Iraqi Journal of Science, 2026, Vol. 67, No. 1, pp: 276-285
DOI: 10.24996/ijs.2026.67.1.22

ISSN: 0067-2904

Ultra-Formulas for Conjugate Gradient Impulse Noise Reduction from Images

Basim A. Hassan *, Mohammed W. Taha
Department of Mathematics, College of Computer Sciences and Mathematics, University of Mosul, Nineveh, Iraq

Received: 11/5/2024 Accepted: 17/10/2024 Published: 30/1/2026

Abstract
In this research, a new coefficient of conjugate gradient technique is identified which is used for solving issues related to image restoration. This coefficient is derived using Perry's conjugacy condition and the quadratic model. The algorithms have been shown to exhibit global convergence and possess the descent property. Through numerical testing, the new method demonstrated a significant improvement. It has been shown that the innovative conjugate gradient method works better than the conventional FR conjugate gradient technique.

Keywords: Adjusting parameters gradient, Theoretical analysis, Image restoration problems, Optimization, Gradient methods.

صيغ فائقة للتدرج المترافق لتقليل التشويش من الصور

باسم عباس حسن*, محمد وعد الله طه
قسم الرياضيات، كلية علوم الحاسوب والرياضيات، جامعة الموصل، محافظة نينوى، العراق

الخلاصة
تستمد هذه الدراسة معاملات مترافقة جديدة لطرائق التدرج المترافق، والذي يستخدم لتقليل التشويش من الصور، باستخدام شرط الترافق لبيري والنموذج التربيعي. تُظهر الخوارزميات تقاربًا شاملاً وتمتلك خاصية الانحدار. ومن خلال الاختبار العددي، أظهرت الطريقة الجديدة تحسنا كبيرا. لقد ثبت أن طريقة التدرج المترافق الجديدة تعمل بشكل أفضل من طريقة التدرج المترافق التقليدية FR.

1. Introduction

A two-phase approach to the fundamental problem of image processing, which is removing noise from data [1] . Phase one consists of precisely identifying the source of impulse noise through the application of a filter that calculates the median value. Phase two is dedicated to eliminating the noise from the image [2].
Assume x is a real image consisting of M-by-N pixels, with XNM representing the gray level of x. Let y be the noisy observed image of x, which has been affected by salt-and-pepper noise, and $\bar{y}$ the image created in the first phase using the noisy image's adaptive median

*Email: basimah@uomosul.edu.iq

filter approach y. The noise pixels can be restored by reducing the functional described below:

$$f_\alpha(u) = \sum_{(i,j)\in N}\left[\left|u_{i,j} - y_{i,j}\right| + \frac{\beta}{2}(S_{i,j}^1 + S_{i,j}^2)\right] \quad \text{.........(1)}$$

where $u_{i,j} = \left[u_{i,j}\right]_{(i,j)\in N}$ is a column vector of length $|N|$, β is parameter of the regularization and : $S_{i,j}^1 = 2\sum_{(m,n)\in P_{i,j}\cap N^c} \varphi_\alpha(u_{i,j} - y_{m,n})$, $S_{i,j}^2 = \sum_{(m,n)\in P_{i,j}\cap N} \phi_\alpha(u_{i,j} - y_{m,n})$ the Noise Candidate Indices set N^c is used to calculate a noisy pixel's maximum s_{max} and minimum s_{min}. $A = \{1,2,3,\ldots\ldots M\} \times \{1,2,3,\ldots\ldots N\}$ and $V_{i,j} = (V_{i,j} \cap N^c) \cup (V_{i,j} \cap N)$ represent (i,j) neighbourhood, while $\phi_\alpha = \sqrt{\alpha + x^2}, \alpha > 0$ is expresses of example to an edge preserving potential function using the parameter α. Similar optimization difficulties arise when $F_\alpha(u)$ is of type (1), when $S_{i,j}^1 + S_{i,j}^2$ smooth but $\left|u_{i,j} - y_{i,j}\right|$non-smooth set to zero,[3] reduces the function to a smooth half-quadratic approximation of $F_\alpha(u)$:

$$f_\alpha(u) = \sum_{(i,j)\in N}\left[(2 \times S_{i,j}^1 + S_{i,j}^2)\right] \quad \text{.........(2)}$$

The conjugated gradients method was used to minimize (2), resulting in:$Min f_\alpha(u)$, $u \in R^n$,$f_\alpha(u): R^n \to R$ is the smooth function. It is essential to keep in mind that conjugate gradient algorithms adjust their sequence of points by using the recursive formula that is shown in the following sentence:

$$u_{k+1} = u_k + \alpha_k d_k \quad \text{.........(3)}$$

where search direction is denoted by d_k, and the typical step length obtained from an adequate exact line search is denoted by α_k, as in:

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T Q d_k} \quad \text{.........(4)}$$

see,[4] . In order to meet the conditions of the Wolfe line search, which are as follows, the step length is often chosen:

$$f(x_k + \alpha_k d_k) \le f(x_k) + \delta \alpha_k g_k^T d_k \quad \text{.........(5a)}$$

$$d_k^T g(x_k + \alpha_k d_k) \ge \sigma\, d_k^T g_k \quad \text{.........(5b)}$$

where $0 < \delta < \sigma < 1$. For more details see [5]. Exploring the convergence behaviour of the aforementioned equations when used with certain line search settings has taken a large amount of time and effort from numerous writers over the course of several years. The conjugate gradient algorithms provide the next set of search axes:

$$d_{k+1} = -g_{k+1} + \beta_k s_k \quad \text{.........(6)}$$

where β_k is a scalar. A study of the original presentation of the FR formula that the Fletcher-Reeves (FR) method [6] made, which may be represented as the following formula:

$$\beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2} \quad \text{.........(7)}$$

The gradient method is used to find the closest local minimum of a function by looking at the function's gradient. This finding served as the impetus for research into conjugate gradient techniques for solving iterative issues. Hence, large-scale iterative challenges were efficiently managed [7]. The Fletcher-Reeves (FR) approach converges [1] when certain assumptions about convex functions are taken into consideration.

The conjugate gradient approach has known more sophisticated modifications, although this is merely a prototype, [6], [7], [8], [9]. There have been several attempts to create conjugate gradient algorithms with the crucial adequate descent property. CG methods provided by Wu and Chen [10] include the following examples:

$$\beta_k^{WC} = \frac{y_{k+1}^T g_{k+1}}{d_k^T y_k} + \frac{2(f_k - f_{k+1}) + g_k^T s_k}{d_k^T y_k} \quad \text{.........(8)}$$

which, in developing its concept, was based on a Perry's conjugacy condition, is characterized as follows:

$$d_{k+1}^T y_k = -s_k^T g_{k+1} \quad \dots\dots\dots (9)$$

See, [11]. These strategies are both successful in theory and efficient in practice when it comes to numbers. The search direction calculation is where the conjugate gradient approach and Chen and Wu method diverge the most from one another. Please refer to [12], [13], [14], [15] for further references about the optimization approaches.

To begin, we suggest a whole new category of equations by basing them on the idea of the quadratic function. Next, we consider these formulae, study their theoretical properties, and then report on both their theoretical and numerical performance.

2. Our new formulas

We derive some efficient formula for gradient method. Let us begin with the Taylor expansion:

$$f(u) = f(u_{k+1}) + g_{k+1}^T(u - u_{k+1}) + \frac{1}{2}(u - u_{k+1})^T Q(u_k)(u - u_{k+1}) \quad \dots\dots\dots (10)$$

By imposing $u = u_k$ and find the derivative of (10) as:

$$g_{k+1} = g_k + Q(x_k)s_k \quad \dots\dots\dots (11)$$

Using (11) in (10), we obtain:

$$s_k^T Q(u_k)s_k = 2/3 s_k^T y_k + 2/3(f_k - f_{k+1}) \quad \dots\dots\dots (12)$$

It follows from (α_k) and (12) that:

$$s_k^T Q(x_k)s_k = \frac{2}{3}\frac{(g_k^T s_k)^2}{(s_k^T y_k + 2/3(f_{k+1} - f_k))} = \omega_k s_k^T y_k \quad \dots\dots\dots (13)$$

where:

$$\omega_k^1 = \frac{2}{3}\frac{(g_k^T s_k)^2}{s_k^T y_k(s_k^T y_k + 2/3(f_{k+1} - f_k))} \quad \dots\dots\dots (14)$$

By using (9) and (14), we have:

$$d_{k+1}^T y_k = -\omega_k s_k^T y_k - s_k^T g_k \quad \dots\dots\dots (15)$$

Putting $d_{k+1} = -g_{k+1} + \beta_k s_k$in (15) suggest that:

$$\beta_k s_k^T y_k = g_{k+1}^T y_k - \omega_k s_k^T y_k - s_k^T g_k \quad \dots\dots\dots (16)$$

As a result

$$\beta_k = \frac{g_{k+1}^T y_k}{s_k^T y_k} - \omega_k \frac{s_k^T y_k}{s_k^T y_k} - \frac{s_k^T g_k}{s_k^T y_k} \quad \dots\dots\dots (17)$$

Utilized exact line search in (12), then (14) reduces to:

$$\omega_k^2 = \frac{2}{3}\frac{(g_k^T s_k)^2}{s_k^T y_k(-s_k^T g_k + 2/3(f_{k+1} - f_k))} \quad \dots\dots\dots (18)$$

and

$$\omega_k^3 = \frac{2}{3}\frac{(g_k^T s_k)^2}{s_k^T y_k(\alpha_k g_k^T g_k + 2/3(f_{k+1} - f_k))} \quad \dots\dots\dots (19)$$

This formula is referred to as the BP. Below is the BP conjugate gradient algorithm.

3. Convergence analysis

Assumptions are needed to demonstrate the new algorithm's global convergence:

1. The level set $\Omega = \{x \in R^n / f(x) \le f(x_1)\}$ is bounded.
2. The Lipschitz gradient g is continuous, that is there exists a positive constant L > 0 such that:

$$\|g(\kappa) - g(\varsigma)\| \le L \ \ \|\kappa - \varsigma\|, \forall \kappa, \varsigma \in R \quad \dots\dots\dots (20)$$

It is called strongly monotone if:

$$(g(\kappa) - g(\varsigma))^T(\kappa - \varsigma) \ge \mu\|\kappa - \varsigma\|^2 \forall \kappa, \varsigma \in R \quad \dots\dots\dots (21)$$

See,[16], [17].

3.1 Theorem

If d_{k+1} is generate by via a new technique, then $d_{k+1}^T g_{k+1} \leq -c\|g_{k+1}\|^2$ holds.

Proof:

As $d_0 = -g_0$, we get $g_0^T d_0 = -\|g_0\|^2 \leq 0$. Let $g_k^T d_k < 0$ for all $k \in n$. By taking the inner product (6) by g_{k+1}^T, , we get:

$$d_{k+1}^T g_{k+1} = -\|g_{k+1}\|^2 + \left[\frac{g_{k+1}^T y_k}{s_k^T y_k} - \omega_k \frac{s_k^T y_k}{s_k^T y_k} - \frac{s_k^T g_k}{s_k^T y_k}\right] s_k^T g_{k+1} \quad \text{..........(22)}$$

From (22) , (11) and (13), it obtains

$$d_{k+1}^T g_{k+1} = -\|g_{k+1}\|^2 + \left[\frac{g_{k+1}^T y_k}{s_k^T y_k} - \frac{s_k^T g_{k+1}}{s_k^T y_k}\right] s_k^T g_{k+1} \quad \text{..........(23)}$$

Applying Lipschitz condition, lead to the:

$$y_k^T g_{k+1} \leq L s_k^T g_{k+1} \quad \text{..........(24)}$$

But it holds from (24) and (25), implies that:

$$d_{k+1}^T g_{k+1} \leq -\|g_{k+1}\|^2 + \left[L \frac{s_k^T g_{k+1}}{s_k^T y_k} - \frac{s_k^T g_{k+1}}{s_k^T y_k}\right] s_k^T g_{k+1} \quad \text{..........(25)}$$

So that we obtain:

$$d_{k+1}^T g_{k+1} \leq -\|g_{k+1}\|^2 + [L-1] \frac{(s_k^T g_{k+1})^2}{s_k^T y_k} \quad \text{..........(26)}$$

Clearly, it gets:

$$d_{k+1}^T g_{k+1} \leq -\|g_{k+1}\|^2 < 0 \quad \text{..........(27)}$$

This completes the proof.

In [1], a general result was established for any conjugate gradient method that adheres to the Wolfe conditions.

3.1 Lemma

Let assumptions hold. Any iteration method, where search direction is a descent and α_k is computed using the Wolfe conditions. If:

$$\sum_{k \geq 0} \frac{1}{\|d_{k+1}\|^2} = \infty \quad \text{..........(28)}$$

Then:

$$\lim_{k \to \infty} \inf \|g_k\| = 0 \quad \text{..........(29)}$$

3.2 Theorem

Let be generated by new algorithm and that assumptions holds If:

$$\lim_{k \to \infty} \inf \|g_k\| = 0 \quad \text{..........(30)}$$

Proof:

By the definition the direction of line search by (9), we have:

$$\|d_{k+1}\| = \left\|-g_{k+1} + \beta_k^{\mathrm{BP1}} s_k\right\| \quad \text{..........(31)}$$

Substituting (17) in (6), we obtain:

$$\|d_{k+1}\| = \left\|-g_{k+1} + \left(\frac{g_{k+1}^T y_k}{s_k^T y_k} - \omega_k \frac{s_k^T y_k}{s_k^T y_k} - \frac{s_k^T g_k}{s_k^T y_k}\right) s_k\right\| \quad \text{..........(32)}$$

From (11), (15) and (32), we obtained:

$$\|d_{k+1}\| = \left\|-g_{k+1} + \frac{g_{k+1}^T y_k}{d_k^T y_k} s_k - \frac{s_k^T g_{k+1}}{d_k^T y_k} s_k\right\| \quad \text{..........(33)}$$

By using (20) and (21), we get:

$$\|d_{k+1}\| \leq \|g_{k+1}\| + \frac{\|g_{k+1}\| L \|s_k\|^2}{\mu \|s_k\|^2} + \frac{\|g_{k+1}\| \|s_k\|^2}{\mu \|s_k\|^2}$$
$$\leq \left(1 + \frac{L}{\mu} + \frac{1}{\mu}\right) \|g_{k+1}\| \leq \left[\frac{\mu + L + 1}{\mu}\right] \|g_{k+1}\| \quad \text{..........(34)}$$

This inequality implies:

$$\sum_{k \geq 1} \frac{1}{\|d_k\|^2} \geq \left(\frac{\mu}{\mu + L + 1}\right) \frac{1}{\Gamma} \sum_{k \geq 1} 1 = \infty \quad \text{..........(35)}$$

It follows from (33) that $\lim_{k \to \infty} \inf \|g_k\| = 0$. Similarly, idea we can test BP2 and BP3 method.

4. Numerical Results

In this part, we present numerical results demonstrating the effectiveness of new in eliminating salt-and-pepper impulse noise. Our experiments compare the new method with the FR technique. We follow a specific approach. The MATLAB r2017a software includes all the necessary code for our study. Then a computer runs them. The halting criteria for both techniques are as follows:

$$\|f(u_k)\| \leq 10^{-4}(1+|f(u_k)|) \text{ and } \frac{|f(u_k)-f(u_{k-1})|}{|f(u_k)|} \leq 10^{-4} \qquad \ldots\ldots\ldots(36)$$

Lena, House, the Cameraman, and Elaine are included in the test photographs, in addition to the test text. We qualitatively evaluate the restoration performance by using the PSNR (peak signal to noise ratio) in a way that is comparable to [3], [18], [19], [20]. The restoration performance is defined as follows:

$$PSNR = 10\, log_{10} \frac{255^2}{\frac{1}{MN}\sum_{i,j}(u^r_{i,j}-u^*_{i,j})^2} \qquad \ldots\ldots\ldots(37)$$

the values of $u^r_{i,j}$ and $u^*_{i,j}$respectively, indicate the pixel values of the original picture as well as the image that has been restored to its previous state. In this study, we show the number of iterations (NI) and evaluations of functions (NF) needed to finish the denoising process, along with the PSNR of the resulting image. There are several iterative methods explained in [21-25]. The FR technique is slow compared to the new method, as indicated in Table 1. The PSNR values that are produced via the use of either the new approach or the FR method are also quite similar to one another.

Table 1: Numerical results of FR, BP1, BP2 and BP3 algorithms.

Image	Noise level r (%)	FR-Method			BP1-Method			BP2-Method			BP3-Method		
		N.I	N.F	PSNR (dB)	N.I	N.F	PSNR (dB)	N.I	N.F	PSNR (dB)	N.I	N.F	PSNR (dB)
Le	50	82	153	30.5529	68	71	30.4161	75	78	30.4322	73	76	30.4026
	70	81	155	27.4824	70	73	27.4228	72	75	27.5111	65	68	27.4234
	90	108	211	22.8583	69	72	22.7644	70	73	23.038	66	69	22.7646
Ho	50	52	53	30.6845	36	40	34.7662	47	49	34.9512	50	51	34.7377
	70	63	116	31.2564	56	59	31.122	53	55	31.0223	51	52	31.1241
	90	111	214	25.287	71	74	25.0109	73	76	24.9974	68	71	25.0125
El	50	35	36	33.9129	34	38	33.8871	34	38	33.8811	36	38	33.8854
	70	38	39	31.864	38	41	31.811	40	43	31.8175	41	43	31.8077
	90	65	114	28.2019	52	55	28.1588	55	57	28.2171	56	58	28.1539
c512	50	59	87	35.5359	39	46	35.4177	44	47	35.400	42	47	35.3973
	70	78	142	30.6259	49	58	30.6629	50	52	30.6118	50	52	30.6516
	90	121	236	24.3962	60	68	24.9243	71	73	24.7334	67	71	24.9046

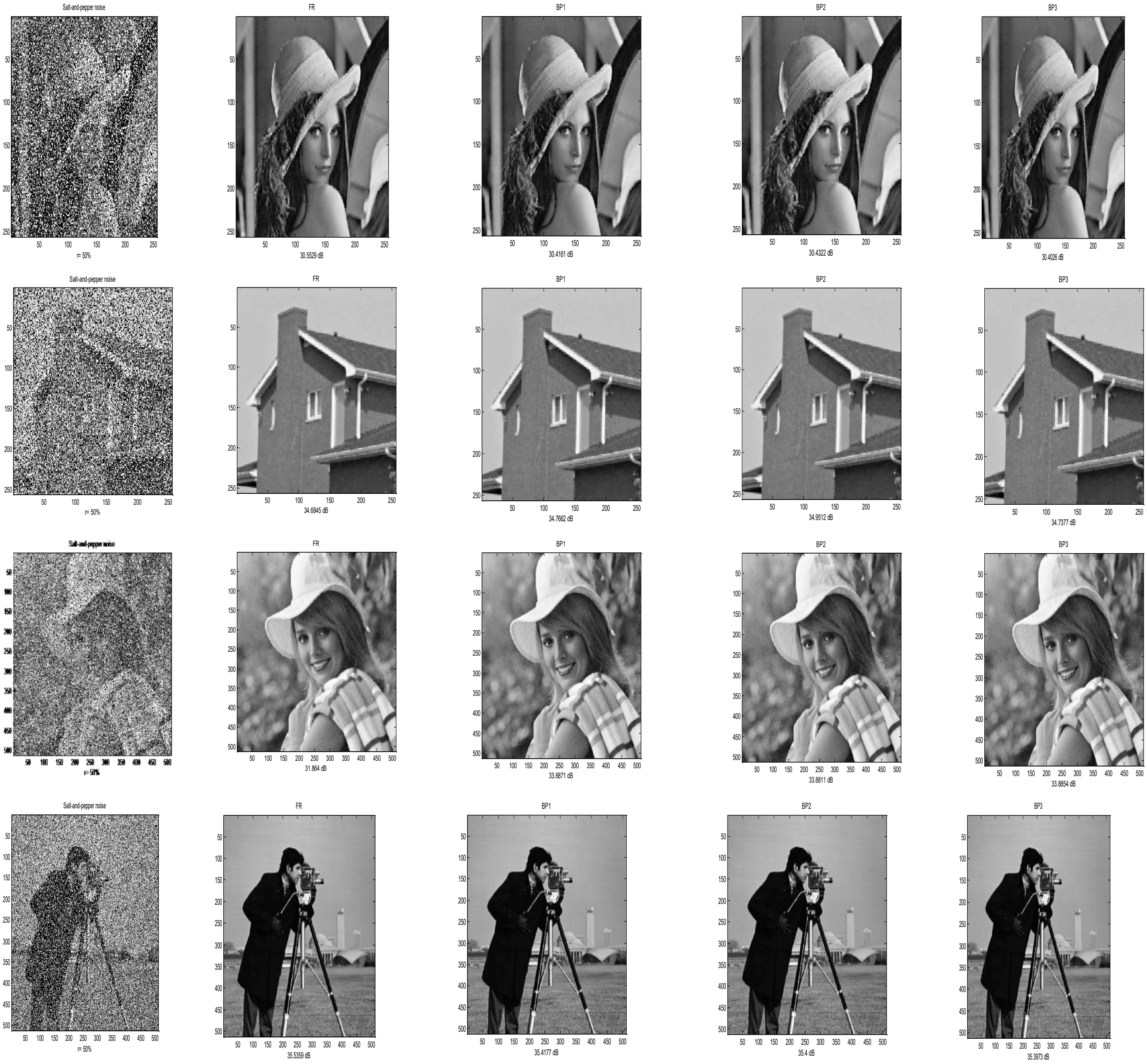

Figure 1: The results of the methods FR, BP1, BP2 and BP3 for 256 * 256 for 50% noise.

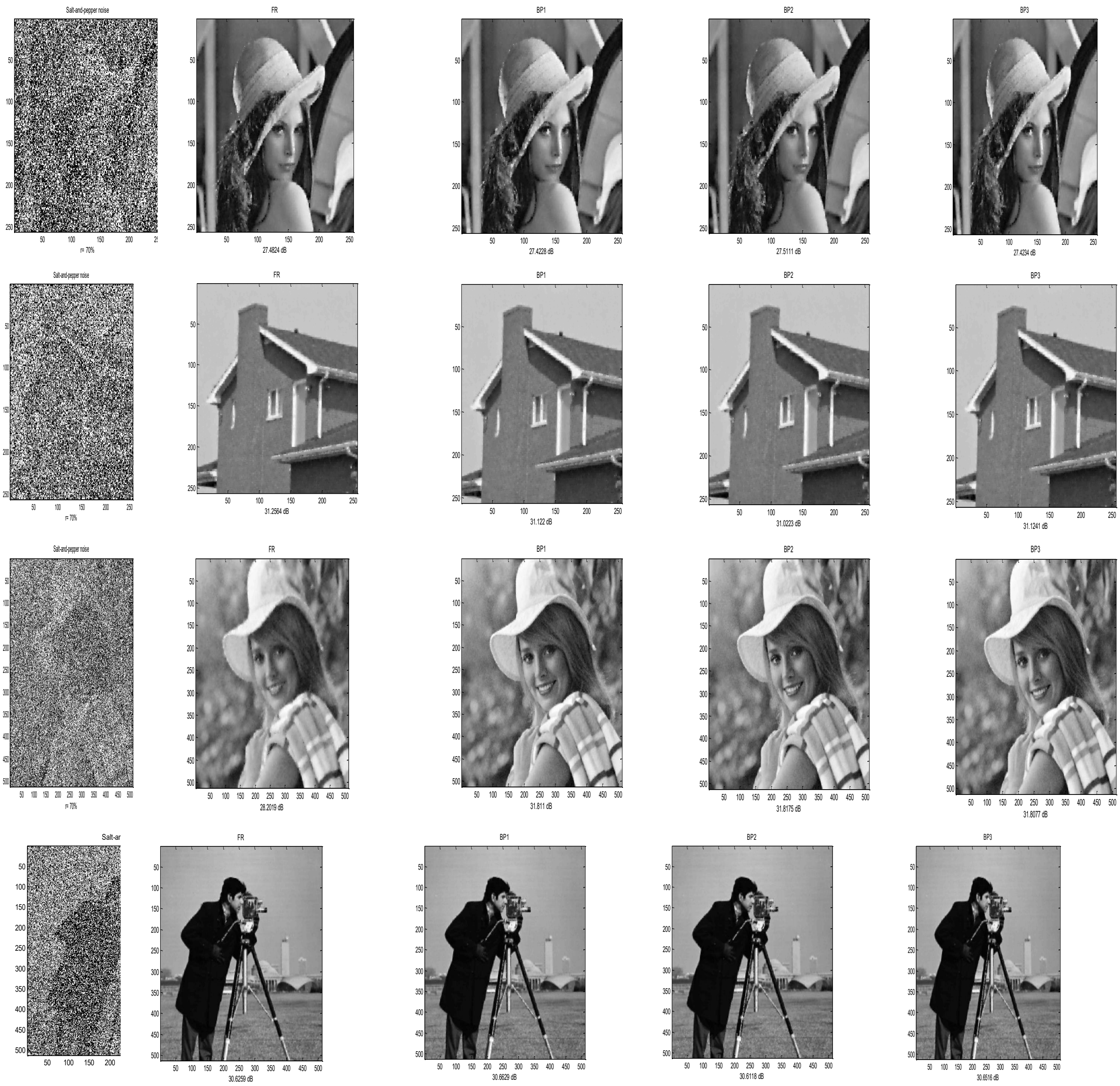

Figure 2: The results of the methods FR, BP1, BP2 and BP3 for 256 * 256 for 70% noise.

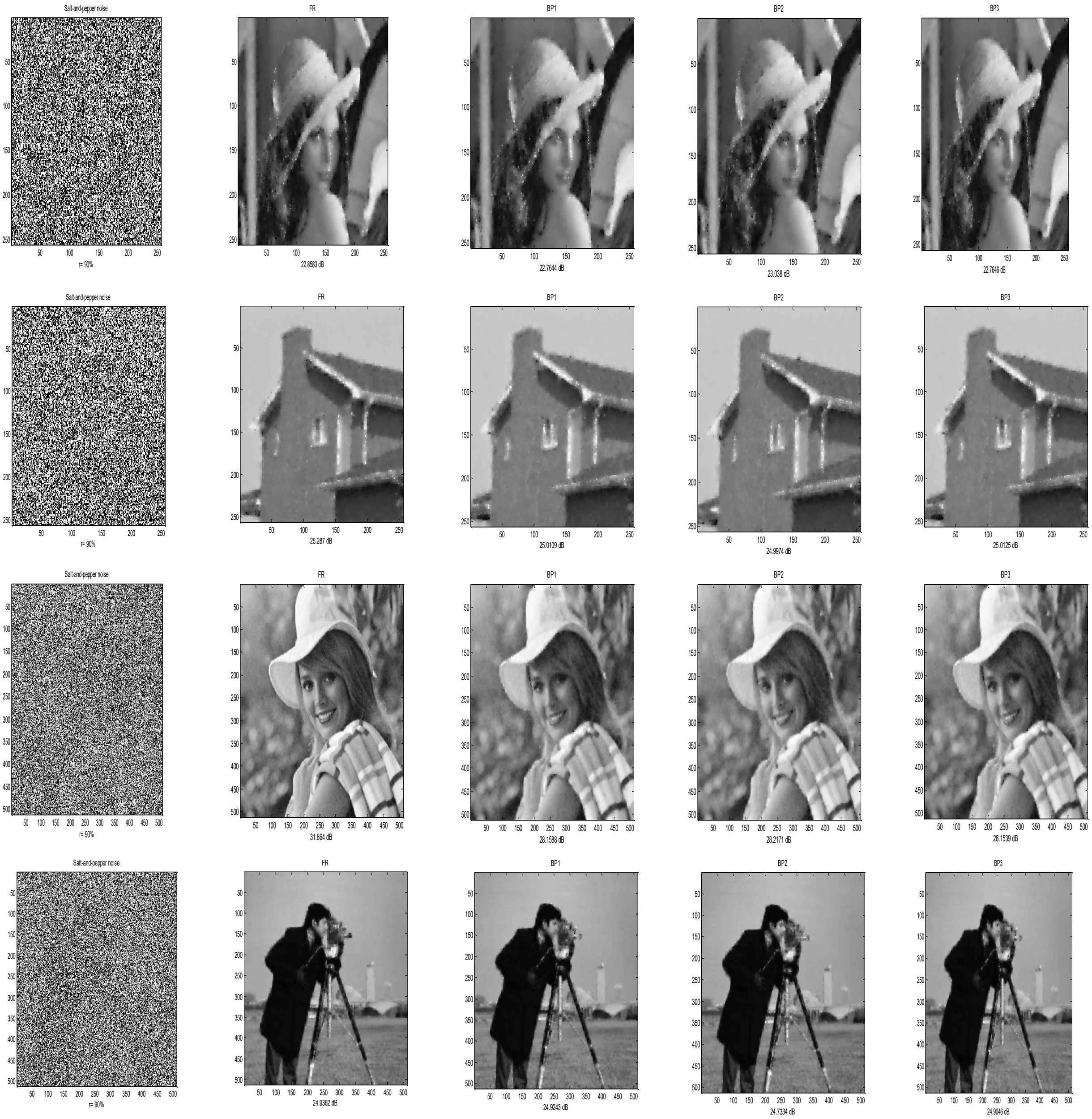

Figure 3: The results of the methods FR, BP1, BP2 and BP3 for 256 * 256 for 90% noise.

5. Conclusions

Based on the Taylor expansion, new ultra-formulas for conjugate gradient impulse noise reduction from images. The search direction always satisfies the descent condition independent of choices of conjugate parameter and line searches. Under some mild conditions, the global convergence for the BP1, BP2, and BP3 are obtained. By embedding the classical FR conjugate parameters in the BP1, BP2, and BP3, respectively, the numerical comparison results associated with the resulting methods show that the BP1, BP2, and BP3 are very promising. Moreover, the numerical results illustrate the encouragement and efficiency.

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