DOI: 10.24996/ijs.2026.67.1.23

ISSN: 0067-2904

Closed-Supplemented Module

Esraa H. Yahyaa, Sahira M. Yaseen
The University of Baghdad, Department of mathematics, College of Science, Iraq

Received: 17/5/2024 Accepted: 6/11/2024 Published: 30/1/2026

Abstract
The purpose of this study is to introduce a new idea for module Q over ring R, let Q be well a unitary left R-module and let R be any ring with one. We refer to Q as a closed-supplemented module, if each submodule of Q has closed-supplement submodule where X is called the closed-supplement submodule of C in Q ,if $Q = X + C$ and $X \cap C \ll_c X$, $C \subseteq Q$ .On the other hand we introduce the concept of closed-weakly supplemented module, whenever that each submodule within Q has closed-weak supplement in Q , such that a submodule X of Q is named closed-weak supplement of C , if $Q = X + C$ and $X \cap C \ll_c Q$.We prove some properties of these class of modules. Many Several of about these ideas are given, additionally the relations involving these notions and other modules related with them are presented.

Keywords: small submodules, closed submodule, closed-small submodule, closed-supplemented module, closed-weakly supplemented module.

مقاس مكمل من النمط المغلق

أسراء هاشم يحيى* , ساهره محمود
قسم الرياضيات, كلية العلوم , جامعة بغداد,بغداد,العراق

الخلاصه:
الغرض من هذه الدراسه هو تقديم فكرة المقاس المكمل Q على الحلقه R , لتكن Q مقاسا وحدويا الى اليسار ولتكن R حلقه مع العنصر المحايد . سنشير الى المقاس Q انه مكمل من النمط المغلق اذا كان كل مقاس جزئي فيه يمتلك مقاسا مكملا جزئيا من النمط المغلق عندما X يدعى مقاسا جزئيا مكملا من النمط المغلق اذا تحقق ان $Q = X + C$ و $X \cap C \ll_c X$, $C \subseteq Q$. من جهه ثانيه قدمنا مفهوم المقاس المكمل الضعيف من النمط المغلق , عندما يكون كل مقاس جزئي من Q يملك مقاس جزئيا مكملا ضعيفا من النمط المغلق في Q , بحيث ان المقاس الجزئي X من Q يسمى مقاس جزئي مكمل من النمط المغلق ل C , اذ كان $Q = X + C$ و $X \cap C \ll_c Q$.اثبتنا عدة مبرهنات لهذا النوع من المقاسات. تم اعطاء العديد من النتائج , اضافتا الى العلاقات التي تنطوي على هذه المفاهيم والمقاسات الاخرى المتعلقه بها.

Introduction

Each ring has its own identity in this article, and all modules will be until left R-module. In Q a submodule X is named as small, (indicated by $X \ll Q$) if for any submodule C of Q

*Email: ideaforevents@gmail.com

such that $X + C = Q$ implies that $C = Q$ [1]. Several authors on this subject generalized small submodule, see [2], [3], [4] , [5].

A proper submodule C of an R-module Q is named essential submodule in Q $(X \leq_e Q)$, if for each nonzero submodule C of X $X \cap C \neq 0$ [6].In R-module Q, a submodule X is referred to a closed, if it has no proper essential extension in Q [1].The Closed-small submodule of Q were developed as an extension of the small submodule, in which a proper submodule X is named closed-small of Q and denoted by $(X \ll_c Q)$,if whenever $X + C = Q$,implies that C is closed submodule of Q [7]. For two submodules X and C of Q ,X is referred to as a supplement of C in Q, when we have the property $Q = X + C$ and $X \cap C \ll X$ [8], [9] .If there is a supplement in a module Q for each of its submodules, then Q is said to be supplemented module [10].Many authors generalized supplemented module see [11] , [12] , [13], [14], [15] .In this paper we will introduce the concepts closed-supplemented and closed-weakly supplemented modules by using the concept closed-small submodules as a broad generalizations of supplemented and weakly supplemented modules , respectively [16], [17], [18].It is evident the fact that every closed-supplemented module is closed-weakly supplemented module, but still the convers is not true. We have proven the main properties of these concepts.

Now we need to introduce the concept of **CSP** as a dual of **CIP** properties, since we use modules which have this type of properties .As if the intersection of any two closed submodules is again closed. The sum of any two closed submodules is again closed see [19].

Lemma 1.1: [7] Let Q be a module in R , if $C \leq X \leq Q$ and $C \ll_c X$ then $C \ll_c Q$.
Proof: Let $i: X \to Q$ inclusion map. Since $\ll_c X$, then by (proposition 2.7, [7]), $f(C) \ll_c Q$ but $f(C) = C$ identity map, then $C \ll_c Q$.
Proposition 1.2: [7] Let $f: Q \to X$ be an isomorphism where Q and X be R-modules such that $C \ll_c X$, then $f^{-1}(C) \ll_c Q$
Proposition 1.3: [7] Let Q be an R-module has the CSP property. If $X_1 \ll_c Q$ and $X_2 \ll_c Q$ then $X_1 + X_2 \ll_c Q$.

2. Closed-Supplemented modules

In this section, we present the idea of closed-supplemented module as a generalization of the supplemented module, and we give some of its properties.

Definition 2.1: Let Q be an R-module and X, C are submodules of Q,then X is called closed- supplement of C in Q, if $Q = X + C$ and $X \cap C \ll_c X$.If every submodule of Q has closed-supplement then Q is called closed-supplemented module.

Remarks and Illustrations 2.2:
1. Since every small submodule is a closed-small submodule, then every supplemented module is a closed-supplemented module.
2. The opposite of (1) is untrue, for example: $\mathbb{Z}_{24}$ as $\mathbb{Z}$-module is closed-supplemented. Since $\mathbb{Z}_{24} = \mathbb{Z}_{24} + (\bar{2})$ and $\mathbb{Z}_{24} \cap (\bar{2}) \ll_c \mathbb{Z}_{24}$, similarly for $X = (\bar{0}), (\bar{3}), (\bar{4}), (\bar{0}), (\bar{6}), (\overline{12})$. Also $(\bar{8}) + (\bar{3}) = \mathbb{Z}_{24}$ and $(\bar{8}) \cap (\bar{3}) = (\bar{0}) \ll_c (\bar{8})$. So, every submodule of $\mathbb{Z}_{24}$ has closed-supplement but it is not supplemented module, since $(\bar{2}), (\bar{3}), (\bar{4})$ has no supplement submodule in $\mathbb{Z}_{24}$, and not every submodule has supplement submodule.
3. Every semi simple module is closed-supplemented for example $\mathbb{Z}_6$ as $\mathbb{Z}$-module, such that $\mathbb{Z}_6 = \{\bar{0}, \bar{3}\} \oplus \{\bar{0}, \bar{2}, \bar{4}\}$ then $\{\bar{0}, \bar{3}\}$ is closed-supplement of $\{\bar{0}, \bar{2}, \bar{4}\}$ since $\mathbb{Z}_6 = \{\bar{0}, \bar{3}\} + \{\bar{0}, \bar{2}, \bar{4}\}$ and $\{\bar{0}, \bar{3}\} \cap \{\bar{0}, \bar{2}, \bar{4}\} = (\bar{0}) \ll_c \{\bar{0}, \bar{3}\}$, also $\{\bar{0}, \bar{2}, \bar{4}\}$ is closed-supplement of $\{\bar{0}, \bar{3}\}$.

4. $(\bar{0})$ is a closed-supplement of Q since $(\bar{0}) + Q = Q$ and $(\bar{0}) \cap Q = (\bar{0}) \ll_c (\bar{0})$, but any $A \lneqq Q$, A is not closed-supplement of Q, since $A + Q = Q$ and $A \cap Q = A$ which is not closed-small in A.

5. Let Q be a uniform module, then Q is supplemented if and only if Q is closed-supplemented module.

Theorem 2.3: If X and C are submodules of an R-module Q, so the following are equivalent:
1. C is closed-supplement of X in Q.
2. $Q = X + C$ and for any non-closed submodule E of C, then $Q \neq X + E$.
Proof: $(\mathbf{1}) \Rightarrow (\mathbf{2})$ Assume C is closed-supplement of X in Q, so we have $Q = X + C$ and $X \cap C \ll_c C$ and suppose $Q = X + E$ where E is non-closed submodule of C, so $C = C \cap Q = C \cap (X + E) = E + (X \cap C)$ by modular law [1] , and since $X \cap C \ll_c C$ so we have $E \leq_c C$, so Contradiction , thus $Q \neq X + E$.
$(\mathbf{2}) \Rightarrow (\mathbf{1})$ from (2) $Q = X + C$, we must show $X \cap C \ll_c C$. Let $U \leq C$ such that $(X \cap C) + U = C$, if U is non-closed submodule of C, then by assumption $Q \neq X + U$, so $Q = X + C = X + (X \cap C) + U = X + U$ and this contradiction, so $U \leq_c C$ and hence $X \cap C \ll_c C$, and we obtain C is closed-supplement of X in Q.

Proposition 2.4: Let Q be an R-module and Q_1, E are submodules of Q, such that Q_1 is closed-supplemented module, if $Q_1 + E$ has closed-supplement in Q then E has closed-supplement in Q.
Proof: Assuming $Q_1 + E$ has closed-supplement in Q, so there exists $U \leq Q$ to the extent that $Q_1 + E + U = Q$ and $(Q_1 + E) \cap U \ll_c U$, since Q_1 is closed-supplemented then $(E + U) \cap Q_1 \leq Q_1$ has closed-supplement in Q_1, so there exists $V \leq Q_1$ such that $\big((E + U) \cap Q_1\big) + V = Q_1$ and $(E + U) \cap V \ll_c V$. Now $Q = Q_1 + E + U = \big((E + U) \cap Q_1\big) + V + E + U = E + (V + U)$. One can easily show $E \cap (V + U) \leq \big((E + V) \cap U\big) + ((E + U) \cap V) \leq \big((E + Q_1) \cap U\big) + \big((E + U) \cap V\big) \ll_c U + V$ by (proposition 2.10 , [7]), so $E \cap (V + U) \ll_c U + V$ and $V + U$ is closed-supplement of E in Q by Definition (2.1), hence E has closed-supplement in Q.

Definition 2.5: [7] An R-module Q is called closed-hollow module, if every proper submodule of Q is closed-small submodule in Q.

Proposition 2.6: Assume Q is an R-module and X is closed-hollow of Q, then X is closed-supplement of each proper submodule C of Q such that $Q = X + C$.
Proof: Assume that C is an appropriate sub-module of Q in the sense that $Q = X + C$. Clear $X \cap C \neq X$, since if $X \cap C = X$, then $X \leq C$ hence $C = Q$ and this contradiction. Since X is closed-hollow, then $X \cap C \ll_c X$ so X is closed-supplement of C in Q.
Proposition 2.7: Let Q be an R-module, then every closed-small submodule of Q has closed-supplement in Q.
Proof: Let X be closed-small submodule of Q, so $Q = X + Q$ and $X \cap Q = X \ll_c Q$ hence Q is closed-supplement of X in Q.
The convers is untrue, for instance $3\mathbb{Z}_{12}$ as $\mathbb{Z}$-module has closed-supplement $(\bar{4})$ in $\mathbb{Z}_{12}$,but not closed-small in $\mathbb{Z}_{12}$, see [7].

Proposition 2.8: Let X and C are two submodules of Q with C is a closed-supplement of. If $Q = E + C$ for some submodule E of X, then C is closed-supplement of E in Q.

Proof: Assume $Q = E + C$ for some submodule E of X and C is closed-supplement of X in Q, so we have $Q = X + C$ and $X \cap C \ll_c C$ and since $E \cap C \leq X \cap C \ll_c C$, then $E \cap C \ll_c C$ by (proposition 2.3, [7]) hence C is closed-supplement of E in Q.

Proposition 2.9: Let Q be an R-module with X , C and U as its submodules, such that $X \leq C$, if X is closed-supplement of U in Q then X is closed-supplement of $U \cap C$ in C.
Proof: Considering that X is closed- supplement of U in Q then we have, $Q = X + U$ and $X \cap U \ll_c X$. Now $C = Q \cap C = (X + U) \cap C = X + (U \cap C)$ by modular law, and since $X \cap (U \cap C) \leq X \cap U \ll_c X$, so we get $X \cap (U \cap C) \ll_c X$ by (proposition 2.3, [7]), hence X is closed-supplement of $U \cap C$ in C.

Proposition 2.10: Let $Q = Q_1 \oplus Q_2$, if X_1 is closed-supplement of X_2 in Q_1 and C_1 is closed-supplement of C_2 in Q_2 , then $X_1 \oplus C_1$ is closed-supplement of $X_2 \oplus C_2$ in Q.
Proof: Since X_1 is closed-supplement of X_2 in Q_1 and C_1 is closed-supplement of C_2 in Q_2, then we have $Q_1 = X_1 + X_2$ and $X_1 \cap X_2 \ll_c X_1$, also $Q_2 = C_1 + C_2$ and $C_1 \cap C_2 \ll_c C_1$, so $Q = Q_1 \oplus Q_2 = (X_1 + X_2) \oplus (C_1 + C_2) = (X_1 \oplus C_1) + (X_2 \oplus C_2)$, since $X_1 \cap X_2 \ll_c X_1$ and $C_1 \cap C_2 \ll_c C_1$ then by(proposition 2.10, [7]) , we have $(X_1 \cap X_2) \oplus (C_1 \cap C_2) \ll_c X_1 \oplus C_1$. Clearly $(X_1 \oplus C_1) \cap (X_2 \oplus C_2) = (X_1 \cap X_2) \oplus (C_1 \cap C_2) \ll_c X_1 \oplus C_1$, hence $X_1 \oplus C_1$ is closed-supplement of $X_2 \oplus C_2$ in Q.

Definition 2.11: [20] An R-module M is called multiplication if for each N is a submodule of M, there exists an ideal I of R such that $N = IM$.

Proposition 2.12: Let Q be a faithful, finitely generated and multiplication module over commutative ring R and X be a submodule, of Q , if $X = JQ$ is closed-supplement of IQ in Q, then J is closed-supplement of I in R, where I , J are ideals of R.
Proof: Let $X = JQ$ is closed supplement of IQ in Q, then we have $Q = X + IQ$ and $X = JQ \cap IQ \ll_c X$. Now $= RQ = IQ + JQ = (I + J)\text{Q}$, as well as being cancellation because Q is faithful, finitely created, and multiplication by [20]. So $R = I + J$ also we have $IQ \cap X = IQ \cap JQ = (I \cap J)Q \ll_c X = JQ$, hence $(I \cap J)Q \ll_c JQ$. To show $I \cap J \ll_c J$, let E be an ideal of R such that $(I \cap J) + E = J$, so $(I \cap J)\, Q + EQ = JQ$ and since $(I \cap J)Q \ll_c JQ$, then $EQ \leq_c JQ$.So $E \leq_c J$ by (lemma 2.13, [7]) and we get the result, and hence J is closed-supplement of I in R.

3. Closed-Weakly Supplemented.

In this work we present the idea of closed-weakly supplemented module, and we discuss some of the basic properties of these modules and other related concepts.
Definition 3.1: Let Q be an R-module and X, C are submodules of Q, then $X$is called closed-weak supplement of C in Q, if $Q = X + C$ and $X \cap C \ll_c Q$, if every submodule of Q has closed-weak supplement, then Q is called closed-weakly supplemented module.

Notes and Illustrations 3.2:
1. Every weakly supplemented module is closed-weakly supplemented module while the converse is not holed. The example below shows that the converse is untrue: $\mathbb{Z}_{24}$ as $\mathbb{Z}$-module is closed-weakly supplemented, since all of them submodules has closed weak supplement, $(\bar{0}), (\bar{6}), (\overline{12})$ is closed-wake supplement submodules in Z_{24} and for a submodule $(\bar{2})$ such that $(\bar{2}) + \mathbb{Z}_{24} = \mathbb{Z}_{24}$ and $(\bar{2}) \cap \mathbb{Z}_{24} = (\bar{2}) \ll_c \mathbb{Z}_{24}$ [7], and for $(\bar{3})$ such that $(\bar{3}) + (\bar{8}) = \mathbb{Z}_{24}$ and $(\bar{3}) \cap (\bar{8}) = (\bar{0}) \ll \mathbb{Z}_{24}$ also for $(\bar{4}), (\bar{8})$.But not weak supplement module since for $(\bar{3})$ such that $(\bar{3}) + \mathbb{Z}_{24} = \mathbb{Z}_{24}$and $(\bar{3}) \cap \mathbb{Z}_{24} = (\bar{3})$ is not small in $\mathbb{Z}_{24}$.Also for $(\bar{2}), (\bar{4}), (\bar{8})$. As a result, not every $\mathbb{Z}_{24}$ submodules have a weak supplement sub-module.

2. $\mathbb{Z}_4$ as $\mathbb{Z}$- module is closed-weakly supplemented, since $\mathbb{Z}_4$ is supplemented module so it is weakly supplemented. Thus, it is a closed-weakly supplemented module.
3. Every supplemented module is weakly supplemented by [8] , hence closed-weakly supplemented by (1).

Proposition 3.3: Every closed-supplemented is closed-weakly supplemented.
Proof: Let Q be closed-supplemented and X, C are submodules of Q, so X is closed-supplement of C in Q, hence $Q = X + C$ and $X \cap C \ll_c C$, then by lemma (1.1), $X \cap C \ll_c Q$.

The following shows under a certain condition closed-weak supplement be closed supplement
Proposition 3.4: Let Q an R-module has CIP and C be a closed-weakly supplement submodule of Q, if C is direct summand of Q then C is closed-supplement submodule in Q.
Proof: Let C is closed-weak supplement, so there exists $E \leq Q$ such that $C + E = Q$ and $C \cap E \ll_c Q$, since $C \cap E \leq C \leq Q$ and C is direct summand of Q then by (proposition 2.8, [7]) we get $C \cap E \ll_c C$, hence $C$is closed-supplement of E in Q.

Proposition 3.5: Let Q be an R-module and Q_1, X are submodules of Q such that Q_1 is closed-weakly supplemented module, if $Q_1 + X$ has closed-weak supplement in Q, then X has closed-weak supplement in Q.
Proof: Let $Q_1 + X$ has closed-weak supplement submodule in Q, so there exists $C \leq Q$ such that $C + (Q_1 + X) = Q$ and $C \cap (Q_1 + X) \ll_c Q$, where $Q_1 + X \leq Q$ and since Q is closed-weakly supplemented, so there exists $E \leq Q_1$ such that $(C + X) \cap Q_1 + E = Q_1$ and $(C + X) \cap E \ll_c Q_1$, where $(C + X) \cap Q_1 \leq Q_1$ so $Q = C + (Q_1 + X) = C + \big((C + X) \cap Q_1 + E + X\big) = C + X + E$. Now to show that $(C + X) \cap E \ll_c Q$, since $(C + X) \cap E \ll_c Q_1$, then by lemma (1.1) ,we get $(C + X) \cap E \ll_c Q$, so E is closed-weak supplement of $C + X$ in Q. We will show $C + E$ is closed-weak supplement of X in Q, clear $(C + E) + X = Q$, we must show that $(C + E) \cap X \ll_c Q$, since $(C + E) \cap X \leq C \cap (Q_1 + X) + (C + X) \cap E \ll_c Q$ by (proposition 2.3, [7]) , so $(C + E) \cap X \ll_c Q$ and hence, $C + E$ is closed-weak supplement of X in Q.

Proposition 3.6: Let $Q = Q_1 + Q_2$ and Q_1, Q_2 are submodules in Q, if Q_1 and Q_2 are closed-weakly supplemented module, then Q is closed-weakly supplemented module.
Proof: Let $C \leq Q$, since $Q_1 + Q_2 + C = Q$, it is trivially has closed-weak supplement in Q, by proposition (3.5), we get $Q_2 + C$ has closed-weak supplement in Q and again by proposition (3.5), C has closed-weak supplement in Q, hence Q is closed-weakly supplemented module.

Proposition 3.7: Let Q be closed-weakly supplemented module, then for every X and C are submodules of Q with $Q = X + C$, there exists a closed-weak supplement E of X in Q, where $E \leq C$.
Proof: Let Q is closed-weakly supplemented, and $X \cap C \leq Q$ such that $X \cap C$ has closed-weak supplement W in Q where $W \leq Q$, hence $Q = (X \cap C) + W$ and $(X \cap C) \cap W \ll_c Q$, so $Q = X + C = (X \cap C) + W = X + (C \cap W)$ by modular law. Let $E = C \cap W$ so, $Q = X + E$ and $X \cap E = X \cap (C \cap W) \ll_c Q$ and hence E is closed-weak supplement of X in Q.

Proposition 3.8: Given an R-module Q with CSP and X and C as its submodules, such that C is closed-weak supplement of X in Q, if $E \leq C$ and $E \ll_c Q$, then C is closed-weak supplement of $X + E$ in Q.

Proof: Let C is closed-weak supplement of X in Q, so $Q = X + C$ and $X \cap C \ll_c Q$, now $Q = X + C = X + C + E = C + X + E$, we must show $C \cap (X + E) \ll_c Q$. Since $X \cap C \ll_c Q$ and $E \ll_c Q$ then $(X \cap C) + E \ll_c Q$ by proposition (3.5), so by modular law $\cap (X + E) = (X \cap C) + E \ll_c Q$, and hence C is closed-weak supplement of $X + E$ in Q.

Proposition 3.9: Let X and C be submodules of Q, an R-module, with X being a closed-supplement of C in Q, if $Q = E + X$ for some submodule E of C, then X is closed -weak supplement of E in Q.
Proof: Let X is closed-weak supplement of C in Q, so $Q = C + X$ and $C \cap X \ll_c Q$, now to show that $E \cap X \ll_c Q$, since $E \cap X \leq C \cap X \ll_c Q$ then $E \cap X \ll_c Q$ by (proposition 2.3, [7]) , and hence X is closed-weak supplement of E in Q.

Proposition 3.10: Let $f: Q \to X$ be an isomorphism where Q and X be R-modules, if X is closed-weakly supplemented then Q is closed-weakly supplemented.
Proof: Let $E \leq Q$ then, $f(E) \leq X$. Since X is closed-weakly supplemented there exists C is closed-weak supplement of $f(E)$ in X , so $X = C + f(E)$ and $C \cap f(E) \ll_c X$. Now $f^{-1}\big(C + f(E)\big) = f^{-1}(X)$ hence $f^{-1}(C) + E = Q$ and since $C \cap f(E) \ll_c X$ then $f^{-1}\big(C \cap f(E)\big) \ll_c f^{-1}(X)$ by proposition (1.2) , hence $f^{-1}(C) \cap E \ll_c f^{-1}(X) = Q$.So ,$f^{-1}(C)$ is closed-weak supplement of E in Q and therefor, Q is closed-weakly supplemented module.

Conclusion:
Recently we developed the closed-supplement module as a covered in this work along how it relates to another concept. Finely, the notion of a closed weakly supplemented module is presented.

Acknowledgment:

The referees provided insightful feedback and recommendations for enhancing the manuscrip t,
which the authors much appreciated.

Conflict of Interest:
The authors declare that they have no conflicts of interest.

References

[1] F. Kasch, *Modules and Rings: a translation of Module und Ringe, london: Academic Press INC,* 1982.

[2] B. Talaee, "A Generalization of M-Small Modules," *Journal of Sciences, Islamic Republic of Iran,* vol. 26, no. 2, pp. 179-185, 2015.

[3] A.Kabban and W.Khalid, "On Jacobson - Small Submodules," *Iraqi Journal of Science,* vol. 60, no. 7, pp. 1584–1591, 2019.

[4] H. Baanoon and W.Khalid, "E∗-Essential Small Submodules and E∗-Hollow Modules," *European Journal of Pure and Applied Mathematics,* vol. 15, no. 2, pp. 478–485, 2022.

[5] R. Beyranva and F. Moradsmal, "SUBMODULES WITH RESPECT TO AN ARBITRARY SUBMODULE," *Journal of Algebra and Related Topics,* vol. 3, no. 2, pp. 43-51, 2015.

[6] K. R. Goodearl, *An Introduction to Noncommutative Noetherian Rings, Cambridge university Press: Marcel Dekkl,* 2004.

[7] E.Esraa and M.Sahira, "Closed-Small Submodule and Closed-Hollow Module," *Wasit Journal of Pure Science,* vol. 2, no. 4, pp. 1-4, 2023.

[8] C.Clark and N. Wisbauer, *Lifting Modules Supplements and Projectivity in module theory,* *Birkhauser Verlag, Basel: Series Frontiers in mathematics,* 2006.

[9] R. Wisbauer, Foundation of Module and rings theory, Philadelphia: Gorden and Breach, 1994.

[10] A. Ali and W. Khalid, "⊕-J-Supplemented Modules," *J Phys Conf Ser,* vol. 1530, no. 012056, 2020.

[11] Y.Wang, "A Generalization of Supplemented Modules," *Journal Algebra Discrete Mathematics,* vol. 11, no. 1, pp. 59-74, 2011.

[12] Y. Talebi and B. Talaee, "On Generalized δ-Supplemented Modules," *Vietnam J Math,* vol. 37, no. 4, pp. 515-525, 2009..

[13] A. A. Hussein, "⊕ Jacobson Supplemented Modules," *Iraqi Journal of Science,* vol. 62, no. 5, pp. 1627-1634, 2020.

[14] E. Mermut and E. Zdemir, "⊕-Rad-Supplemented Modules," *Univ. Padova,* vol. 124, 2010.

[15] R. S.Wadbude and M.R.Aloney , "Characterization of δ-small submodule," *Mathematical Theory and Modeling,* vol. 5, no. 7, pp. 2224-5804, 2015.

[16] B.KOŞAR, "Some Properties of Amply Weak Essential Supplemented Modules," *Erzincan Üniversitesi Fen Bilimleri Enstitüsü Dergisi,* vol. 15, no. 1, pp. 387–393, 2022.

[17] N. Burcu Turkmen, "Nisuncit Weakly Locally Artinan Supplemented Mdules," *Journal of Universal Mathematics,* vol. 4, no. 1, pp. 102–108, 2021.

[18] A. Musa and S. Al-Saad, "Strongly Weakly Supplement Extending Modules," *Al- Mustansiriyah Journal of Science,* vol. 30, no. 4, pp. 682, 2020.

[19] S. Neamah and H. Mohammed, "Closed Multiplication Modules," *Italian Journal of Pure and Applied Mathematics,* vol. 46, no. 1, pp. 82–90, 2021.

[20] Z.A.E.lbast and P.F.Smith, "Multiplication Modules," *Communication in Algebra,* vol. 29, no. 6, pp. 2597-2609, 2001.