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A New Convex Estimator for Linear Regression Model

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Abstract:
In this article, we propose for a linear regression model a new biased estimator aimed at mitigating the impact of multicollinearity on the estimation of unknown parameters. The idea of the proposed estimator is constructed as a combination of the unbiased ridge estimator (URR) and the ordinary ridge regression estimator (ORR). The statistical properties for the new estimator and other existing estimators are found. We demonstrate the advantages of this new estimator using the mean squared error criterion, showing that its mean squared error does not exceed that of either the URR or ORR individually. This simulation study supports these theoretical results, revealing that the new estimator outperforms both the URR and ORR. This suggests that combining two estimators linearly can leverage the strengths of both, resulting in a superior estimator. To further validate the new approach, we also present a numerical example.

Keywords: Linear Regression Model, Unbiased Ridge Estimator, Multicollinearity, Ridge Regression Estimator, Convex estimator.

مقدر محدب جديد لنموذج الانحدار الخطي

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الخلاصة
في هذه المقالة، نقترح لنموذج الانحدار الخطي مقدرًا متحيزًا جديدًا يهدف إلى التخفيف من تأثير التعدد الخطي على تقدير المعلمات غير المعروفة. تم بناء فكرة المقدر المقترح كمزيج من مقدر انحدار الحرف الغير المتحيز (URR) ومقدر الانحدار الحرف العادي (ORR). تم ايجاد الخصائص الإحصائية للمقدر الجديد والمقدرين الآخرين الموجودين. افضلية هذا المقدر الجديد تم دراستها باستخدام معيار الخطأ التربيعي المتوسط، حيث تبين أن خطأه التربيعي المتوسط لا يتجاوز خطأ URR أو ORR بشكل فردي. تدعم دراسة المحاكاة هذه النتائج النظرية، وتكشف أن المقدر الجديد يتفوق على كل من URR وORR. يشير هذا إلى أن الجمع بين مقدرين خطيًا يمكن أن يستفيد من نقاط قوة كليهما، مما يؤدي إلى مقدر متفوق. لمزيد من التحقق من صحة النهج الجديد، نقدم أيضًا مثالًا توضيحيا.

1. Introduction

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In the realm of statistics, the multicollinearity problem and its effects on the estimation of the unknown parameters of the linear regression model are widely recognized. Notably, multicollinearity significantly affects the least squares estimator (OLS), leading to inflated sample variances in estimated regression coefficients. Consequently, this inflation increases the risk of erroneous exclusion of significant coefficients from the model. To address this challenge, researchers have proposed various strategies to deal with it. One of these strategies is to use the biased estimation. Several biased estimators have been introduced as an alternative to the OLS have been introduced. [1] [2] [3] [4]

The ordinary ridge regression (ORR) is one of the most famous biased estimators that deals with the multicollinearity problem. Although the estimator is always biased, it outperforms the OLS estimator by the criterion of sampling variance. However, the sampling variance is not an appropriate criterion for evaluating the performance of a biased estimator. The unbiased ridge regression (URR) is one of the most important estimators and the most popular due to its many features. It aims to reduce the variance. Therefore, the resulting estimator has a lower mean squared error matrix (MMSE). [5] [6]

The method of combining two estimators generates a new estimator having the useful statistical properties found in the two estimators. For this reason, a convex combination of two estimators can be useful when both estimators appear to be appropriate in a specific situation. The MSE of the convex combination is limited by the maximum of MSE of the individual estimators.

The aim of this article is to obtain some improvements by using the convex combination of URR and ORR estimators. Thus, the structure of this paper is organized as follows: In section 2, we discuss some estimation methods and their statistical properties. In section 3, we suggest a new version of convex estimator by combining ORR and URR. We also show that the MSE of the new convex estimator cannot exceed the individual MSE of ORR and URR estimator in section 4. We have submitted a numerical example to illustrate the theoretical results.

2. Methodology:

Consider the multiple linear regression model

$$Y = X\beta + u, \tag{1}$$

where Y is a $(t \times 1)$ vector of observations, X *is a* $(t \times p)$ decision matrix of p independent variables, β is a $(p \times 1)$ vector of unknown parameters and u is an $(t \times 1)$ vector of disturbances. It is assumed that $E(u) = 0$ and $E(u\,ú) = \sigma^2 I$. We know that using the method of the least squares, we are searching for the vector β that minimizes the sum of squared residuals $(u'u)$ to get:

$$\hat{\beta}_{OLS} = (X'X)^{-1}\ X'Y \tag{2}$$

When there is linear correlation (multicollinearity) between the independent variables, the least squares estimator loses the property of least variance, as this correlation between the independent variables will cause an inflation of the variance that increases with the strength of the correlation.

For this reason, Hoerl and Kennard [7] proposed an estimator by minimizing $(Y - X\beta)'(Y - X\beta)$ subject to $\beta'\beta = e$, where e is a constant, is given as follows:

$$(Y - X\beta)'\ (Y - X\beta) + k(\beta'\beta - e), \tag{3}$$

where k is a Lagrangian multiplier. The solution of Eq. (3) gives the ORR estimator:

$$\hat{\beta}_{\mathrm{ORR}} = (X'X + \mathrm{kI})^{-1}\,\mathrm{X'Y}, k \geq 0 \tag{4}$$

where k is the ridge parameter. Crouse et al. [8] introduced the unbiased ridge estimator (URR) as a linear combination of prior information with OLS estimator as follows:

$$\hat{\beta}_{URR} = (X'X + kI)^{-1}(X'Y + kJ), k \geq 0 \tag{5}$$

where J is a random vector with $J \sim N\,(\beta, \sigma^2/k\ \ I)$ for $k > 0$ and independent of $\hat{\beta}$.
The convex estimator for the linear model can be written as follows [9]:

$$\bar{\beta} = A\beta_1^* + (I - A)\beta_2^*,$$

where β_1^*, β_2^*be any two estimators and A is a square matrix. The matrix of mean square error (MMSE) is one of the measures that is used to find out how close an estimator is to the true value of the parameter. In addition, it can be used as a criterion to determine when one estimator is better than another. The MMSE is defined as:

$$M = M\,(\beta^*, \beta) = E(\beta^* - \beta)(\beta^* - \beta)'.$$

Where β^* is an estimator ,while, the covariance matrix of an estimator β^* is given by

$$Cov(\beta^*) = E(\beta^* - E(\beta^*))\,(\beta^* - E(\beta^*))'.$$

If $E(\beta^*) = \beta,$ then β^* is unbiased estimator for β, β^* is biased estimator for β. The difference between $E(\beta^*)$ and β is :

$$Bias\,(\beta^*, \beta) = E(\beta^*) - \beta.$$

We can rewrite the MSE as follows:

$$M = M(\beta^*, \beta) = Cov(\beta^*) + (Bias\,(\beta^*, \beta))\,(Bias\,(\beta^*, \beta))'.$$

For all matrices and F , we will write $H \leq F$ when $F - H$ is a matrix that will be nonnegative definite (n.n.d., while if $\mathrm{H} < \mathrm{F}$ then $\mathrm{F} - \mathrm{H}$ will be called a positively definite (p.d.). Let β_1^* and β_2^* any two estimators of β . Then β_2^* is called MMSE- superior to β_1^* if :

$$\Delta(\beta_1^*, \beta_2^*) = M(\beta_1^*, \beta) - M(\beta_2^*, \beta) \geq 0.$$

A scalar mean squared error valued version of the MMSE is given by

$$msem = R(\beta^*, \beta) = E(\beta^* - \beta)'(\beta^* - \beta)$$

and

$$R(\beta^*, \beta) = tr(M(\beta^*, \beta))$$

The symbol $tr(M(\beta^*, \beta)$) denotes the trace of the matrix $M(\beta^*, \beta)$.
Convex estimators incorporate penalty terms into the optimization problem, which encourages simpler models with smoother coefficient profiles. This regularization helps prevent overfitting by penalizing overly complex models, which is especially beneficial when dealing with noisy or correlated expantary variables. Regularization can lead to more stable and generalizable models. Also, convex estimators can effectively deal with multicollinearity by shrinking the coefficients of correlated predictors towards each other. This reduces the sensitivity of coefficient estimates to small changes in the data and mitigates the instability associated with collinear expantary variables. As a result, convex estimators provide more robust and reliable coefficient estimates in the presence of multicollinearity. By adding a penalty term to the objective function, some convex estimators strike a balance between bias and variance. While it increasing bias slightly, they reduce variance, leading to more stable coefficient estimates. This bias-variance tradeoff often results in improved predictive performance, especially in situation when the explanatory variables are highly correlated.
For this reason, the following section gives the proposed estimator to improve predictive performance.

3. A New Estimator

Now, we can suggest a new version of convex estimator by combining ORR and URR as follows [10]:

$$\bar{\beta}_{KM} = A\hat{\beta}_{URR} + (I - A)\hat{\beta}_{ORR} \tag{6}$$

We call it as KM estimator. The following notations will be used.

$$D = M(\bar{\beta}_{KM}, \beta) \quad , \quad D_{ij} = E(\beta_i^* - \beta)(\beta_j^* - \beta)', i, j = 1,2, \tag{7}$$

D is minimized when: [7]

$$A = [D_{22} - D_{21}][D_{11} + D_{22} - D_{12} - D_{21}]^{-1} \tag{8}$$

Hence , the minimal value of D of KM estimator is given by

$$D_0 = D_{22} - [D_{22} - D_{21}][D_{11} + D_{22} - D_{12} - D_{21}]^{-1}[D_{22} - D_{12}] \tag{9}$$

Or

$$D_0 = D_{11} - [D_{11} - D_{12}][D_{11} + D_{22} - D_{12} - D_{21}]^{-1}[D_{11} - D_{21}] \tag{10}$$

Clearly, if β_1^* and β_2^* are independent ,then $D_{12} = 0$. Note also that:
$D_0 < D_{11}$ and $D_0 < D_{22}$.
The following theorem shows the benefit of KM estimator.

Theorem: - Under linear regression mode, the MSE of KM estimator cannot exceed the individual MSE of URR and ORR, where

$$A = k^2 S_k^{-1} \beta\beta' [\sigma^2(I - SS_k^{-1}) + k^2 S_k^{-1} \beta\beta']^{-1}$$

Proof: -
For all A in D_p, the set of all $p \times p$ matrices. Using (7), we get $D_{11} = \sigma^2 S_k^{-1}$ and

$$D_{22} = \sigma^2 X'X(X'X + kI)^{-2} + k^2(X'X + kI)^{-1}\beta\beta'(X'X + kI)^{-1}$$
$$= \sigma^2 S S_k^{-2} + k^2 S_k^{-1}\beta\beta' S_k^{-1} ,$$

where $S = X'X$, and $S_k = X'X + kI$.

$$D_{12} = \sigma^2 X'X \ (X'X + kI)^{-2} = \sigma^2 S S_k^{-2} = D_{21}$$
$$D_{22} - D_{21} = \sigma^2 S S_k^{-2} + k^2 S_k^{-1}\beta\beta' S_k^{-1} - \sigma^2 SS_k^{-2} = k^2 S_k^{-1}\beta\beta' S_k^{-1}$$

$D_{11} - D_{12} = \sigma^2 S_k^{-1} - \sigma^2 S S_k^{-2}$. Then we can write the MSE of $\bar{\beta}_{KM}$ as:

$$D = E(\bar{\beta}_{KM} - \beta)(\bar{\beta}_{KM} - \beta)'$$
$$= ATA' - AW_1' - W_1A' + D_{22},$$

where

$$T = D_{11} + D_{22} - D_{12} - D_{21} \text{ and } W_1 = D_{22} - D_{21} .$$

By taking the derivative of D w.r.t A and then setting the result to be equal to zero, we get
$2AT - 2W_1 = 0$.
D is minimized if A is the solution to:

$$AT = W_1$$
$$A[D_{11} + D_{22} - D_{12} - D_{21}] = [D_{22} - D_{21}]$$

Or

$$A = W_1 T^{-1}$$

So, $A = [D_{22} - D_{21}][D_{11} + D_{22} - D_{12} - D_{21}]^{-1}$ and the minimal value of $D = D_0$, where $D_0 = D_{22} - AW_1'$

$$= D_{22} - [D_{22} - D_{21}][D_{11} + D_{22} - D_{12} - D_{21}]^{-1}[D_{22} - D_{21}] \tag{11}$$
$$= D_{11} - [D_{11} - D_{12}][D_{11} + D_{22} - D_{12} - D_{21}]^{-1}[D_{11} - D_{12}] \tag{12}$$

Now, $T = D_{11} + D_{22} - D_{12} - D_{21}$

$$= \sigma^2 S_k^{-1} + \sigma^2 SS_k^{-2} + k^2 S_k^{-1} \beta \beta' S_k^{-1} - \sigma^2 S S_k^{-2} - \sigma^2 SS_k^{-2}$$
$$= \sigma^2 S_k^{-1} + k^2 S_k^{-1} \beta\beta' S_k^{-1} - \sigma^2 S S_k^{-2}$$
$$= \sigma^2(I - S S_k^{-1}) S_k^{-1} + k^2 S_k^{-1} \beta\beta' S_k^{-1}$$
$$= [\sigma^2(I - S S_k^{-1}) + k^2 S_k^{-1} \beta \beta'] S_k^{-1}.$$

So, $T^{-1} = [D_{11} + D_{22} - D_{12} - D_{21}]^{-1}$

$$= ([\sigma^2(I - S S_k^{-1}) + k^2 S_k^{-1} \beta\beta'] S_k^{-1})^{-1}$$
$$= (S_k^{-1})^{-1} [\sigma^2(I - S S_k^{-1}) + k^2 S_k^{-1} \beta\beta']^{-1}$$
$$= S_k[\sigma^2(I - S S_k^{-1}) + k^2 S_k^{-1} \beta \beta']^{-1} \tag{13}$$

Using (12) and (13), we get

$$D_0 - D_{11} = -\sigma^4 [I - S S_k^{-1}][\sigma^2(I - S S_k^{-1}) + k^2 S_k^{-1}\beta\beta']^{-1}[I - S S_k^{-1}]S_k^{-1}.$$

Since $I - SS_k^{-1} > 0$, $k > 0$ and $S_k^{-1} > 0$, $D_0 - D_{11} < 0$, where 0 denotes zero matrix. Therefore, we can say that $D_0 < D_{11}$. Using (11) and (13), we get:

$$D_0 - D_{22} = -k^4 S_k^{-1} \beta\beta'[\sigma^2 S^{-1}(I - S^2 S_k^{-2})S_k + k^2 S_k^{-1}\beta\beta']^{-1} S_k^{-1} \beta\beta' S_k^{-1}$$

Since $(I - S^2 S_k^{-2}) > 0$, $k > 0$ and $S_k^{-1} > 0$, then $D_0 - D_{22} < 0$. So we have $D_0 < D_{22}$
Thus, the proof is completed.

It can be seen that the new estimator depends on unknown parameters β and σ², and thus the estimator will be inapplicable. To eliminate this problem, we obtain the adaptive KM estimator as follows:

$$\bar{\beta}_{KM} = \hat{A}\hat{\beta}_{URR} + (I - \hat{A})\hat{\beta}_{ORR},$$

where $\hat{A} = k^2 S_k^{-1}\hat{\beta}\hat{\beta}'[\hat{\sigma}^2(I - S S_k^{-1}) + k^2 S_k^{-1}\hat{\beta}\hat{\beta}']^{-1}$ and

$$\hat{\sigma}^2 = \frac{1}{t-p}(Y - X\hat{\beta})'(Y - X\hat{\beta}).$$

3.1 Estimated Parameter of KM

For practical use, since k is unknown, we need to estimate it. Hoerl and Kennard [7] concluded that, the total variance decreases and the squared bias increases as k increases**.** Therefore, different estimated k in the literature using different techniques have been suggested. We can present some of estimate of k as follows:

- Hoerl and Kennard [7] introduced k as follows:

$$k_{HK} = \frac{\hat{\sigma}^2}{\hat{\beta}^2_{max}OLS}, \tag{14}$$

Where $\hat{\beta}^2_{max}$ is the maximum element of $\hat{\beta}_{OLS}$. Hoerl et al. [11], proposed k denoted by:

$$k_{HKB} = \frac{t\hat{\sigma}^2}{\hat{\beta}_{OLS}'\hat{\beta}_{OLS}}. \tag{15}$$

- Lawless and Wang [12], proposed the following estimator:

$$k_{LW} = \frac{t\hat{\sigma}^2}{\hat{\beta}_{OLS}'X'X\hat{\beta}_{OLS}}. \tag{16}$$

- Hocking [13], suggested the following estimator of k as follows:

$$k_{HSL} = \hat{\sigma}^2 \frac{\left(\sum_{i=1}^{p}(\lambda_i\hat{\beta}_{OLS})^2\right)}{\sum_{i=1}^{p}(\lambda_i\hat{\beta}^2_{OLS})^2}. \tag{17}$$

- Nomura [14], proposed the estimator of k as follows:

$$k_{HMO} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\left(\frac{\hat{\beta}_{iOLS}}{1+\left(1+\lambda_i\left(\frac{\hat{\beta}^2_{iOLS}}{\hat{\sigma}^2}\right)^{\frac{1}{2}}\right)}\right)}. \tag{18}$$

- Kibria et al. [15], proposed the estimators for k by taking the geometric mean (GM), arithmetic mean (AM), and median (MED) of $\frac{\hat{\sigma}^2}{\hat{\beta}_i^2}$ given as follows:

$$k_{GM} = \frac{\hat{\sigma}^2}{\left(\prod_{i=1}^{p}\hat{\beta}^2_{OLS}\right)^{1/t}}, \tag{19}$$

$$k_{AM} = \frac{1}{t}\sum_{i=1}^{p}\frac{\hat{\sigma}^2}{\hat{\beta}_{iOLS}}, \tag{20}$$

$$k_{MED} = midian\left\{\frac{\hat{\sigma}^2}{\hat{\beta}^2_{OLS}}\right\}. \tag{21}$$

- Khalaf and Shukur [16], proposed the estimator for k based on k_{HK} as follows:

$$k_{KS} = \frac{\lambda_{max}\hat{\sigma}^2}{(t-p)\hat{\sigma}^2 + \lambda_{max}\hat{\beta}^2_{max}OLS}, \tag{22}$$

where λ_{max} the maximum eigenvalues of $X'X$.

- Alkhamisi [17], proposed the following estimators of *k* as follows:

$$k_{s_{airth}} = \frac{1}{t}\sum_{i=1}^{p} \frac{\lambda_i\hat{\sigma}^2}{(t-p)\hat{\sigma}^2 + \lambda_i\,\hat{\beta}_i^2 OLS}\,. \tag{23}$$

$$k_{s_{md}} = midian\left(\frac{\lambda_i\hat{\sigma}^2}{(t-p)\hat{\sigma}^2 + \lambda_i\hat{\beta}_i^2 OLS}\right). \tag{24}$$

The *msem* of KM estimator is given by :

$$msem(\bar{\alpha}_{KM}) = \sigma^2 \sum_{i=1}^{p} \frac{(1-k)\sigma^2 + k\alpha_i^2}{(\lambda_i + k)(\sigma^2 + k\alpha_i^2)} \quad ,i = 1, \dots, p \tag{25}$$

where $\alpha = Q'\beta$ and Q is an orthogonal matrix.

From (25), the optimal value for *k* can be derived by minimizing $mse(\bar{\alpha}_{KM})$ with respect to *k.* Therefore, by differentiating $msem(\bar{\alpha}_{KM})$ with respect to k

$$\frac{\partial msem(\bar{\alpha}_{KM})}{\partial k} = \sigma^2 \sum_{i=1}^{p} \frac{(\lambda_i+k)(\sigma^2+k\alpha_i^2)\,(-\sigma^2+\alpha_i^2) - (\sigma^2-\sigma^2k+k\alpha_i^2)[(\lambda_i+k)(\alpha_i^2)+(\sigma^2+k\alpha_i^2)]}{(\lambda_i+k)^2(\sigma^2+k\alpha_i^2)^2}$$

and equating $\frac{\partial msem(\bar{\alpha}_{KM})}{\partial k}$ to zero. Since the equation of k depends on unknown parameters for practical side, we replace the parameters σ^2 and α by their unbiased estimators to get the optimal estimator of k as:

$$\hat{k}_i = \frac{\hat{\alpha}_i\hat{\sigma}^2 + \hat{\sigma}^2\sqrt{\hat{\sigma}^2\lambda_i - \hat{\alpha}_i^2\,\lambda_i + \hat{\sigma}^2}}{\hat{\sigma}^2\,\hat{\alpha_i} - \hat{\alpha}_i^3} \tag{26}$$

From (26), we suggest the following estimators for k:

$$k_{MM} = \frac{\hat{\sigma}^2 * \sum_{i=1}^{p}\hat{k}_i}{p*n*\min(\hat{k}_i)} \tag{27}$$

$$k_{HMM} = \frac{\min(\hat{k}_i)}{n*\hat{\sigma}^2} \tag{28}$$

$$k_{KMM} = \frac{p^2*\hat{\sigma}^2}{n*\min(\hat{k}_i)} \tag{29}$$

4. A simulation study

To cover the performance of the proposed estimator, we performed a simulation study using Matlab program. To obtain different degrees of collinearity between multiple independent variables (Xs) where this is very important point, Kibria [18], Arumairajan and Wijekoon [19], Alheety [20] used the following equation to generate Xs:

$$X_{ij} = (1-\varphi)^{\frac{1}{2}} z_{ij} + \varphi z_{ip} \quad , \quad i = 1,2,3, \dots, t, \quad j = 1,2, \dots, p. \tag{30}$$

where the z_{ij} represents the numbers distributed as independent standard normal pseudo-random and φ represents the correlation between any two X's. To get X'X in a correction forms, X's are standardized. The response variable y is considered by :

$$y_i = \beta_1 X_{i1} + \beta_2 X_{i2} + \dots + \beta_p X_{ip} + e_i\,, \tag{31}$$

where the e_i is i.i.d. N(0, σ^2). Therefore, the zero intercept of (31) will be assumed. Also, the number of independent variables p=4, while the values of σ are chosen as (0.001, 0.01, 0.1,1,5). The correlation φ are chosen as (0.80, 0.90, 0.95, 0.99) and sample size t are (20, 60, 180). The values of k will be estimated by : k_{HKB}, k_{LW}, k_{HSL}, k_{HMO}, k_{AM}, k_{GM}, k_{MED}, k_{KS}, $k_{s_{arith}}$.

We use the eigenvectors corresponding to the largest eigenvalue of the matrix X'X as the coefficients β_1, β_2, …, β_p subject to constraint $\beta'\beta = 1$. The experiment was re-preformed

10000 times. The estimated mean squared error (EMSE) for each estimator is calculated as follows:

$$mse(\beta_1) = \frac{1}{10000}\sum_{i=1}^{10000}(\beta^* - \beta)'(\beta^* - \beta) \tag{32}$$

where β^* would be any of the estimators (URR, ORR and KM).

4.1 Discussion of the Simulation Results

From Tables 1-6, the simulation results show that when the values t, φ and k are fixed, the EMSE values also increase with the increase of σ values. While when the values of t increase, the EMSE values decrease for fixed values of σ, φ and k. In addition, when the value of φ increases, the MSE values increase. In Table 1, the KM estimator dominates the ridge and unbiased ridge regression estimators when ($\sigma = 0.001$ and $\varphi = 0.8, 0.9$) while the unbiased ridge regression estimator dominates the KM and ORR estimators when ($\sigma = 0.1$ and $\varphi = 0.8, 0.9$).

In Table 2, when ($\varphi = 0.95$ and $\sigma = 0.01, 0.1, 1$) the suggested estimator KM is the best because it has minimum MSE. When ($\varphi = 0.99$ and $\sigma = 0.001, 0.1$), the best estimator is the KM because it has a minimum MSE. In Table 3, the KM estimator dominates the ORR and URR estimators when ($\varphi = 0.8$ and $\sigma = 0.1, 1$) also in the case (φ =0.9 and σ=0.01,5). While, in the case ($\varphi = 0.8$ and $\sigma = 5$) the ridge regression estimator dominates the KM and URR estimators. In Table 4, when ($\varphi = 0.95$ and $\sigma = 0.001, 0.01, 0.1, 5$) the proposed KM estimator outperforms both the ridge and unbiased ridge estimators. While in the $case$ ($\varphi = 0.99$ and $\sigma = 0.1, 1$) the ridge estimator outperforms both the KM and URR estimators. In Table 5, the proposed KM estimator is better than both ridge and unbiased ridge estimators when ($\varphi = 0.8$ and $\sigma = 0.1, 1$). while, in the case ($\varphi = 0.9$ and $\sigma = 0.001, 0.01, 0.1, 1$) the unbiased ridge regression estimator is better than both proposed and ridge estimators. In Table 6, the proposed estimator dominates the ridge and unbiased ridge regression estimators when ($\varphi = 0.95$ and $\sigma = 0.1, 5$). So, we can say that the simulation results supported the theoretical result.

5. Numerical Example:

To support our theoretical findings, we examined a dataset concerning Portland cement, initially presented by Woods et al. [21], and extensively studied thereafter by researchers such as Lukman et al. [22], Özkale [23], Toker and Kaçıranlar [24], and others. The information originates from an experimental study examining the heat generated during the solidification and strengthening processes of Portland cements with diverse compositions. It explores the relationship between this heat and the proportions of four compounds present in the clinkers used to produce the cement. In this instance, the variable Y represents the dependent variable, indicating the heat produced and measured in calories per gram of cement. The independent variables are amounts of the following compounds: tricalcium aluminate (X_1), tricalcium silicate (X_2), tetracelcium alumino femrrite (X_3), and dicalcium silicate (X_4). The model includes the intercept item.

The scale mean square error matrix (smse) of the ORR, URR and KM are respectively given by:

$$mse(\hat{\alpha}_{ORR}) = tr\big(MSE(\hat{\alpha}_{ORR})\big) = \sum_{i=1}^{P}\frac{\lambda_i\sigma^2 + k^2\alpha_i{}^2}{(\lambda_i + k)^2}\ , \tag{33}$$

$$mse(\hat{\alpha}_{URR}) = tr\big(MSE(\hat{\alpha}_{URR})\big) = \sum_{i=1}^{P}\frac{\sigma^2}{(\lambda_i + k)}\ , \tag{34}$$

$$mse(\bar{\alpha}_{KM}) = tr\big(MSE(\bar{\alpha}_{KM})\big) = \sigma^2\sum_{i=1}^{p}\frac{(1-k)\sigma^2 + k\alpha_i{}^2}{(\lambda_i + k)(\sigma^2 + k\alpha_i{}^2)}\ , \tag{35}$$

where tr is the trace of the matrix.

The five eigenvalues of X' X are λ_1 = 211.367, λ_2 = 77.236, λ_3 = 28.459, λ_4 = 10.267 and λ_5 = 0.0349. The condition number of X' X is K = $\lambda_{max}/\lambda_{min}$ =6056.37 and so X' X may be considered as being quite "ill-Conditioned". To get the data that X' X is in the form of a correlation matrix, we recommend to standardize it. The benefit of standardization is that the regression coefficients can be expressed in comparable numerical units [25].
Therefore, the four eigenvalues of the correlation matrix X' X are 2.2357, 1.57606, 0.18661, and 0.00162 while the estimated value of σ^2 is 0.00196.
For the standardized data, we obtain $\hat{\sigma}^2 = 0.003932$. The eigenvalues of X' X are 2.95743, 0.91272, 0.10984, and 0.02021. The $X'X$ matrix will be as follows:

$$X'X = \begin{bmatrix} 1.000 & 0.888 & .925 & 0.309 \\ 0.888 & 1.000 & 0.962 & 0.157 \\ 0.925 & 0.962 & 1.000 & 0.328 \\ 0.309 & 0.157 & 0.328 & 1.000 \end{bmatrix}$$

The variables in $X'X$ matrix suffer from sever correlations. Another method for diagnosing multicollinearity in linear regression is the condition index (C.I.) which is defined as follows:
$C.I. = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}}$,
where λ_{max} and λ_{min} are the largest and the smallest eigenvalues of X' X. Therefore, in this data $, 10 < C.I. = \sqrt{\frac{2.95743}{0.02021}} = 12.1 < 30$, which indicates that there is a moderate multicollinearity and may be corrected.

Table 1: The mse for estimators and different estimated value of k

	k_{HKB}	k_{LW}	k_{HSL}	k_{MM}	k_{HMM}	k_{KMM}
OLS	1.218629	1.218629	1.218629	1.218629	1.218629	1.218629
ORR	0.1663935	0.158478	0.145758	0.194666	0.193161	0.187584
KM	0.2930769	0.261189	0.158810	0.011439	0.011811	0.013379

From Table 1, the proposed k estimators had the best performance in the sense of a smaller MMSE. When the estimated value of k is closed to 1, the MMSE of KM estimator will be better than OLS and ORR and vice versa.

6. Conclusion

The presence of multicollinearity makes the performance of OLS estimator inefficient and unstable. To address this problem, we developed new convex estimator named (KM) estimator for the linear regression model. Theoretically, we established the sufficient conditions under which the KM estimator outperforms the other classical estimators that presented in this paper using mean squared error. In order to evaluate the performance of the proposed KM estimator, a simulation study was conducted with different values of shrinkage parameter k. According to the results, we indicated that the KM estimator is more efficient than traditional ORR and URR estimators. Additionally, we compared the KM estimator and others using a numerical example. We proposed new estimated values of k which show more efficiency others that given in this study.

Appendix

Table 1: Estimated MSEs when t=20 and φ=0.8 and 0.9

φ		0.8			0.9		
σ	k	MK	ORR	URR	MK	ORR	URR
0.001	k_{HK}	0.250316545143	0.250316546711	0.250316545924	0.261585443062	0.261585448314	0.261585447628
	k_{HKB}	0.250316567493	0.250316575236	0.250316568799	0.261585452832	0.261585479030	0.261585473335
	k_{LW}	0.250316546640	0.250316548621	0.250316547512	0.261585443543	0.261585449824	0.261585448930
	k_{HSL}	0.250316545130	0.250316546694	0.250316545910	0.261585443062	0.261585448313	0.261585447627
	k_{HMO}	0.250326091643	0.250328674762	0.250326055088	0.261595885749	0.261617020059	0.261611401959
	k_{AM}	0.250317548237	0.250317826373	0.250317549952	0.261586950331	0.261590159057	0.261589320947
	k_{GM}	0.250317960004	0.250318351306	0.250317961507	0.261590810424	0.261601976878	0.261599035975
	k_{MED}	0.250316545143	0.250316546711	0.250316545924	0.261585443062	0.261585448314	0.261585447628
	k_{KS}	0.250325848040	0.250328366674	0.250325813550	0.261595657747	0.261616355522	0.261610855700
	$k_{S_{arith}}$	0.250433557521	0.250458414179	0.250427753953	0.261646079936	0.261744939007	0.261716533800
σ	k	MK	ORR	URR	MK	ORR	URR
0.01	k_{HK}	0.1987994482	0.1988230503	0.1988107260	0.2872999206	0.2871928536	0.2871915280
	k_{HKB}	0.1991462975	0.1992579096	0.1991571100	0.2877661440	0.2872324835	0.2872201390
	k_{LW}	0.1988491542	0.1988859703	0.1988623727	0.2873109261	0.2871937130	0.2871921920
	k_{HSL}	0.1987990840	0.1988225884	0.1988103412	0.2872998396	0.2871928473	0.2871915231
	k_{HMO}	0.3653065185	0.3736349962	0.3295072421	0.3327904718	0.3078532375	0.3012585274
	k_{AM}	0.2167090699	0.2170588982	0.2128973875	0.3022960840	0.2904982828	0.2894135214
	k_{GM}	0.2272363744	0.2268567725	0.2203888917	0.3119613108	0.2945123141	0.2921304179
	k_{MED}	0.1987993594	0.1988229377	0.1988106322	0.2872998801	0.2871928504	0.2871915255
	k_{KS}	0.2064786447	0.2072352212	0.2053412064	0.2928461224	0.2879721899	0.2877193484
	$k_{S_{arith}}$	0.2436398950	0.2422729623	0.2321036584	0.3257277300	0.3026450494	0.2976858171
σ	k	MK	ORR	URR	MK	ORR	URR
0.1	k_{HK}	0.2164845939	0.2164741833	0.2164703108	0.2915098177	0.2915009017	0.2914994844
	k_{HKB}	0.2166516986	0.2165988773	0.2165668169	0.2916008327	0.2915500253	0.2915381956
	k_{LW}	0.2165030528	0.2164880148	0.2164814276	0.2915112471	0.2915017156	0.2915001557
	k_{HSL}	0.2164845336	0.2164741381	0.2164702739	0.2915098120	0.2915008985	0.2914994817
	k_{HMO}	0.2713365390	0.2610638677	0.2495807590	0.3499264731	0.3296265458	0.3204076652
	k_{AM}	0.228031869	0.224590680	0.222556523	0.305777499	0.296765103	0.295488654

		7	6	6	8	2	5
	k_{GM}	0.2427263092	0.2357977478	0.2309073430	0.3323165039	0.3133400875	0.3080630283
	k_{MED}	0.2164845820	0.2164741744	0.2164703035	0.2915098141	0.2915008997	0.2914994827
	k_{KS}	0.2244946379	0.2220860854	0.2206840309	0.2940077153	0.2924400944	0.2922171401
	$k_{S_{arith}}$	0.2509534117	0.2426343375	0.2359806902	0.3229143524	0.3062620508	0.3026896395
σ	k	MK	ORR	URR	MK	ORR	URR
1	k_{HK}	0.3864485716	0.3712117243	0.3757254121	0.6333054280	0.4839371649	0.5812892575
	k_{HKB}	0.3851575857	0.3731268012	0.3745973683	0.6262065445	0.4335674161	0.5713961557
	k_{LW}	0.3831966889	0.3675723128	0.3724773988	0.6318684389	0.4746445002	0.5793099291
	k_{HSL}	0.3872755084	0.3722442902	0.3766034929	0.6346853130	0.4926773920	0.5832341568
	k_{HMO}	0.4705303812	0.4839098841	0.4494455615	0.6425313621	0.3859716360	0.5615733799
	k_{AM}	0.4090814315	0.4050245636	0.3954944107	0.6241003269	0.3954405628	0.5646588579
	k_{GM}	0.4181899800	0.4166594794	0.4033644478	0.6274650398	0.4438156192	0.5732686430
	k_{MED}	0.3877227544	0.3728154240	0.3770861876	0.6339958393	0.4883292563	0.5822559192
	k_{KS}	0.3876011054	0.3726592453	0.3769543633	0.6373585089	0.5092648198	0.5871816871
	$k_{S_{arith}}$	0.4780406122	0.4937298664	0.4562029232	0.6318310856	0.3797711024	0.5612997501
σ	k	MK	ORR	URR	MK	ORR	URR
5	k_{HK}	0.8273396495	0.8028187769	0.8173402027	1.4832660624	1.3874038907	1.4254389888
	k_{HKB}	0.7230979512	0.6976003707	0.7187768198	1.2303261817	1.0645499240	1.1684401964
	k_{LW}	0.7109877235	0.6914496350	0.7097598199	1.0701215595	0.9568304634	1.0525564510
	k_{HSL}	0.7388505956	0.7106701538	0.7325432869	1.4241094306	1.3028686268	1.3594503004
	k_{HMO}	0.7300988243	0.7612991547	0.7545980345	0.8332215215	0.9899050047	1.0348964972
	k_{AM}	0.7089134976	0.7068445631	0.7158567619	0.9120419916	0.9681282506	1.0273735304
	k_{GM}	0.7075127023	0.6936762420	0.7089330709	0.8392167205	0.9881078234	1.0341078356
	k_{MED}	0.8393775363	0.8162742801	0.8293883549	1.4917006841	1.3998118548	1.4352417338
	k_{KS}	0.823986	0.799093	0.814003	1.292872	1.13339	1.226302

Table 2: Estimated MSEs when t=20 and φ=0.95 and 0.99

Φ		0.95			0.99		
σ	k	MK	ORR	URR	MK	ORR	URR
0.001	k_{HK}	0.2867711306	0.2867711303	0.2867711298	0.3022537820	0.3022537921	0.3022537918
	k_{HKB}	0.2867711505	0.2867711485	0.2867711446	0.3022537563	0.3022538067	0.3022538039

	k_{LW}	0.2867711310	0.2867711306	0.2867711301	0.3022537819	0.3022537922	0.3022537918
	k_{HSL}	0.2867711306	0.2867711303	0.2867711298	0.3022537820	0.3022537921	0.3022537918
	k_{HMO}	0.2868502963	0.2868401088	0.2868256871	0.3032959600	0.3025765157	0.3025173144
	k_{AM}	0.2867827238	0.2867817117	0.2867794997	0.3022412982	0.3022619976	0.3022605263
	k_{GM}	0.2868391667	0.2868308315	0.2868183497	0.3022454275	0.3022589752	0.3022580462
	k_{MED}	0.2867711306	0.2867711303	0.2867711298	0.3022537820	0.3022537921	0.3022537918
	k_{KS}	0.2868447445	0.2868354961	0.2868220389	0.3022549272	0.3023230895	0.3023105775
	$k_{s_{arith}}$	0.2870597358	0.2869975920	0.2869502284	0.3022308758	0.3023007764	0.3022923127
σ	k	MK	ORR	URR	MK	ORR	URR
0.01	k_{HK}	0.2652744614	0.2652753177	0.2652752003	0.3023458586	0.3023469640	0.3023469346
	k_{HKB}	0.2652762788	0.2652805320	0.2652795597	0.3023427764	0.3023482023	0.3023479570
	k_{LW}	0.2652745438	0.2652755550	0.2652754048	0.3023458477	0.3023469683	0.3023469383
	k_{HSL}	0.2652744613	0.2652753173	0.2652751999	0.3023458586	0.3023469640	0.3023469346
	k_{HMO}	0.2772029300	0.2756172918	0.2737242536	0.3410833828	0.3188734494	0.3156751085
	k_{AM}	0.2655450408	0.2658304424	0.2657308165	0.3058493179	0.3029659774	0.3028443441
	k_{GM}	0.2657357863	0.2661170696	0.2659657635	0.3121027343	0.3039629547	0.3036462991
	k_{MED}	0.2652744614	0.2652753177	0.2652752003	0.3023458586	0.3023469640	0.3023469346
	k_{KS}	0.2663284081	0.2668388544	0.2665568709	0.3032191894	0.3025918386	0.3025440221
	$k_{s_{arith}}$	0.2681372723	0.2685539185	0.2679594374	0.3165877078	0.3049302260	0.3044260181
σ	k	MK	ORR	URR	MK	ORR	URR
0.1	k_{HK}	0.2871691525	0.2874325586	0.2874391777	0.3102633399	0.3104539696	0.3104653486
	k_{HKB}	0.2859889585	0.2870866978	0.2871276439	0.3090333298	0.3097532451	0.3098318765
	k_{LW}	0.2871233752	0.2874187954	0.2874266286	0.3102562266	0.3104501816	0.3104618703
	k_{HSL}	0.2871700240	0.2874328208	0.2874394172	0.3102635414	0.3104540768	0.3104654471
	k_{HMO}	0.2985188797	0.2978537481	0.2962334971	0.8840407269	0.9956124922	0.9016312363
	k_{AM}	0.2851079760	0.2878887135	0.2877109602	0.3340113290	0.3237130338	0.3215578881
	k_{GM}	0.2899309734	0.2913293340	0.2906236946	0.3149641730	0.3086951293	0.3084578222
	k_{MED}	0.2871695718	0.2874326848	0.2874392929	0.3102634498	0.3104540281	0.3104654023
	k_{KS}	0.2840326623	0.2866723239	0.2867311583	0.3076702108	0.3088123638	0.3089811495
	$k_{s_{arith}}$	0.3052256169	0.3034186972	0.3010519571	0.3161927025	0.3093524959	0.3090145314

σ	k	MK	ORR	URR	MK	ORR	URR
1	k_{HK}	0.4627099762	0.5384609651	0.5071869775	1.8311092165	1.8545512370	1.8829565784
	k_{HKB}	0.4062471035	0.4760663664	0.4423721746	1.1174597871	1.0698727468	1.1122034628
	k_{LW}	0.4243786760	0.4979355713	0.4644154511	0.6366631282	0.5148483402	0.5483207297
	k_{HSL}	0.4382454879	0.5132517826	0.4803375115	0.6518510796	0.5313481051	0.5658378904
	k_{HMO}	0.5662060047	0.5719760461	0.6357479819	0.6371173332	0.5153383143	0.5488426116
	k_{AM}	0.3778334508	0.4083175997	0.3868960244	0.8268322401	0.7310830918	0.7722961075
	k_{GM}	0.4267600176	0.5006350104	0.4671962679	0.8211161091	0.7244149362	0.7655107388
	k_{MED}	0.4682969380	0.5439778940	0.5131496199	1.8343493820	1.8579528583	1.8862765534
	k_{KS}	0.5492912876	0.6179117274	0.5949873818	1.1305133521	1.0849102600	1.1271334959
	$k_{s_{arith}}$	0.3811260742	0.4043799798	0.3861980601	0.7699838362	0.6650074313	0.7048068801
σ	k	MK	ORR	URR	MK	ORR	URR
5	k_{HK}	1.4832660624	1.3874038907	1.4254389888	2.1828369360	2.1824044694	2.1690243446
	k_{HKB}	1.2303261817	1.0645499240	1.1684401964	1.5456465180	1.4980038865	1.4838074041
	k_{LW}	1.0701215595	0.9568304634	1.0525564510	0.9458196023	0.9562158975	0.9554943727
	k_{HSL}	1.4241094306	1.3028686268	1.3594503004	1.3597705059	1.3024856675	1.2893191032
	k_{HMO}	0.8332215215	0.9899050047	1.0348964972	0.8875111907	0.9814066988	0.9820985048
	k_{AM}	0.9120419916	0.9681282506	1.0273735304	1.0090295796	0.9689136858	0.9641609370
	k_{GM}	0.8392167205	0.9881078234	1.0341078356	1.2648341091	1.2048298105	1.1926872738
	k_{MED}	1.4917006841	1.3998118548	1.4352417338	2.2003691489	2.2012628992	2.1879537935
	k_{KS}	1.2928721368	1.1333903878	1.2263018224	1.0652478788	1.0123955651	1.0048695836

Table 3: Estimated MSEs when t=60 and φ=0.8 and 0.9

		0.8			0.9		
σ	k	MK	ORR	URR	MK	ORR	URR
0.001	k_{HK}	0.2258544693	0.2258544693	0.2258544691	0.2531547848	0.2531547851	0.2531547849
	k_{HKB}	0.2258544749	0.2258544747	0.2258544734	0.2531547934	0.2531547951	0.2531547930
	k_{LW}	0.2258544698	0.2258544698	0.2258544695	0.2531547853	0.2531547857	0.2531547854
	k_{HSL}	0.2258544693	0.2258544693	0.2258544691	0.2531547848	0.2531547851	0.2531547849
	k_{HMO}	0.2258555472	0.2258555274	0.2258552840	0.2531590950	0.2531597602	0.2531587435
	k_{AM}	0.2258547065	0.2258547021	0.2258546485	0.2531555070	0.2531556190	0.2531554486
	k_{GM}	0.225854984	0.225854975	0.225854859	0.253156930	0.253157262	0.253156756

		9	5	0	8	6	3
	k_{MED}	0.2258544693	0.2258544693	0.2258544691	0.2531547848	0.2531547851	0.2531547849
	k_{KS}	0.2258555465	0.2258555268	0.2258552835	0.2531590776	0.2531597402	0.2531587275
	$k_{s_{arith}}$	0.2259313453	0.2259298424	0.2259124972	0.2532428602	0.2532558227	0.2532351671
σ	k	MK	ORR	URR	MK	ORR	URR
0.01	k_{HK}	0.2364450640	0.2364450223	0.2364449988	0.2701560143	0.2701561560	0.2701561207
	k_{HKB}	0.2364459848	0.2364457777	0.2364455834	0.2701569093	0.2701576160	0.2701573230
	k_{LW}	0.2364451403	0.2364450849	0.2364450493	0.2701560491	0.2701562128	0.2701561692
	k_{HSL}	0.2364450638	0.2364450222	0.2364449987	0.2701560142	0.2701561559	0.2701561207
	k_{HMO}	0.2368457398	0.2367723524	0.2366915975	0.3247176748	0.3157401332	0.3066412186
	k_{AM}	0.2365092490	0.2364976415	0.2364846682	0.2704686915	0.2706128745	0.2705248454
	k_{GM}	0.2365365225	0.2365199784	0.2365014972	0.2704386116	0.2705725714	0.2704923181
	k_{MED}	0.2364450640	0.2364450223	0.2364449988	0.2701560143	0.2701561560	0.2701561207
	k_{KS}	0.2367774597	0.2367167584	0.2366497279	0.2706484173	0.2708425099	0.2707101499
	$k_{s_{arith}}$	0.2377939210	0.2375374981	0.2372675030	0.2719004088	0.2721584617	0.2717713041
σ	k	MK	ORR	URR	MK	ORR	URR
0.1	k_{HK}	0.2212581528	0.2212670241	0.2212653301	0.2524813910	0.2524820216	0.2524787901
	k_{HKB}	0.2213079030	0.2213515772	0.2213376621	0.2525971205	0.2525995442	0.2525726988
	k_{LW}	0.2212634507	0.2212760614	0.2212732341	0.2524884597	0.2524892335	0.2524847727
	k_{HSL}	0.2212581288	0.2212669832	0.2212652941	0.2524813830	0.2524820134	0.2524787832
	k_{HMO}	0.4533576688	0.4813563237	0.4332797791	0.4131268252	0.4177798909	0.3779139623
	k_{AM}	0.2282868310	0.2304507207	0.2289776367	0.2797126749	0.2748283564	0.2698290802
	k_{GM}	0.2348895902	0.2375453243	0.2349087487	0.3081457677	0.3001832631	0.2893318908
	k_{MED}	0.2212581455	0.2212670116	0.2212653191	0.2524813698	0.2524820000	0.2524787720
	k_{KS}	0.2228287204	0.2236902585	0.2233062429	0.2544197810	0.2543339462	0.2539270236
	$k_{s_{arith}}$	0.2364118460	0.2391148126	0.2362185770	0.2894103466	0.2830569086	0.2761773644
σ	k	MK	ORR	URR	MK	ORR	URR
1	k_{HK}	0.3695161800	0.3711742855	0.3713050616	0.4205995971	0.4068199659	0.4130477473
	k_{HKB}	0.3674062043	0.3720521898	0.3719128051	0.4091791079	0.3784754609	0.3955169001
	k_{LW}	0.3681551077	0.3704474177	0.3706139051	0.4192833891	0.4036727720	0.4110831285
	k_{HSL}	0.3696880454	0.3712760557	0.3714011989	0.4208798225	0.4074883703	0.4134687337

	k_{HMO}	0.5438561565	0.5742749736	0.5527368939	0.5234252089	0.4989121081	0.4379399343
	k_{AM}	0.4142048191	0.4256150678	0.4195963567	0.4114073963	0.3563501991	0.3750125465
	k_{GM}	0.4342004904	0.4480439147	0.4396814516	0.4107347384	0.3559738629	0.3750464595
	k_{MED}	0.3699542047	0.3714366849	0.3715528753	0.4209679159	0.4076983873	0.4136013261
	k_{KS}	0.3698921827	0.3713989460	0.3715172432	0.4209845226	0.4077379727	0.4136263355
	$k_{s_{arith}}$	0.4628515729	0.4805154540	0.4687796578	0.4160490441	0.3595261738	0.3753260416
σ	k	MK	ORR	URR	MK	ORR	URR
5	k_{HK}	0.9491678307	0.8925103338	0.9325906212	0.9416546060	0.9434855443	0.9421940808
	k_{HKB}	0.8949203768	0.8153961469	0.8881433626	0.8309717379	0.8365970363	0.8346008135
	k_{LW}	0.8635132898	0.7858075751	0.8654506962	0.7500620395	0.7590688777	0.7569523081
	k_{HSL}	0.9015156197	0.8235897573	0.8932165789	0.8142143046	0.8205173602	0.8184532892
	k_{HMO}	0.8359623803	0.8110821753	0.8605745573	0.7034705469	0.7156900544	0.7145494865
	k_{AM}	0.8546170760	0.7833819693	0.8602206784	0.7575774448	0.7662611112	0.7641250414
	k_{GM}	0.8534103620	0.7834915295	0.8596277145	0.8266546003	0.8324527947	0.8304376575
	k_{MED}	0.9610680393	0.9116482043	0.9435473158	0.9595654043	0.9609344124	0.9597806166
	k_{KS}	0.9646825240	0.9175818670	0.9470103435	0.9655216466	0.9667482747	0.9656408583

Table 4: Estimated MSEs when t=60 and φ=0.95 and 0.99

		0.95			0.99		
σ	k	MK	ORR	URR	MK	ORR	URR
0.001	k_{HK}	0.2871144736	0.2871144743	0.2871144742	0.3021684819	0.3021684826	0.3021684824
	k_{HKB}	0.2871144774	0.2871144812	0.2871144799	0.3021684861	0.3021684893	0.3021684879
	k_{LW}	0.2871144737	0.2871144745	0.2871144743	0.3021684819	0.3021684826	0.3021684824
	k_{HSL}	0.2871144736	0.2871144743	0.2871144742	0.3021684819	0.3021684826	0.3021684824
	k_{HMO}	0.2871240646	0.2871313340	0.2871281915	0.3023736660	0.3022718524	0.3022512638
	k_{AM}	0.2871158644	0.2871169485	0.2871164874	0.3021739963	0.3021759497	0.3021744643
	k_{GM}	0.2871216421	0.2871271193	0.2871247624	0.3022153858	0.3022077609	0.3021999441
	k_{MED}	0.2871144736	0.2871144743	0.2871144742	0.3021684819	0.3021684826	0.3021684824
	k_{KS}	0.2871235665	0.2871304697	0.2871274883	0.3022123066	0.3022059556	0.3021984983
	$k_{s_{arith}}$	0.2871814765	0.2872240837	0.2872036501	0.3023361278	0.3022594213	0.3022413114
σ	k	MK	ORR	URR	MK	ORR	URR
0.01	k_{HK}	0.270156014	0.270156156	0.270156120	0.302010931	0.302010742	0.302010727

		3	0	7	1	7	3
	k_{HKB}	0.2701569093	0.2701576160	0.2701573230	0.3020122724	0.3020113260	0.3020111976
	k_{LW}	0.2701560491	0.2701562128	0.2701561692	0.3020109360	0.3020107448	0.3020107291
	k_{HSL}	0.2701560142	0.2701561559	0.2701561207	0.3020109311	0.3020107427	0.3020107273
	k_{HMO}	0.3247176748	0.3157401332	0.3066412186	0.3501133782	0.3251914511	0.3202102191
	k_{AM}	0.2704686915	0.2706128745	0.2705248454	0.3058085967	0.3026863540	0.3025424227
	k_{GM}	0.2704386116	0.2705725714	0.2704923181	0.3099631448	0.3033164346	0.3030378456
	k_{MED}	0.2701560143	0.2701561560	0.2701561207	0.3020109311	0.3020107427	0.3020107273
	k_{KS}	0.2706484173	0.2708425099	0.2707101499	0.3023558133	0.3021158131	0.3020935550
	$k_{s_{arith}}$	0.2719004088	0.2721584617	0.2717713041	0.3131623306	0.3038666750	0.3034705623
σ	k	MK	ORR	URR	MK	ORR	URR
0.1	k_{HK}	0.2921595689	0.2921718911	0.2921713725	0.3059423471	0.3059374851	0.3059410922
	k_{HKB}	0.2921392445	0.2921983037	0.2921940007	0.3058424537	0.3058052462	0.3058329139
	k_{LW}	0.2921591940	0.2921723364	0.2921717650	0.3059418082	0.3059368445	0.3059405488
	k_{HSL}	0.2921595712	0.2921718884	0.2921713701	0.3059423528	0.3059374919	0.3059410980
	k_{HMO}	0.3588228971	0.3498478266	0.3402584582	0.3117807858	0.3045594351	0.3046704820
	k_{AM}	0.2946924884	0.2944744868	0.2940920248	0.3068684870	0.3046287828	0.3048439600
	k_{GM}	0.2958256708	0.2951373878	0.2946430458	0.3118643799	0.3045672783	0.3046749365
	k_{MED}	0.2921595708	0.2921718889	0.2921713706	0.3059423545	0.3059374939	0.3059410997
	k_{KS}	0.2920892797	0.2924210650	0.2923814669	0.3057209908	0.3055451388	0.3056222070
	$k_{s_{arith}}$	0.3046050189	0.3003514417	0.2989826042	0.3196704050	0.3060590221	0.3057141156
σ	k	MK	ORR	URR	MK	ORR	URR
1	k_{HK}	0.3606456818	0.3594314365	0.3602421252	1.2979921322	1.2639330065	1.2864816206
	k_{HKB}	0.3574298688	0.3514995868	0.3530451491	0.8716765268	0.8016930902	0.8683465330
	k_{LW}	0.3603330686	0.3590154249	0.3598791490	0.7154370583	0.6353211053	0.7116220887
	k_{HSL}	0.3606776049	0.3594734838	0.3602788337	0.7246472935	0.6451170644	0.7210620525
	k_{HMO}	0.3974642367	0.3860542491	0.3792673544	0.5187773618	0.4088932866	0.4245449491
	k_{AM}	0.3729813857	0.3612833176	0.3598052437	0.5004625090	0.3986838800	0.4620598973
	k_{GM}	0.3828908340	0.3706480608	0.3670721666	0.6204812953	0.5339295767	0.6113911904
	k_{MED}	0.3608572548	0.3597087009	0.3604842718	1.2991654995	1.2652251199	1.2876467070
	k_{KS}	0.363835153	0.363349665	0.363699309	1.132370530	1.082646695	1.123527922

		8	9	2	6	9	3
	$k_{s_{arith}}$	0.4138123990	0.4046844177	0.3941541362	0.6084884806	0.5210065132	0.5981781717
σ	k	MK	ORR	URR	MK	ORR	URR
5	k_{HK}	1.4341540475	1.6649972022	1.5093427752	5.7725980110	6.2941579993	6.0008245993
	k_{HKB}	1.1088629836	1.4039725507	1.1583213746	2.6007002487	3.7125129750	2.7379427092
	k_{LW}	0.9669951162	0.9368907730	0.9528489121	2.2093230044	0.7574593396	1.4343521790
	k_{HSL}	1.0963252445	1.3921250224	1.1441276014	1.6737468959	0.8333384202	1.0839860841
	k_{HMO}	1.0302732976	0.9101603782	1.0288952949	2.2609680296	0.7542468802	1.4709474877
	k_{AM}	0.9018790443	1.1388601298	0.9086066624	1.1327505882	1.0692218015	0.7795998123
	k_{GM}	1.0031321216	1.2948805062	1.0362334430	0.9555624080	1.6377491550	0.8275480879
	k_{MED}	1.4628926323	1.6854452376	1.5389304311	5.7814310112	6.3008276207	6.0094508591
	k_{KS}	1.0458425206	1.3418343519	1.0862729427	0.9940229254	1.2380463370	0.7324607372

Table 5: Estimated MSEs when t=180 and φ=0.8 and 0.9

		0.8			0.9		
σ	k	MK	ORR	URR	MK	ORR	URR
0.001	k_{HK}	0.2252916041	0.2252916042	0.2252916041	0.2652602285	0.2652602285	0.2652602284
	k_{HKB}	0.2252916074	0.2252916077	0.2252916069	0.2652602312	0.2652602312	0.2652602305
	k_{LW}	0.2252916045	0.2252916045	0.2252916044	0.2652602286	0.2652602286	0.2652602285
	k_{HSL}	0.2252916041	0.2252916042	0.2252916041	0.2652602285	0.2652602285	0.2652602284
	k_{HMO}	0.2253205907	0.2253227642	0.2253157602	0.2652663519	0.2652663541	0.2652650021
	k_{AM}	0.2252927111	0.2252927949	0.2252925272	0.2652608793	0.2652608798	0.2652607360
	k_{GM}	0.2252940287	0.2252942120	0.2252936259	0.2652620874	0.2652620887	0.2652616781
	k_{MED}	0.2252916041	0.2252916042	0.2252916041	0.2652602285	0.2652602285	0.2652602284
	k_{KS}	0.2253176970	0.2253196553	0.2253133501	0.2652661079	0.2652661101	0.2652648120
	$k_{s_{arith}}$	0.2254074060	0.2254158623	0.2253879294	0.2653427655	0.2653423271	0.2653242057
σ	k	MK	ORR	URR	MK	ORR	URR
0.01	k_{HK}	0.2028332748	0.2028332921	0.2028332819	0.2650376472	0.2650376564	0.2650376497
	k_{HKB}	0.2028335996	0.2028336862	0.2028336013	0.2650378703	0.2650379164	0.2650378601
	k_{LW}	0.2028333164	0.2028333426	0.2028333241	0.2650376571	0.2650376680	0.2650376594
	k_{HSL}	0.2028332748	0.2028332921	0.2028332819	0.2650376472	0.2650376564	0.2650376496
	k_{HMO}	0.2085416889	0.2092341919	0.2079023984	0.2665090341	0.2665607093	0.2662432898

	k_{AM}	0.2030978925	0.2031527361	0.2030866782	0.2651495114	0.2651664652	0.2651396604
	k_{GM}	0.2037775030	0.2039595399	0.2037264487	0.2660498965	0.2661170272	0.2658921803
	k_{MED}	0.2028332748	0.2028332921	0.2028332819	0.2650376472	0.2650376564	0.2650376497
	k_{KS}	0.2035260445	0.2036631844	0.2034914801	0.2652818919	0.2653151540	0.2652573979
	$k_{s_{arith}}$	0.2041871214	0.2044372716	0.2041051449	0.2660612170	0.2661282333	0.2659010494
σ	k	MK	ORR	URR	MK	ORR	URR
0.1	k_{HK}	0.2186660960	0.2186675271	0.2186667282	0.2666802833	0.2666801789	0.2666794414
	k_{HKB}	0.2186917243	0.2186988347	0.2186922109	0.2667080171	0.2667074251	0.2667012904
	k_{LW}	0.2186688398	0.2186708809	0.2186695412	0.2666814864	0.2666813632	0.2666804286
	k_{HSL}	0.2186660912	0.2186675213	0.2186667233	0.2666802827	0.2666801783	0.2666794409
	k_{HMO}	0.4326978915	0.4530095571	0.3984506603	0.3052599752	0.2964211976	0.2898393216
	k_{AM}	0.2271595400	0.2278049508	0.2259315871	0.2736807921	0.2722450055	0.2710344763
	k_{GM}	0.2246673090	0.2252885941	0.2239347401	0.3051073897	0.2962908925	0.2897383842
	k_{MED}	0.2186660922	0.2186675224	0.2186667243	0.2666802798	0.2666801754	0.2666794385
	k_{KS}	0.2194840375	0.2196479019	0.2194487013	0.2671537339	0.2671317215	0.2670340626
	$k_{s_{arith}}$	0.2313983861	0.2319432942	0.2292101261	0.2862724240	0.2813245693	0.2781156197
σ	k	MK	ORR	URR	MK	ORR	URR
1	k_{HK}	0.2851749479	0.2854754220	0.2854465517	0.3274146032	0.3268700037	0.3268498235
	k_{HKB}	0.2860973307	0.2873884543	0.2871512067	0.3302308916	0.3276787302	0.3274170579
	k_{LW}	0.2852521405	0.2856631860	0.2856166966	0.3275422235	0.3268975927	0.3268707458
	k_{HSL}	0.2851736845	0.2854722798	0.2854436893	0.3274110548	0.3268692529	0.3268492486
	k_{HMO}	0.3874012887	0.3985819513	0.3838067876	0.7969466694	0.9352106092	0.7159430642
	k_{AM}	0.3153090028	0.3208223941	0.3163832886	0.4115220285	0.3977960318	0.3743223990
	k_{GM}	0.3417762573	0.3487290316	0.3406481007	0.3836041411	0.3674825038	0.3540060253
	k_{MED}	0.2851658410	0.2854527209	0.2854258576	0.3273823539	0.3268632125	0.3268446126
	k_{KS}	0.2853295269	0.2858439832	0.2857793082	0.3273745953	0.3268615897	0.3268433634
	$k_{s_{arith}}$	0.3650609699	0.3738797894	0.3624506341	0.4088809299	0.3947617876	0.3722921375
σ	k	MK	ORR	URR	MK	ORR	URR
5	k_{HK}	0.6678723124	0.6671301646	0.6687497472	0.7523482164	0.7367361453	0.7455010412
	k_{HKB}	0.6631751288	0.6635979395	0.6651958265	0.7462627641	0.7239363037	0.7379097598
	k_{LW}	0.665126827	0.664601856	0.666498878	0.746911572	0.725244281	0.739000456

		3	5	9	5	8	1
	k_{HSL}	0.6685476223	0.6677909486	0.6693213212	0.7482944380	0.7281580635	0.7407672595
	k_{HMO}	0.8502077289	0.9354251746	0.8724145813	0.7961284030	0.8577591016	0.7894420298
	k_{AM}	0.6930426145	0.7070908632	0.6967042960	0.7563899641	0.7495923050	0.7456116964
	k_{GM}	0.6789908803	0.6874434523	0.6819727636	0.7492066336	0.7313977038	0.7391846925
	k_{MED}	0.6729794846	0.6723061396	0.6731975163	0.7562215205	0.7448972956	0.7501845259
	k_{KS}	0.6780210358	0.6776594211	0.6779034201	0.7616066972	0.7561890547	0.7576770617

Table 6: Estimated MSEs when t=180 and φ=0.95 and 0.99

		0.95			0.99		
σ	k	MK	ORR	URR	MK	ORR	URR
0.001	k_{HK}	0.2860277114	0.2860277115	0.2860277114	0.3020999187	0.3020999188	0.3020999188
	k_{HKB}	0.2860277133	0.2860277140	0.2860277135	0.3020999202	0.3020999205	0.3020999202
	k_{LW}	0.2860277114	0.2860277115	0.2860277115	0.3020999187	0.3020999188	0.3020999188
	k_{HSL}	0.2860277114	0.2860277115	0.2860277114	0.3020999187	0.3020999188	0.3020999188
	k_{HMO}	0.2863193622	0.2863493029	0.2862848422	0.3021078412	0.3021082447	0.3021065897
	k_{AM}	0.2860334226	0.2860348396	0.2860334113	0.3021009873	0.3021011498	0.3021009051
	k_{GM}	0.2860360591	0.2860381150	0.2860360303	0.3021074232	0.3021078444	0.3021062690
	k_{MED}	0.2860277114	0.2860277115	0.2860277114	0.3020999187	0.3020999188	0.3020999188
	k_{KS}	0.2860904680	0.2861037277	0.2860884941	0.3021059651	0.3021064188	0.3021051268
	$k_{s_{arith}}$	0.2861226484	0.2861409432	0.2861182509	0.3021391360	0.3021320552	0.3021256653
σ	k	MK	ORR	URR	MK	ORR	URR
0.01	k_{HK}	0.2858222427	0.2858222487	0.2858222430	0.3014137272	0.3014137177	0.3014137119
	k_{HKB}	0.2858224423	0.2858224720	0.2858224245	0.3014139947	0.3014139467	0.3014138982
	k_{LW}	0.2858222469	0.2858222533	0.2858222469	0.3014137283	0.3014137187	0.3014137127
	k_{HSL}	0.2858222427	0.2858222487	0.2858222430	0.3014137272	0.3014137177	0.3014137119
	k_{HMO}	0.4245986361	0.4231057805	0.3928700056	0.3565001764	0.3338081177	0.3270803368
	k_{AM}	0.2890285649	0.2881189518	0.2876488022	0.3056599972	0.3023545621	0.3021626099
	k_{GM}	0.3101960390	0.3004762689	0.2974509932	0.3203118711	0.3057785073	0.3048860611
	k_{MED}	0.2858222427	0.2858222487	0.2858222430	0.3014137272	0.3014137177	0.3014137119
	k_{KS}	0.2860244630	0.2860326724	0.2859896699	0.3014811027	0.3014583439	0.3014492632
	$k_{s_{arith}}$	0.287712134	0.287312192	0.287007383	0.308077101	0.302789430	0.302508610

		0	7	4	6	2	0
σ	k	MK	ORR	URR	MK	ORR	URR
0.1	k_{HK}	0.3491598406	0.3499787335	0.3500333658	0.3041165315	0.3041185574	0.3041189810
	k_{HKB}	0.3450703674	0.3473757765	0.3475958276	0.3040954941	0.3041015139	0.3041048541
	k_{LW}	0.3489943075	0.3498759812	0.3499371202	0.3041164376	0.3041184907	0.3041189238
	k_{HSL}	0.3491682071	0.3499839223	0.3500382298	0.3041165319	0.3041185577	0.3041189813
	k_{HMO}	0.3609452614	0.3612104856	0.3596626308	0.3152711218	0.3053438488	0.3050169750
	k_{AM}	0.3480574868	0.3498216170	0.3495280772	0.3061805640	0.3039910308	0.3039932928
	k_{GM}	0.3533023481	0.3542591815	0.3534491258	0.3108831524	0.3045220050	0.3043843344
	k_{MED}	0.3492405061	0.3500287430	0.3500802603	0.3041165337	0.3041185590	0.3041189824
	k_{KS}	0.3499456027	0.3504642018	0.3504905900	0.3040829623	0.3040874103	0.3040932979
	$k_{s_{arith}}$	0.3805356948	0.3807160475	0.3772480047	0.3207347102	0.3068512216	0.3062008586
σ	k	MK	ORR	URR	MK	ORR	URR
1	k_{HK}	0.3286098747	0.3277831175	0.3277701376	0.5470307296	0.5226958386	0.5391631588
	k_{HKB}	0.3324956746	0.3284821337	0.3282698016	0.4959254761	0.4494652125	0.4856177415
	k_{LW}	0.3286796085	0.3277912130	0.3277764160	0.5308870535	0.4998021108	0.5226087446
	k_{HSL}	0.3286091276	0.3277830320	0.3277700710	0.5320087620	0.5013968307	0.5237610887
	k_{HMO}	0.4314804628	0.4171249056	0.3899284040	0.4484411371	0.3592571777	0.4005969912
	k_{AM}	0.3783656820	0.3564519733	0.3477308308	0.4527274799	0.3795918486	0.4267833258
	k_{GM}	0.4005257554	0.3797639151	0.3640152320	0.4486470369	0.3590530338	0.4000748243
	k_{MED}	0.3285805966	0.3277797845	0.3277675388	0.5481675020	0.5243042762	0.5403307422
	k_{KS}	0.3282889568	0.3277488139	0.3277428884	0.5784348243	0.5670587925	0.5722744059
	$k_{s_{arith}}$	0.4231957270	0.4067855963	0.3827835161	0.4501740382	0.3737511035	0.4206330068
σ	k	MK	ORR	URR	MK	ORR	URR
5	k_{HK}	0.7433631558	0.7441830516	0.7448033811	0.9892314612	0.9618740376	0.9827465878
	k_{HKB}	0.7134192415	0.7146112854	0.7153546664	0.8347753117	0.8011517819	0.8260966315
	k_{LW}	0.7086908546	0.7099856483	0.7107039925	0.6984775470	0.6575216028	0.6735785294
	k_{HSL}	0.7130985486	0.7142964350	0.7150389775	0.7008632033	0.6600248438	0.6768342980
	k_{HMO}	0.7026211019	0.7045999477	0.7050359653	0.6926415740	0.6516644021	0.6650552768
	k_{AM}	0.7045705270	0.7060251907	0.7066790778	0.7404982300	0.7021649913	0.7248024156
	k_{GM}	0.7084687645	0.7097696480	0.7104859425	0.8169519262	0.7825331705	0.8075208745

	k_{MED}	0.7597407358	0.7603942662	0.7608837135	1.0096491918	0.9830946854	1.0031487150
	k_{KS}	0.7952004812	0.7955279243	0.7957248517	0.9677551602	0.9395537435	0.9612415643

References

[1] M. M. Abdelwahab, M. R. Abonazel, A. T. Hammad, and A. M. El-Masry, "Modified Two-Parameter Liu Estimator for Addressing Multicollinearity in the Poisson Regression Model," *Axioms,* vol. 13, no. 1, pp. 1-46, 2024.

[2] N. K. Rashad, and Z. Y. Algamal, "A New Ridge Estimator for the Poisson Regression Model," *Iran. J. Sci. Technol. Trans. A Sci.,* vol. 43, no. 6, pp. 2921-2928, 2019.

[3] I. Dawoud, F. A. Awwad, E. T. Eldin, and M. R. Abonazel, "New Robust Estimators for Handling Multicollinearity and Outliers in the Poisson Model: Methods, Simulation and Applications," *Axioms,* vol. 11, pp. 1-11, 2023.

[4] M. R. Abonazel, A. R. R. Alzahrani, A. A. Saber, I. Dawoud, E. Tageldin, and A. R. Azazy, "Developing Ridge Estimators for the Extended Poisson-Tweedie Regression Model: Method, Simulation, and Application," *Elsevier,* vol. 23, pp. 1-15, 2024.

[5] B. M. Kibria, A. F. Lukman, "A new ridge-type estimator for the linear regression model: Simulations and applications," *Scientifica,* vol. 2020, no. 4, pp. 1-16, 2020.

[6] A. F. Lukman, K. Ayinde, S. Binuomote, and O. A. Clement, "Modified ridge-type estimator to combat multicollinearity: Application to chemical data," *Journal of Chemometrics,* vol. 33, no. 5, pp. 1-12, 2019.

[7] A. Hoerl, R. Kennard, "Ridge regression: Biased estimation for nonorthogonal problems," *Technometrics,* vol. 12, no. 1, pp. 55--67, 1970.

[8] R. Crouse, C. Jin and R. Hanumara, "Unbiased ridge estimation with prior information and ridge trace," *Communications in Statistics-Theory and Methods,* vol. 24, no. 9, pp. 2341-2354, 1995.

[9] P. L. Odell, D. Dorsett, D. Young and J. Igwe, "Estimator models for combining vector estimators," *Mathematical and Computer Modelling*, vol. 12, no. 12, pp. 1627–1642, 1989.

[10] J. Wu, "An unbiased two-parameter estimation with prior information in linear regression model," *The Scientific World Journal*, vol. 2014, pp. 1-8, 2014.

[11] A. Hoerl, R. Kannard and K. Baldwin, "Ridge regression: some simulations," *Communications in Statistics-Theory and Methods*, vol. 4, no. 2, pp. 105–123, 1975.

[12] J. F. Lawless, P. Wang, "A simulation study of ridge and other regression estimators," *Communications in Statistics-theory and Methods*, vol. 5, no. 4, pp. 307–323, 1976.

[13] R. Hocking, "A Biometrics invited paper. The analysis and selection of variables in linear regression," *Biometrics*, vol 32, pp. 1–49, 1976.

[14] M. Nomura, "On the almost unbiased ridge regression estimator," *Communications in Statistics-Simulation and Computation*, vol. 17, no. 3, pp. 729–743, 1988.

[15] B. Kibria, S. Banik, "Some ridge regression estimators and their performances," *Journal of Modern Applied statistical methods*, vol. 15, pp. 206–238, 2016.

[16] G. Khalaf and G. Shukur, “Choosing Ridge Parameter for Regression Problems,” *Communications in Statistics-Theory and Methods,* vol. 34, no. 5. Informa UK Limited, pp. 1177–1182, May 2005.

[17] M. Alkhamisi, G. Khalaf and G. Shukur, "Some modifications for choosing ridge parameters," *Communications in Statistics-Theory and Methods*, vol. 35, no. 11, pp. 2005–2020, 2006.

[18] B. Kibria, "Performance of some new ridge regression estimators," *Communications in Statistics-Simulation and Computation*, vol. 32, no. 2, pp. 419–435, 2003.

[19] S. Arumairajan, P. Wijekoon, "Modified almost unbiased liu estimator in linear regression model," *Communications in Mathematics and Statistics*, vol. 5, pp. 261–276, 2017.

[20] M. I. N. Alheety, "New Versions of Liu-type Estimator in Weighted and non-weighted Mixed Regression Model," *Baghdad Science Journal,* vol. 17, no. 1, pp. 361-370, 2020.

[21] H. Woods, H. Steinour and H. Starke, "Effect of composition of Portland cement on heat evolved during hardening," *Industrial & Engineering Chemistry*, vol. 24, no. 11, pp. 1207–1214, 1932.

[22] A. F. Lukman, K. Ayinde, S. S. Kun and E. T. Adewuyi, "A modified new two-parameter estimator in a linear regression model," *Modelling and Simulation in Engineering*, vol. 2019, pp. 1-10, 2019.

[23] M. Ozkale, "A stochastic restricted ridge regression estimator," *Journal of multivariate analysis*, vol. 100, no. 8, pp. 1706–1716, 2009.

[24] S. Toker, S. Kaçıranlar, "On the performance of two parameter ridge estimator under the mean square error criterion," *Applied Mathematics and Computation*, vol. 219, no. 9, pp. 4718–4728, 2013.

[25] F. Akdeniz, H. Erol, "Mean squared error matrix comparisons of some biased estimators in linear regression," *Communications in Statistics-Theory and Methods*, vol. 32, no. 12, pp. 2389–2413, 2003.