تم تدقيق البحث من قبل
الاستاذ:
بتاريخ:
و قد تم تصحيح كافة الاخطاء و كان البحث وفق متطلبات النشر.
توقيع الاستاذ:

CERTAIN TYPES OF SEPARATION AXIOMS IN TRI-TOPOLOGICAL SPACES

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Abstract
In this paper, we introduce and study new types of separation axioms in tri-topological spaces, where we defined it using new types of closed set but not necessarily closed sets. Several properties of these concepts are proved.

بَعْض أنواعِ بديهياتِ الفصل في الفضاءات التبولوجية الثلاثية

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الخلاصة
في هذه الورقةِ، نُقدّمُ ونَدْرسُ أنواعَ جديدةَ مِنْ بديهياتِ الفصل في الفضاءات التبولوجية الثلاثية، حيث عرّفنَاه بإستعمال أنواعَ جديدةَ مِنْ المجموعةِ المُغلقةِ لكن لَيستْ مجموعاتَ مُغلقةَ بالضرورة. كما سنثبت مجموعة من القضايا المقترحة على تلك المفاهيم.

1. Introduction

In 2007, M. Ganster and M. Steiner [1] introduce the concept of $b\tau$-closed set, which the complement of it is called $b\tau$-open set where he defined a subset A of a topological space X to be $b\tau$-closed if $cl_b(A)\subset U$ whenever $A\subset U$ and U is open, where $cl_b(A)$ denoted to the intersection of all b-closed sets containing a subset A.

It is now the propose of this paper to introduce and investigate the corresponding concept, i.e. $123b\tau$-closeness in tri-topological spaces, where the study of tri-topological spaces was first initiated by Martin M. Kovar [2] in 2000, where anon empty set X with three topologies is called a tri-topological space, also we use the main concept of this paper to introduce and investigation some new types of separation axioms in tri-topological, we call these axioms as $123b\tau - T_k$ Spaces, k=0, 1, 2.

We recall some definitions and concepts which are useful in the following section. The symbol □ will indicate the end or omission of a proof.

2. Preliminaries

Definition (2.1)

A subset A of a topological space (X,τ) is called b open set if $A\subset cl(int\ A)\cup int(clA)$, where clA (int A) is the closure of A (the interior of A) [3], [4].

The complement of b open set is called b closed set. Thus $A\subset X$ is b closed if and only if $cl(int\ A)\cap int(clA)\subset A$. [3], [4].

Definition (2.2)

A map $f:(X,\tau)\to(Y,\sigma)$ is called irresolute if $f^{-1}(A)$ is semi-open set in X for each semi-open set A of Y [5]. where a subset A

of a topological space (X,τ) is said to be semi-open if A $\subset$ cl(int A) [6].
In what follows, by a space X and Y we mean tri-topological spaces (X,τ_1,τ_2,τ_3) and (Y, σ_1, σ_2, σ_3) respectively, where X and Y are nonempty set, τ_1, τ_2 and τ_3 are topologies on X and σ_1, σ_2, and σ_3 are topologies on Y.

Definition (2.3)
A subset A of a space X is called 123 open set if $A\in\tau_1\bigcup\tau_2\bigcup\tau_3$, and the complement of 123 open set is 123 closed set.

Example (2.4)
Let X={a, b, c}, $\tau_1=\{X,\phi,\{a\}\}$, $\tau_2=\{X,\phi,\{b\}\}$, $\tau_3=\{X,\phi,\{c\}\}$, the sets $X,\phi,\{a\},\{b\},\{c\}$ are all 123-open sets and $X,\phi,\{b,c\},\{a,c\},\{a,b\}$ are all 123-closed sets in (X,τ_1,τ_2,τ_3).

Definition (2.5)
A subset A of a tri-topological space (X,τ_1,τ_2,τ_3) is called 123 neighborhood of a point x$\in$X if and only if there exists an 123 open set U such that x$\in$U$\subset$A.

Example (2.6)
Let X={a, b, c}, $\tau_1=\{X,\phi,\{a,b\}\}$, $\tau_2=\{X,\phi,\{a\}\}$, $\tau_3=\{X,\phi,\{c\},\{a,c\}\}$, since the set {a} is 123 open and a$\in${a}$\subseteq${a, b}, then {a, b} is 123 neighborhood of a.

Remark (2.7)
We will denoted to the 123 interior (resp. 123 closure) of any subset, say A of X by 123 int A (resp. 123clA), where 123int A is the union of all 123 open sets contained in A, and 123clA is the intersection of all 123 closed sets containing A.

Definition (2.8)
A subset A of a space X is said to be 123b open set if $A\subset 123cl(123\text{int}\,A)\bigcup 123\text{int}(123clA)$.

Remarks (2.9)
① The complement of 123b open set is called 123b closed set. Thus A$\subset$X is 123b closed if and only if $123cl(123\text{int}\,A)\bigcap 123\text{int}(123clA)\subset A$.
② The intersection of all 123b closed sets of X containing a subset A of X is called 123b closure of A and is denoted by $123cl_b$(A). Analogously the 123b interior of A is the union of all 123b open sets contained in A denoted by 123int_b(A)$123\text{int}_b(A)$.

Definition (2.10)
A subset A of a tri-topological space X is called to be $123b\tau$-closed if $123cl_b$(A)$\subset$U whenever A$\subset$U and U is 123 open.

Remark (2.11)
① The complement of $123b\tau$-closed is $123b\tau$- open.
② The intersection of all $123b\tau$-closed sets of X containing a subset A of X is called $123b\tau$-closure of A and is denoted by $123cl_{b\tau}(A)$ Analogously the $123b\tau$-interior of A is the union of all $123b\tau$-open sets contained in A denoted by $123\text{int}_{b\tau}(A)$.

Remark (2.12)
The relationships between the concepts 123 closed set, 123b closed set and $123b\tau$-closed summarized in the following diagram:

123-closed set $\rightarrow$ $123b$-closed set $\rightarrow$ $123b\tau$-closed set

Diagram (2-1)

Now, we will prove every pointed in the above diagram in the following propositions:

Proposition (2.13)
Every 123 closed subset of a tri-topological space X is 123b closed.

Proof:
Let A$\subseteq$X be 123 closed set, since A°$\subset$123clA°, hence 123int A°$\subset$123int(123clA°), but 123int A$\subset$A for any subset A, hence A°$\subset$123int (123clA°), and A°$\subset$123int (123clA°)Y123cl(123int A°), hence A° is 123b open set, hence A is 123b open set .□

Proposition (2.14)
Every 123b closed subset of a tri-topological space X is $123b\tau$-closed.

Proof:
Let A be a $123b$-closed subset of X, and let $A\subseteq U$, where U is 123-open, since A is $123b$-closed set, hence $123\text{int}(123cl(A))\bigcap 123cl(123\text{int}(A))\subset A$, but $A\subset U$, hence

$123\text{int}(123cl(A)) \cap 123cl(123\text{int}(A)) \subset U$, since $123cl_b(A)$ is the smallest $123b$-closed set containing A, so,

$$123cl_b(A) = A \cup (123\text{int}(123cl(A)) \cap 123cl(123\text{int}(A)))$$
$$\subset A \cup U$$
$$\subset U,$$

i.e. A is $123b\tau$-closed.□

Now, we will give some examples to show that the inverse pointed in the diagram (2.1) is not true

Example (2.15)

$123b$-closed set $\nrightarrow$ 123-closed set.

Let $X = \{a,b,c\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a,b\}\}$, $\tau_2 = \{X, \phi, \{b\}\}$, $\tau_3 = \{X, \phi, \{a\}\}$, then the sets in $\{X, \phi, \{b,c\}, \{a,c\}, \{c\}\}$ are 123-closed sets and the family $\{X, \phi, \{b,c\}, \{a,c\}, \{c\}, \{a\}, \{b\}\}$ is the $123b$-closed sets, so $\{a\}, \{b\}$ are $123b$-closed but not 123-closed sets.

Example (2.16)

$123b\tau$-closed set $\nrightarrow$ $123b$-closed set.

In example (2.15), the set $\{a,b\}$ is $123b\tau$-closed but it is not $123b$-closed set.

3. $123b\tau - T_k$ **Spaces,** $k = 0,1,2$

In this section we will introduce new types of separation axioms which we called $123b\tau - T_k$ spaces for $k = 0,1,2$, for the sake of convenience, we begin with definition the concepts $123 - T_k$ spaces for $k = 0,1,2$.

Definition (3.1)

A tri-topological space X is called:

① $123 - T_0$ if and only if to each pair of distinct points x, y in X, there exists an 123-open set containing one of the points but not the other.

② $123 - T_1$ if and only if to each pair of distinct points x, y of X, there exist a pair of 123-open sets one containing x but not y and the other containing y but not x.

③ $123 - T_2$ if and only if to each pair of distinct points x, y of X, there exist a pair of disjoint 123-open sets one containing x and the other containing y.

④ 123-regular if and only if to each 123-closed set F and each point $x \notin F$, there exist disjoint 123-open sets U and V such that $x \in U$ and $F \subset V$.

Definition (3.2)

A tri-topological spaces X is said to be $123b\tau - T_0$ space if and only if to each pair of distinct points x, y in X, there exists a $123b\tau$-open set containing one of the points but not the other.

Now we proceed to prove that every tri-topological space is $123b\tau - T_0$ space.

Proposition (3.3)

If $\{x\}$ is $123b\tau$-open for some $x \in X$, then $x \notin 123cl_{b\tau}(\{y\})$, for all $y \neq x$.

Proof:

Let $\{x\}$ be $123b\tau$-open for some $x \in X$, then $X - \{x\}$ is $123b\tau$-closed, and $x \notin X - \{x\}$. If $x \in 123cl_{b\tau}(\{y\})$ for some $y \neq x$, then y, x both are in all the $123b\tau$-closed sets containing y, so $x \in X - \{x\}$ which is contraction, hence $x \notin 123cl_{b\tau}(\{y\})$.□

Proposition (3.4)

In any tri-topological space X, any distinct points have distinct $123b\tau$-closures.

Proof:

Let $x, y \in X$ with $x \neq y$, and let $A = \{x\}^c$, hence $123cl(A) = A$ or X. Now, if $123cl(A) = A$, then A is 123-closed, hence it is $123b\tau$-closed, so $X - A = \{x\}$ is $123b\tau$-open and not containing y. So by proposition (3.3), $x \notin 123cl_{b\tau}(\{y\})$ and $y \in 123cl_{b\tau}(\{y\})$, which implies that $123cl_{b\tau}(\{y\})$ and $123cl_{b\tau}(\{x\})$ are distinct. If $123cl(A) = X$, then A is $123b\tau$-open, hence $\{x\}$ is $123b\tau$-closed, which mean that $123cl_{b\tau}(\{x\}) = \{x\}$ which is not equal to $123cl_{b\tau}(\{y\})$.□

Proposition (3.5)

In any tri-topological space X, if distinct points have distinct $123b\tau$-closures then X is $123b\tau - T_0$ space.

Proof:

Let $x, y \in X$ with $x \neq y$, with $123cl_{b\tau}(\{y\})$ is not equal to $123cl_{b\tau}(\{x\})$, hence there exists $z \in X$ such that $z \in 123cl_{b\tau}(\{x\})$ but $z \notin 123cl_{b\tau}(\{y\})$ or $z \in 123cl_{b\tau}(\{y\})$ but $z \notin 123cl_{b\tau}(\{x\})$. Now, without loss of generality, let $z \in 123cl_{b\tau}(\{x\})$ but $z \notin 123cl_{b\tau}(\{y\})$. If $x \in 123cl_{b\tau}(\{y\})$, then $123cl_{b\tau}(\{x\})$ is contained in $123cl_{b\tau}(\{y\})$, hence $z \in 123cl_{b\tau}(\{y\})$, which is a contradiction, this mean that $x \notin 123cl_{b\tau}(\{y\})$ hence $x \in 123cl_{b\tau}(\{y\}^c)$, hence X is $123b\tau - T_0$ space. □

Proposition (3.6)

Every tri-topological space is $123b\tau - T_0$ space.

Proof:

Follows from propositions (3.5) and (3.4).

Definition (3.7)

A tri-topological space X is said to be $123b\tau - T_1$ space if and only if to each pair of distinct points x, y in X with $x \neq y$, there exist two $123b\tau$-open sets U, V such that $x \in U, y \notin U$ and $y \in V, x \notin V$.

Proposition (3.8)

Every $123b\tau - T_1$ space is $123b\tau - T_0$ space.

Proof:

Follows from the definition of $123b\tau - T_1$ space.

Now, we will give an example to show that the converse of proposition (3.8) is not true.

Example (3.9)

Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$, $\tau_2 = \{X, \phi, \{a\}\}$, $\tau_3 = \{X, \phi, \{a, c\}\}$, all $123b\tau$-open sets are $X, \phi, \{a\}, \{a, b\}$ and $\{a, c\}$, so $(X, \tau_1, \tau_2, \tau_3)$ is $123b\tau - T_0$ space but it is not $123b\tau - T_1$ space.

Proposition (3.10)

In a tri-topological space X, the following statements are equivalent:

① X is $123b\tau - T_1$ space.

② For each $x \in X$, $\{x\}$ is $123b\tau$-closed in X.

③ Each subset of X is the intersection of all $123b\tau$-open sets containing it.

④ The intersection of all $123b\tau$-open sets containing the point $x \in X$ is $\{x\}$.

Proof:

①⇒② X is $123b\tau - T_1$ space and let $x, y \in X$ and $x \neq y$, then there exists an $123b\tau$-open set, say U_y such that $y \in U_y$. Hence $y \in U_y \subset \{x\}^c$, so $\{x\}^c = \bigcup\{U_y : y \in \{x\}^c\}$ which is $123b\tau$-open, so $\{x\}$ is $123b\tau$-closed in X.

②⇒③ Let $A \subset X$ and $y \notin A$. Hence $A \subset \{y\}^c$ and $\{y\}^c$ is $123b\tau$-open in X and $A = \bigcap\{\{y\}^c : y \in A^c\}$ which is the intersection of all $123b\tau$-open sets containing A.

③⇒④ Obvious.

④⇒① Let $x, y \in X$ and $x \neq y$. By assumption, there exist at least an $123b\tau$-open set containing x but not y also an $123b\tau$-open set containing y but not x. i.e. X is $123b\tau - T_1$ space. □

Definition (3.11)

A tri-topological space X is said to be $123b\tau - T_2$ if and only if for $x, y \in X$, $x \neq y$, there exist two disjoint $123b\tau$-open sets U, V in X such that $x \in U$ and $y \in V$.

Proposition (3.12)

Every $123b\tau - T_2$ space is $123b\tau - T_1$ space.

Proof:

Let X is $123b\tau - T_2$ space and let x, y in X with $x \neq y$, so by hypothesis there exist two disjoint $123b\tau$-open, say U, V such that $x \in U$ and $y \in V$, but $U \cap V = \phi$, hence $x \notin V$ and $y \notin U$, i.e. X is $123b\tau - T_1$ space.

Definition (3.13)

A subset A of a tri-topological space $(X, \tau_1, \tau_2, \tau_3)$ is called $123b\tau$-neighborhood of a point $x \in X$ if and only if there exists an $123b\tau$-open set U such that $x \in U \subset A$.

Proposition (3.14)

In a tri-topological space X, the following statements are equivalent:

① X is $123b\tau - T_2$ space.

② If $x \in X$, then for each $y \neq x$, there is an $123b\tau$-neighborhood $N(x)$ of x such that $y \notin 123cl_{b\tau}(N(x))$.

③ For each $x \in \{123cl_{b\tau}(N)\} = \{x\}$, where N is an $123b\tau$-neighborhood of x.

Proof:

①$\Rightarrow$② Let $x \in X$, if $y \in X$ with $x \neq y$, then there exist disjoint $123b\tau$-open sets U, V in X such that $x \in U$ and $y \in V$. Then $x \in U \subset X - V$, hence $X - V$ is an $123b\tau$-neighborhood of x, but $X - V$ is an $123b\tau$-closed and $y \notin X - V$. Now let $N(x) = X - V$, i.e. $y \notin 123cl_{b\tau}(N(x))$.

②$\Rightarrow$③ Obvious.

③$\Rightarrow$① Let $x, y \in X$ and $x \neq y$. By assumption, there exist at least an $123b\tau$-neighborhood N of x such that $y \notin 123cl_{b\tau}(N)$, so $x \notin X - 123cl_{b\tau}(N)$ is $123b\tau$-open, but N $123b\tau$-neighborhood of x, hence there exists an $123b\tau$-open set U such that $x \in U \subset N$ and $U \cap X - 123cl_{b\tau}(N) = \phi$. i.e. X is $123b\tau - T_2$ space. □

Definition (3.15)

A tri-topological spaces X is said to be $123b\tau$-regular space if and only if for each $123b\tau$-closed set F and each point $x \notin F$, there exist disjoint $123b\tau$-open sets U and V such that $x \in U$ and $F \subset V$.

Proposition (3.16)

A $123b\tau - T_0$ space is $123b\tau - T_2$ space if it is $123b\tau$-regular space.

Proof:

Let X be $123b\tau - T_0$ space and $123b\tau$-regular space. And let $x, y \in X$ and $x \neq y$, hence there exists an $123b\tau$-open, say U such that U contains one of x and y, say x but not y, so $X - U$ is an $123b\tau$-closed and $x \notin X - U$, but X is $123b\tau$-regular space, hence there exist disjoint $123b\tau$-open sets V_1 and V_2 such that $x \in V_1$ and $X - U \subset V_2$, hence $x \in V_1$ and $y \in V_2$, i.e. X is $123b\tau - T_2$. □

Definition(3.17)

A map $f : (X, \tau_1, \tau_2, \tau_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ is called $123b\tau$-irresolute if the inverse image of every $123b\tau$-open set in Y is $123b\tau$-open in X.

Proposition (3.18)

If $f : (X, \tau_1, \tau_2, \tau_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ is an injective and $123b\tau$-irresolute map and Y is $123b\tau - T_2$ space then X is $123b\tau - T_2$ space.

Proof:

Let $x, y \in X$ and $x \neq y$, since f is injective, then $f(x) \neq f(y)$, and since Y is $123b\tau - T_2$, then there exist disjoint $123b\tau$-open sets U and $_V$ such that $f(x) \in U$ and $f(y) \in V$. Now let $G = f^{-1}(U)$ and $H = f^{-1}(V)$, hence $x \in G$, $y \in H$ and G, H are $123b\tau$-open sets, with $G \cap H = f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \phi$, i.e. X is $123b\tau - T_2$ space. □

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