m-2}(z^{1-\rho}\frac{d}{dz})^m f(z)\, dz.$$

Integration by parts, m-1 times, yields

$I_{a+}^{\alpha(t),\rho}\, D_{a+}^{\alpha(t),\rho} f(t) = \frac{\rho^{2-m}}{(m-2)!}\int_a^t z^{\rho-1}(t^\rho - z^\rho)^{m-2}(z^{1-\rho}\frac{d}{dz})^m f(z)\, dz + \frac{\rho^{2-m}}{(m-2)!}(t^\rho - a^\rho)^{m-1}[\left(x^{1-\rho}\frac{d}{dx}\right)^n f(x)\Big]|_{x=a}, = \cdots$

$$= \int_a^t f'(z)dz - \sum_{n=1}^{m-1}\frac{1}{\rho^n n!}(t^\rho - a^\rho)^n[(x^{1-\rho}\frac{d}{dx})^n f(x)]|_{x=a},$$

3. Characteristics of shifted Legendre polynomials

Famous Legendre polynomial are determined by intervals $[-1, 1]$ and the definition can be articulated utilizing the subsequent recurrence formulas:

$$L_{j+1} = \frac{2j+1}{j+1} w L_j(w) - \frac{j}{j+1} L_{j-1}(w),\ \ j = 1,2,\ldots$$

where $L_0(w) = 1$, $L_1(w) = w$, to use these polynomials by interval $t \in [0,1]$,we describe shifted-Legendre polynomial by implementing a variation in the variable, $w = 2t - 1$ where shifted-Legendre polynomial $L_j(2t-1)$ be represented by $P_j(t)$. Then $P_j(t)$ the acquisition procedure is as follows:

$$P_j = \frac{(2j+1)(2t-1)}{(j+1)} P_j(t) - \frac{j}{j+1} P_{j-1}(t),\ \ j = 1,2,\ldots$$

$P_0 = 0$ and $P_1(t) = 2t - 1$, analytic formula of shifted Legendre polynomial $P_i(t)$ of the degrees i are provided by the following:

$P_j(t) = \sum_{k=0}^{j} (-1)^{j+k} \frac{(j+k)!}{(j-k)!} \frac{x^k}{(k!)^2}$.

It is important to remember that $P_j(0) = (-1)^j$ and $P_j(1)=1$, orthogonality conditions are

$$\int_0^1 P_i(t)\, P_j(t) dt = \begin{cases} \frac{1}{2j+1} & \text{for} i = j \\ 0 & \text{for} i \neq j \end{cases}$$

A function $y(t)$, the square integrables in $[0,1]$, may potentially be articulated with respect to the shifted Legendre polynomials as $y(t) = \sum_{i=0}^{\infty} C_i P_i(t)$, where coefficient C_i are provided by the following:

$$ci = (2i + 1) \int_0^1 y(t) P_i(t) dt, \quad i = 1,2, \ldots$$

In practical implementation, only first (m + 1)-term shifted-Legendre polynomial is investigated. After that we have
$y(t) = \sum_{j=0}^{m} C_i P_i(t) = C^T \varphi(t)$,

where shifted Legendre vectors Φ(t) and coefficients vectors C are provided by the following:
$C^T = [c_0, \ldots, c_m]$,
It is possible to express derivative of vectors Φ(t) as

$$\Phi(t) = [P0(t), P1(t), \ldots, Pm(t)]^T,$$

where $D^{(1)}$ is the $(m + 1) * (m + 1)$ operation matrix of derivatives provided by the following:

$$D^{(1)} = (d_{ij}) = \begin{cases} 2(2i+1), & \text{for } i = j - k, \\ 0, & \text{otherwise}, \end{cases} \quad \begin{cases} k = 1,3, \ldots, m-1, \text{if m even} \\ k = 1,3, \ldots, m, \quad \text{if m odd} \end{cases}$$

For instance, in the case of an even value of m, we observe that

$$D^{(1)} = 2 \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 5 & 0 & \cdots & 2m-3 & 0 & 0 \\ 0 & 3 & 0 & 7 & \cdots & 0 & 2m-1 & 0 \end{pmatrix}$$

4.Operational Matrix for Fractional Derivatives

We underline importance of operation matrix associated with fraction derivatives by utilizing it to solve the orders of a fraction-differential equation. The subsistence's, singularity, and constant dependence of the solution to this specifics problem are considered in depth in [19].

$$D_t^{\alpha(t)} u(x) + \beta_1 u'(x) + \beta_2 u(x) = f(x), \quad u(0) = u_0, \tag{1}$$

where $u(x) \in L^2[0,1]$ is unknown function, $f(x) \in L^2[0,1]$ is known that we seek to approximation with great precision, β_1, β_2 and u_0 are all fixed.
So as to resolve eq. (1), we proposed estimated solutions of the eq. (1) :

$$u(x) \approx \sum_{i=0}^{n} c_i P_{i,n}(x) = c^T \Phi(x), \tag{2}$$

where $\Phi(x) = [P_0(x), P_1(x), \dots, P_n(x)]^T$ and $C = [C0, C1, \dots, Cn]T$ .Operating $D_x^{\alpha(x)}$ to both side of equation (2) and use equations:

$$D_x^{\alpha(x)} u(x) = c^T D_x^{\alpha(x)} \Phi(x) = c^T A G A^{-1} \Phi(x) \ , \tag{3}$$

where:

$$G(x) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \rho^{\alpha}\frac{\Gamma(2)}{\Gamma(2-\beta)}(x^{\rho}-a^{\rho})^{-\alpha(x)} & 0 & \cdots & 0 \\ 0 & 0 & \rho^{\alpha}\frac{\Gamma(3)}{\Gamma(3-\beta)}(x^{\rho}-a^{\rho})^{-\alpha(x)} & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \rho^{\alpha}\frac{\Gamma(n+1)}{\Gamma(n-\beta+1)}(x^{\rho}-a^{\rho})^{-\alpha(x)} \end{bmatrix} \tag{4}$$

substitute equations(2) ,(3)and (4) into equation(1), to get:

$$c^T A G A^{-1} \Phi(x) + \gamma_1 c^T A D^{(1)} A^{-1} \Phi(x) + \gamma_2 c^T \Phi(x) = f(x) \tag{5}$$

$c^T \Phi(0) = u_0$,

As such, the values of Φ and f on the [0, 1], use $x_i = \frac{2_i+1}{2(n+1)}$ for i=0,1,…,n,

Then we get followings systems of the algebraic equation:

$$c^T A G A^{-1} \Phi(x_i) + \gamma_1 c^T A D^{(1)} A^{-1} \Phi(x_i) + \gamma_2 c^T \Phi(x_i) = f(x_i) \tag{6}$$

$c^T \Phi(0) = u_0$, the unidentified variable 'c' can successfully determine by resolving system of the algebraic equation given by (6) and by Substitute into eq. (2), required solution is obtained.

5. Illustrative Examples

This subsection presents resolution of three distinct Fraction variable orders problems by employing the advanced operationally-shifted Legendre polynomial approach. For a deeper comprehension of this research, The graph has been meticulously constructed and resultant data have been systematically organize within table for enhanced clarity. We match our numerical result with exact solution and other approaches in the scholarly literatures. Our samples were calculated and programmed using Matlab software.

Example 5.1

For a linear Fractional variable order ,we examine the variable order $\alpha(x)$

$$D^{\alpha(x),\rho} u(x) = f(x), \ \ 0 \le \alpha(x) \le 1 \ , \boldsymbol{\rho} = 1 \text{ and } \boldsymbol{\rho} = 1.01, x \in [0,1], \tag{7}$$

For $u(0) = 0$ and $\alpha(x) = \sin(x)$[14], $f(t) = \frac{\Gamma(3) t^{2-\alpha(x)}}{\Gamma(3-\alpha(x))} + 3\frac{\Gamma(2) x^{1-\alpha(x)}}{\Gamma(2-\alpha(x))}$,

the accurate solutions of Eq.(7) is $u(x) = x^2 + 3x$ we utilize current approach with $M = 3$ and $M = 4$,

$$C^T D^{\alpha(x),\rho} \phi(x) - f(x) = 0,$$

Table 1: Estimated solutions of **ρ**=1,N=3and N=4,for examples (5.1)

T	Exact Solutions	Estimated Solutions M=3	Estimated Solution M=4
0.1	0.3100	0.3099	0.3099
0.2	0.6400	0.6399	0.6399
0.3	0.9900	0.9900	0.9899
0.4	1.3600	1.3600	1.3599
0.5	1.7500	1.7500	1.7500
0.6	2.1600	2.1601	2.1600
0.7	2.5900	2.5900	2.5899
0.8	3.0400	3.0400	3.0399
0.9	3.5100	3.5099	3.5099

Table 2: The Estimated solution of ρ=1.01and M=3,for examples (5.1)

T	accurate	Estimated M=3
0.1	0.3100	0.3100
0.2	0.6400	0.6400
0.3	0.9900	0.9900
0.4	1.3600	1.3600
0.5	1.7500	1.7500
0.6	2.1600	2.1600
0.7	2.5900	2.5900
0.8	3.0400	3.0400
0.9	3.5100	3.5100

Table 3: Comparative Analysis of Absolute Errors in example (5.2)

$\alpha(t)$	$E_\infty(4)$ [20]	$E_\infty(4)$ [21]	$E_\infty(4)$ [22]	$E_\infty(4)$(Present method)
sin (t)	2.47×10^{-2}	2.87×10^{-1}	7.53×10^{-7}	5.36×10^{-10}

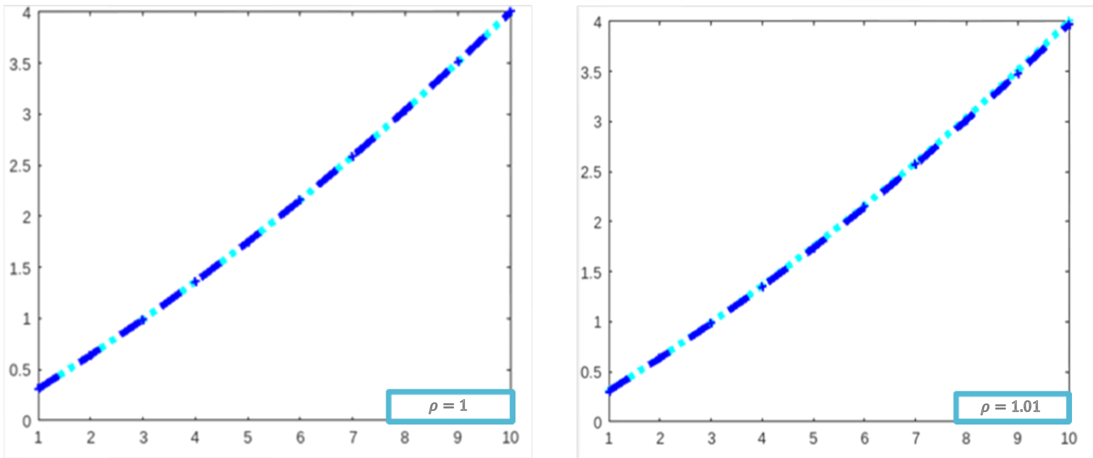

Figure1: Estimated solution of ρ=1&1.01,for example (5.1)

Example 5.2

For a linear Fractional variable order ,we examine the variable order $\alpha(t)$

$$D^{\alpha(x),\rho}u(x) = f(x), \quad 0 \le \alpha(x) \le 1, \boldsymbol{\rho} = 1 \text{ and } \boldsymbol{\rho} = 1.01, x \in [0,1], \quad (8)$$

For $u(0) = 0$, and $\alpha(x) = x/2$ [18] , $f(x) = \frac{\Gamma(3)x^{2-\alpha(x)}}{\Gamma(3-\alpha(x))} + 3\frac{\Gamma(2)x^{1-\alpha(x)}}{\Gamma(2-\alpha(x))}$ .The accurate solutions of Eq.(8) is $u(x) = x^2 + 3x$,we utilize current approach with $M = 2$ and 4

$$C^T D^{\alpha(x),\rho}\phi(x) - f(x) = 0$$

Table 4: Estimated solutions of **ρ**=1,M=2 and M=4,for examples (5.2)

T	accurate Solutions	Estimated Solutions M=2	Estimated Solutions M=4
0.1	0.3100	0.3062	0.3099
0.2	0.6400	0.8370	0.6399
0.3	0.9900	0.9922	0.9899
0.4	1.3600	1.3726	1.3600
0.5	1.7500	1.7762	1.7500
0.6	2.1600	2.2048	2.1600
0.7	2.5900	2.6579	2.5899
0.8	3.0400	3.1355	3.0399
0.9	3.5100	3.6376	3.5099

Table 5: Estimated solution of **ρ**=1.01,M=3,for examples (5.2)

T	accurate Solutions	Estimated Solutions M=3
0.1	−0.0900	−0.0900
0.2	−0.1600	−0.1600
0.3	−0.2100	−0.2100
0.4	−0.2400	−0.2400
0.5	−0.2500	−0.2500
0.6	−0.2400	−0.2400
0.7	−0.2100	−0.2100
0.8	−0.1600	−0.1600
0.9	−0.0900	−0.0900

Table 6: Comparative Analysis of Absolute Errors in example (5.2)

$\alpha(t)$	$E_\infty(4)$ [20]	$E_\infty(4)$ [21]	$E_\infty(4)$ [22]	$E_\infty(4)$(Present method)
t/2	5.98×10^{-3}	2.12×10^{-1}	1.20×10^{-6}	1.06×10^{-9}

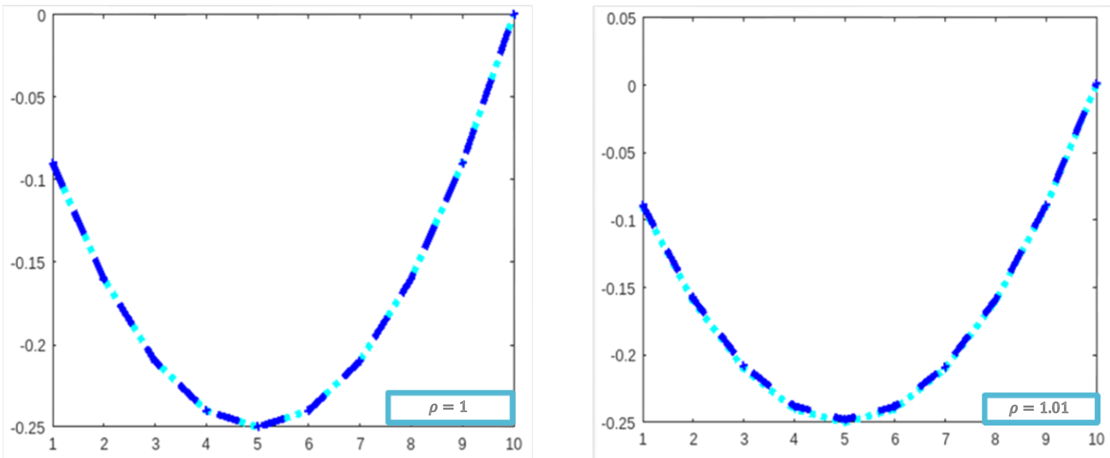

Figure 2: Estimated solution of **ρ**=1and **ρ**=1.01,for example (5.2)

Example5. 3

For a nonlinear Fractional variable order, we examine the variable order $\alpha(x)$

$D^{\alpha(x),\rho}u(x) = 1 - u(x)^2, \quad 0 < \alpha(x) \leq 1$, $\boldsymbol{\rho} = 1$ and $\boldsymbol{\rho} = 1.01, x \in [0,1],$ (9)

for $u(0) = 0$ and $\alpha(x) = 0.98x + 0.02x$ [19] ,the accurate solutions of Eq.(9) is $u(x) = \frac{e^{2x}-1}{e^{2x}+1}$, we utilize current approach with $M = 3$ and $M = 4,$

$$C^T D^{\alpha(x),\rho}\phi(x) + (C^T\phi(x))^2 - f(x) = 0,$$

Table 7: Estimated solutions of ρ=1, M = 3 and M = 4,for examples (5.3)

T	Accurate solutions	Estimated solutions M=3	Estimated solution M=4
0.1	0.0996	0.1041	0.1043
0.2	0.1973	0.2037	0.2040
0.3	0.2913	0.2981	0.2986
0.4	0.3799	0.3866	0.3870
0.5	0.4621	0.4683	0.4685
0.6	0.5370	0.5426	0.5425
0.7	0.6043	0.6087	0.6089
0.8	0.6640	0.6659	0.6677
0.9	0.7162	0.7135	0.7193

Table 8: The Estimated solution of ρ=1.01and M=3,for examples (5.3)

T	accurate solutions	Estimated solutions M=3
0.1	0.0996	0.1041
0.2	0.1973	0.2037
0.3	0.2913	0.2981
0.4	0.3799	0.3866
0.5	0.4621	0.4683
0.6	0.5370	0.5426
0.7	0.6043	0.6087
0.8	0.6640	0.6659
0.9	0.7162	0.7135

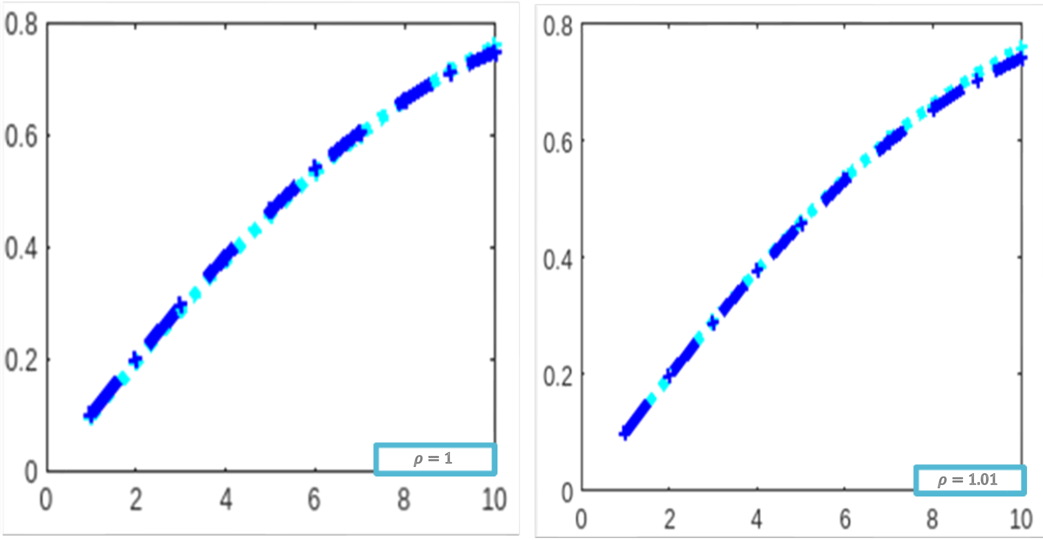

Figure 3: Estimated solutions of ρ=1and ρ=1.01,for examples (5.3)

6. Conclusion

The last numerical resolutions of the problem have successfully addressed the Non-linear and the Linear time variable-order of Fraction. This method uses the operation matrix dependent on the shifted Legendre polynomial in new generalize Caputo-type fraction derivatives sense, that reduce issues to non-linear and linear group of the algebraic equation. New features are offered, and new theorem are advanced. The graphical representation of solutions show cases the results, characterizing both the method employed and its broader application in fraction domain. Our performance is the robust and high dependable in the seek solution to variable orders fractional problem. This method reveal outcome of selected unlike value for fraction orders, parameter, and variable orders, which provide benefits for more accurate analysis of problems. The recent development in differential equations could potentially pave the way for enhanced exploration into the realm of variable order fractional differentiations.

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