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Arcs in the Finite Projective Space of Dimension Five Over F_2

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Abstract
The study of arcs in finite projective spaces involves investigating their cardinality, structure, and relationships with other geometric objects. Different types of arcs can be distinguished based on their properties, such as size and degree. In this paper, the arcs of the finite projective space, $PG(5,q), q = 2$ are studied depending on the orbits that formed from the action of the projective linear group $PGL(6,2)$ on $PG(5,2)$. Each arc is extending by its degree until degree thirty one. Also, for a fixed degree, the size of each arc is extending until to be complete. Finally, the maximum value k for an $(k;r)$-arc, $m_r(5;2)$ is determined for some r.

Keywords: Arc, Complete arc, Finite field, Finite projective space, Vector space.

اقواس في الفضاء الاسقاطي المنتهي على الحقل F2

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الخلاصة
دراسة الاقواس في المساحات الاسقاطية المحدودة تنطوي على التحقق من هيكلتها وعلاقتها مع الاشياء الهندسية الاخرى. انواع مختلفة من الاقواس يمكن تميز بناءها من خلال معرفة خصائصها مثل الحجم والدرجة . في هذا البحث الاقواس في البعد الخامس والرتبة الثانية تم دراستها بالاعتماد على المدارات التي تشكلت من المصفوفة المولدة على الفضاء الاسقاطي ذا البعد الخامس على الحقل المنتهي من الرتبة الثانية.كل قوس تمت توسعت درجته الى الدرجة الواحد والثلاثين.وايضا بدرجة ثابتة يمتد كل قوس حتى يكتمل ، واخيرا يتم تحديد القيمة القصوى للبعد الخامس والرتبة الثانية لبعض الدرجات
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1. Introduction

Finite projective spaces are mathematical structures that generalize the concepts of points, lines and planes. Arcs in finite projective spaces are fundamental objects of study in combinatorial mathematics. An arc refers to a collection of points that satisfies certain geometric properties.

The finite projective space of five dimension denoted as $PG(5,q)$ represents a geometric space composed $q^5 + q^4 + q^3 + q^2 + q + 1$ points, where q is a prime power. Each point in $PG(5,q)$ corresponds to a 6-tuple of elements from a finite field with q elements. In $PG(5,2)$, there are 63 points, 63 hyperplanes, 651 3-dimension, 1395 planes and 651 lines.

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Every points passed through 31 hyperplane, every hyperplane passed through 15 3-dimension, every 3-dimenson passed through 7 planes and passed through 3 lines.
There are many studies presented regarding the arc on certain finite fields, as well as, in general on non-certain finite fields, especially the arc that is complete, see [1] [2]. The study of arcs has gained this interest recently because of the connection of the concept of arcs with linear codes of various types, see for general codes [3] [4], see for MDS codes [5], see for optimal codes [6], and connection with cryptography and networks, see [7]. As for the second-degree arc in the projective plane, it has been classified by many researchers, see [8] [9], where it was classified by finding the different arcs projectively. As for the arcs with degrees higher than two, they are not fully classified, see for example [10] [11] of many degrees, see [12] for degree three, and see [13] for degree four. Likewise, the arc in projective spaces of dimensions higher than two, there is difficulty in classifying them and finding the complete arcs from them. Therefore, those interested in the subject resort to partial classifications, as in [14] [15] for dimension three, [16] [17] for dimension four, [18] for dimension five, and [19] for dimension six.
An arc in $PG(5,q)$ is defined as a set of points that does not lie entirely on a hyperplane (a subspace of dimension four). The size of an arc is the number of points it contains.

The goal of this research is to present many complete and incomplete arcs in the projective space of the fifth dimension and find some of their properties, like T_i-distributions and c_i-distribution. Where the idea of the action of subgroups on space was used (as in [1]) and the resulting orbits were taken and considered as arcs. For theoretical details of $PG(n,q), n \geq 1$ see [20] [21].
Definition 1.1 [21]**:** A group G acts on a set K if there is a map $\varphi: K \times G \longrightarrow K$ such that if e is the identity and ω, ω' are elements in G; then, for any $x \in K$ the action of G on K is
(i) $\varphi\ (x,e) = x$.
(ii) $\varphi(\varphi\ (x,\omega),\omega') = \varphi(x,\omega\ \omega')$.
Definition 1.2 [21]**:** The orbit of a set S is $SG\ =\{\ \varphi\ \ (S,\omega) \mid \omega \in S\}$, a subset of K.
Definition 1.3 [22]**:** An $(k;r)$-arc K (arc of degree r and size k) in $PG(n,q)$ is a set of k points with the property that every hyperplane is incident with at most r points of K and there is some hyperplane incident with exactly r points of K. An arc K is complete if it is not contain in a larger arc.
Definition 1.4 [22]**:** Let K be an arc of degree r, an i-secant of K in $PG(n,q)$ is a hyperplane π such that the number of common points between K and π is i. The number of i-secants of K denoted by τ_i.
Let Q be a point not on the $(k;r)$-arc, the number of i-secant of K passing through Q denoted by $\sigma_i(Q)$. The number $\sigma_r(Q)$ of r secants is called the index of Q with respect to K. The set of all points of index i will denoted by C_i and the cardinality of C_i denoted by c_i. The sequences $(T_{i_1},\dots,T_{i_h}) = (i_1,\dots,i_h)$, $T_{i_j} \neq 0$, $0 \leq i \leq r$ will be represented the secant distribution and the sequences $(c_{i_1},\dots,c_{i_h}) = (i_1,\dots,i_h)$, $c_{i_h} \neq 0$, $0 \leq i \leq r$, will be represented the index distribution. The maximum value k for an $(k;r)$-arc is denoted by $m_r(k;q)$.

Definition 1.5 [21]**:** The group of projectivity of $PG(n,q)$ is called the projective general linear group $PGL(n+1,q)$. The elements of $PGL(n+1,q)$ are non-singular matrices of dimension $n+1$.

2. Arcs in $PG(5,2)$

Let $T_f = [[0,1,0,0,0,0],[0,0,1,0,0,0],[0,0,0,1,0,0],[0,0,0,0,1,0],[0,0,0,0,0,1],[1,1,1,0,0,1]]$ be the 6×6 companion matrix over the Galois field F_2. The elements of $PG(5,2)$ are formed by T_f and it can be referred by an integer $i \coloneqq [1,0,0,0,0,0]{T_f}^{i-1}$, $1 \le i \le 63$. The set π_1 of points $X = (x_0, x_1, x_2, x_3, x_4, x_5)$ of $PG(5,2)$ with $x_5 = 0$ is formed a hyperplane, so $\pi_1 = \{1,2,3,4,5,10,11,13,14,16,18,20,21,24,26,27,28,29,31,32,33,35,38,45,47,51,52,53,56,57,61\}$. The remaining hyperplanes will formed by the formula: $\pi_j = \pi_1 {T_f}^i$, $2 \le i \le 63$.

Lemma 2.1: Let $\Upsilon = \{3,9,7,21\}$ be the set of non-trivial factors of $\theta(5,2)$. Then

(i) There exist four non-trivial cyclic subgroups of $\langle T_f \rangle$ of order t belong to Υ;

(ii) There exist four equivalence classes in the projective space $PG(5,2)$ of order j in Y such that, $t * j = \theta(5,2)$.

Proof:

(i) The matrix T_f (companion matrix) has order $\theta(5,2)$ give a cyclic subgroup, $\langle T_f \rangle$ of $PGL(6,2)$ such that the projective space $PG(5,2)$ invariant with respect to it. For each i in Υ, the group $\zeta^i = \langle {T_f}^i \rangle$ is a cyclic subgroup $\langle T_f \rangle$ of order t such that $i * t = \theta(5,2)$. The others cyclic groups of $PGL(6,2)$ of order t will be isomorphic to ζ^i for each $i \in \Upsilon$.

(ii) For each i in Υ, the action of the group ζ^j on projective space $PG(5,2)$ will divided the space into i orbits of order $t \in \Upsilon$; that is, $t = \frac{\theta(5,2)}{i}$. These i orbits of order t will be projectively equivalent by T_f. Let O_t^ibe the representative of it with the following points:

$O_{21}^3 = \{1,4,7,10,13,16,19,22,25,28,31,34,37,40,43,46,52,55,58,61\}$;

$O_9^7 = \{1,8,15,22,29,36,43,50,57\}$;

$O_7^9 = \{1,10,19,28,37,46,55\}$;

$O_3^{21} = \{1,22,43\}$.■

Theorem 2.2: The orbits ${}_iO_t$ in Lemma 2.1 are formed complete arcs with following details:

(i) The orbit O_{21}^3 is formed a $(21;13)$-complete arc.

(ii) The orbit O_9^7 is formed a $(9;7)$-complete arc.

(iii) The orbit O_7^9 is formed a $(7;7)$-complete arc.

(iv) The orbit O_3^{21} is formed a $(3;3)$-complete arc.

Proof:

(i) The orbit O_{21}^3 has the following the T_i-distribution and c_i-distribution:
$T_9 = 42$, $T_{13} = 21$, $c_9 = 42$. Since the length of O_{21}^3 is 21 and maximum index of T_i is 13, so O_{21}^3 is $(21;13)$-arc, and since $c_0 = 0$, so O_{21}^3 is $(21;13)$-complete arc.

(ii) The orbit O_9^7 has the following the T_i-distribution and c_i-distribution:
$T_3 = 27$, $T_5 = 27$, $T_7 = 9$, $c_3 = 27$, $T_5 = 27$. Since the length of O_9^7 is 9 and maximum index of T_i is 7, so O_9^7 is $(9;7)$-arc, and since $c_0 = 0$, so O_9^7 is $(9;7)$-complete arc.

(iii) The orbit O_7^9 has the following the T_i-distribution and c_i-distribution:
$T_3 = 56$, $T_7 = 7$, $c_3 = 56$. Since the length of O_7^9 is 7 and maximum index of T_i is 7, so O_7^9 is $(7;7)$-arc, and since $c_0 = 0$, so O_7^9 is $(7;7)$-complete arc.

(vi) The orbit O_3^{21} has the following the T_i-distribution and c_i-distribution:
$T_1 = 48$, $T_3 = 15$, $c_7 = 60$. Since the length of O_3^{21} is 3 and maximum index of T_i is 3, so O_3^{21} is $(3;3)$-arc, and since $c_0 = 0$, so O_3^{21} is $(3;3)$-complete arc.

Remarks 2.3:

(i) O_{21}^3 is just the union of seven lines and union of three planes.

(ii) O_7^9 is just plane.

(iii) O_3^{21} is line of cyclic ten. Also, it is intersection of fifteen hyperplanes covered the whole space.

3. Extension of Arcs in $PG(5,2)$ by Size and Degree

In this section, each $(k;r)$-arc in Theorem 2.2 will be extended by its size and degree. The processes that are used to do the extension are as in the following procedures.

1. Determine the first hyperplane π that meets the orbit O_k^i in r points.

2. Find the extension points to $O_k^i = Q_i^{p_0}$ to extend the degree one step, which is the set $E_i = \pi - Q_i^{p_0}$.

3. Find the set of index zero C_0 with respect to $Q_i^{p_j} = Q_i^{p_{j-1}} \cup \{p_j\}$, $p_j \in E_i$ $1 \leq j \leq |E_i|$.

4. Check whether the set $Q_i^{p_j}$ is complete or incomplete arc.

i. If it is incomplete, then make a loop on C_0 until $Q_i^{p_j}$ be complete, and then back to step 2 with respect to $Q_i^{p_j}$ and go through the next steps.

ii. If it is complete, then back to step 2 with respect to $Q_i^{p_j}$ and go through the next steps.

5. If $j = |E_i|$, then we stop.

The GAP program (https://www.gap-system.org.) is using to perform the algorithms in this work.

Examples 3.1:

(i) There are 18 points 3,5,6,12,15,18,20,23,26,29,30,33,35,47,53,54,59,63 can be adding to$O_{21}^3 = Q_3^{p_0}$, (21;13)-complete arc to do the extension as following.

Let $Q_3^3 = Q_3^{p_0} \cup \{3\}$. The T_i-distribution of Q_3^3 is (9,10,13,14)=(20,22,12,9), and the c_i-distribution of Q_3^3 is (3,5) = (26,15). Since $c_0 = 0$, then Q_3^3 is (22; 14)-complete arc.

Let $Q_3^{63} = Q_3^{p_0} \cup \{3,5,6,12,15,18,,20,23,26,29,30,33,35,47,53,54,59,63\}$. The T_i-distribution of Q_3^{63} is (17,19,21,31) = (12,38,12,1), and the c_i-distribution of Q_3^{63} is (0) = (24). Since $c_0 \neq 0$, then Q_3^{63} is (39; 31)-incomplete arc. To be complete, 24 points are adding to Q_3^{63} to be complete. These points are 2, 8, 9, 11, 14, 17, 21, 24, 27, 32, 36, 38, 39, 41, 42, 44, 45, 48, 50, 51, 56, 57, 60, 62. So, $Q_3^{63} \cup \{2, 8, 9, 11, 14, 17, 21, 24, 27, 32, 36, 38, 39, 41, 42, 44, 45, 48, 50, 51, 56, 57, 60, 62\} = PG(5,2)$ is (63; 31)-complete arc.

(ii) There are 24 points 3,6,7,9,10,16,18,19,21,23,25,26,31,32,33,34,37,38,40,52,56,58,61,62 can be adding to$O_9^7 = Q_7^{p_0}$, (9; 7)-complete arc to do the extension as following.

Let $Q_7^3 = Q_7^{p_0} \cup \{3\}$. The T_i-distribution of Q_7^3 is (3,4,5,6,7,8) = (12,15,16,11,4,5), and the c_i-distribution of Q_7^3 is (3,5) =(26,15). Since $c_0 = 0$, then Q_7^3 is (10; 8)-complete arc.

Let $Q_7^{37} = Q_7^{p_0} \cup \{3,6,7,9,10,16,18,19,21,23,25,26,31,32,33,34,37\}$. The T_i-distribution of Q_7^{37} is (10,11,12,13,14,15,16,24) = (3,12,14,16,12,4,1,1), and the c_i-distribution of Q_7^{37} is (0,1) = (30,7). Since $c_0 \neq 0$, then Q_7^{37} is (26;24)-incomplete arc. To be complete, 30 points can be adding to Q_7^{37}, but only 16 of them enough to make Q_7^{37}complete. These points are 2, 4,5,11,12,13,14,17,20,24,27,28,30,35,39,41. So, $Q_7^{37} \cup \{$ 2, 4,5,11,12,13,14,17,20,24,27,28,30,35, 39,41$\}$ is (42; 24)-complete arc.

(iii) There are 24 points 2,6,9,11,12,13,18,21,22,24,26,29 32,34,35,36,39,40,41,43,53,59,60,61 can be adding to$O_7^9 = Q_9^{p_0}$, (7; 7)-ccomplete arc to do the extension as following.

Let $Q_9^2 = Q_9^{p_0} \cup \{2\}$. The T_i-distributions of Q_9^2 is (3,4,7,8) = (28,28,4,3), and the c_i-distribution of Q_9^2 is (1,3) = (48,7). Since $c_0 = 0$, then Q_9^2 is (8; 8)-complete arc.

Let $Q_9^9 = Q_9^{p_0} \cup \{2,6,9\}$. The T_i-distribution of Q_9^9 is (3,4,5,6,7,8,9,10)=(6,22,22,6,2,2,2,1), and the c_i-distribution of Q_9^9 is (0,1) = (32,21). Since $c_0 \neq 0$, then Q_9^9 is (10; 10)-incomplete arc. To be complete, 32 points can be adding to Q_9^9. But only four of them make Q_9^9 complete arc. These points are 3, 4, 5, 7. So, $Q_9^9 \cup \{3, 4, 5, 7\}$ is (14; 10)-complete arc.

(iv) There are 28 points 2, 6, 9, 10, 11, 12, 13, 18, 19, 21, 24, 26, 28, 29, 32, 34, 35, 36, 37, 39, 40, 41, 46, 53, 55, 59, 60, 61 can be adding to$O_3^{21} = Q_{21}^{p_0}$, (3;3)-complete arc to do the extension.
Let $Q_{21}^2 = Q_{21}^{p_0} \cup \{2\}$. The T_i-distribution of Q_{21}^2 is $(1,2,3,4) = (24,24,8,7)$, and the c_i-distribution of Q_{21}^2 is (3,7) = (56,3). Since $c_0 = 0$, then Q_{21}^2 is (4; 4)-complete arc.
Let $Q_{21}^{12} = Q_{21}^{p_0} \cup \{2,6,9,10,11,12\}$. The T_i-distribution of Q_{21}^{12} is $(2,3,4,5,6,7,9) =$ (6,8,20,18,6,4,1), and the c_i-distribution of Q_{21}^{12} is (0,1) = (32,22). Since $c_0 \neq 0$, then Q_{21}^{12} is (9; 9)-incomplete arc. To be complete there are 32 points can be adding to Q_{21}^{12}, but only six of them, 3,4 ,5,7,8,14 make Q_{21}^{12} (15; 9)-complete arc.

The four orbits $Q_3^{p_0}$, $Q_7^{p_0}$, $Q_9^{p_0}$, and $Q_{21}^{p_0}$ gave the following complete arcs as in Table 1:
Let 𝔭 denote the numbers of points which can be add to extension arc;
\# denote the number of points, which that add to make extension arc complete;
$\mathcal{S}$ denote the size of extension arc;
$\mathcal{D}$ denote the degree of extension arc.

Table 1: Orbit as an Arc

No.	Orbit	T_i-distrubution	c_i-distrubution	$\mathcal{S}$	$\mathcal{D}$
1	Q_{21}^3	$(T_9, T_{13}) = (42,21)$	$(c_9) = (42)$	21	13
2	Q_9^7	$(T_3, T_5, T_7) = (27,27,9)$	$(c_3, c_5) = (27,27)$	9	7
3	Q_7^9	$(T_3, T_7) = (56,7)$	$(c_3) = (56)$	7	7
4	Q_3^{21}	$(T_1, T_3) = (48,15)$	$(c_7) = (60)$	3	3

The extension of the arcs in Table 1 with respect to the degree and size are given in Tables 2,3,4, and 5.

Table 2: Extension of (21;13)-complete arc

j	$Q_3^{p_j}$	T_i-distrubution	c_i-distrubution	$\mathcal{S}$.	$\mathcal{D}$.	𝔭	#
1	$Q_3^{p_1}$	$(T_9, T_{10}, T_{13}, T_{14})$=(20,22,12,9)	(c_3, c_5)=(26,15)	22	14	0	0
2	$Q_3^{p_2}$	$(T_9, T_{10}, T_{11}, T_{13}, T_{14}, T_{15})$=(10,20,12,6,12,3)	$(c_0, c_1 c_2, c_3)$=(8, 18,12,2)	23	15	8	2
3	$Q_3^{p_3}$	$(T_9, T_{10}, T_{11}, T_{12}, T_{13} T_{14}, T_{15}, T_{16})$=(4,16,16,6,4,8,8, 1)	(c_0, c_1)=(24,15)	24	16	24	2
4	$Q_3^{p_4}$	$(T_9, T_{11}, T_{13}, T_{15}, T_{17})$=(4,32,10,16,1)	(c_0, c_1)=(24,14)	25	17	24	4
5	$Q_3^{p_5}$	$(T_9, T_{10}, T_{11}, T_{12}, T_{13} T_{14}, T_{15}, T_{16}, T_{18})$=(1,3,15,17,7 ,3,9,7,1)	(c_0, c_1)=(24,13)	26	18	24	4
6	$Q_3^{p_6}$	$(T_9, T_{10}, T_{11}, T_{12}, T_{13} T_{14}, T_{15}, T_{16}, T_{17}, T_{19})$ =(1,1,7, 19,11,7,7,5,4,1)	(c_0, c_1)=(24,12)	27	19	24	4
7	$Q_3^{p_7}$	$(T_{10}, T_{11}, T_{12}, T_{13}, T_{14} T_{15}, T_{16}, T_{17}, T_{18}, T_{20})$=(1,5,1 1,17,9,7,7,3,2,1)	(c_0, c_1)=(24,11)	28	20	24	8
8	$Q_3^{p_8}$	$(T_{10}, T_{11}, T_{12}, T_{13}, T_{14} T_{15}, T_{16}, T_{17}, T_{18}, T_{21})$=(1,1,7, 19,11,7,9,3,4,1)	(c_0, c_1)=(24,10)	29	21	24	8
9	$Q_3^{p_9}$	$(T_{11}, T_{12}, T_{13}, T_{14}, T_{15} T_{16}, T_{17}, T_{18}, T_{19}, T_{22})$=(2,2,14 ,16,8,10,6,2,2,1)	(c_0, c_1)=(24,9)	30	22	24	12
10	$Q_3^{p_{10}}$	$(T_{11}, T_{12}, T_{13}, T_{14}, T_{15} T_{16}, T_{17}, T_{18}, T_{19}, T_{20}, T_{23})$ =(1,1,6,20,12,6,10,4,1,1,1)	(c_0, c_1)=(24,8)	31	23	24	12
11	$Q_3^{p_{11}}$	$(T_{12}, T_{13}, T_{14}, T_{15}, T_{16} T_{17}, T_{18}, T_{19}, T_{20}, T_{24})$=(1,5,1 1,15,11,11,5,1,2,1)	(c_0, c_1)=(24,7)	32	24	24	12
12	$Q_3^{p_{12}}$	$(T_{13}, T_{14}, T_{15}, T_{16} T_{17}, T_{18}, T_{20}, T_{21}, T_{25})$=(3,10,8,14, 18,6,2,1,1)	(c_0, c_1)=(24,6)	33	25	24	12
13	$Q_3^{p_{13}}$	$(T_{13}, T_{14}, T_{15}, T_{16} T_{17}, T_{18}, T_{19}, T_{20}, T_{21}, T_{26})$=(1,7,5, 15,21,7,3,1,2,1)	(c_0, c_1)=(24,5)	34	26	24	14
14	$Q_3^{p_{14}}$	$(T_{14}, T_{15}, T_{16}, T_{17} T_{18}, T_{19}, T_{20}, T_{21}, T_{27})$=(4,6,10,18,	(c_0, c_1)=(24,4)	35	27	24	16

		16,4,2,2,1)					
15	$Q_3^{p_{15}}$	$(T_{15},T_{16},T_{17},T_{18}T_{19},T_{20},T_{21},T_{28})$=(6,10,14,16,10,4,2,1)	(c_0,c_1)=(24,3)	36	28	24	18
16	$Q_3^{p_{16}}$	$(T_{15},T_{16},T_{17},T_{18}T_{19},T_{20},T_{21},T_{29})$=(2,8,12,16,14,8,2,1)	(c_0,c_1)=(24,2)	37	29	24	18
17	$Q_3^{p_{17}}$	$(T_{16},T_{17},T_{18},T_{19}T_{20},T_{21},T_{30})$=(6,6,18,20,6,6,1)	(c_0,c_1)=(24,1)	38	30	24	20
18	$Q_3^{p_{18}}$	$(T_{17},T_{19},T_{21},T_{31})$=(12,38,12,1)	c_0=24	39	31	24	24

Table 3: Extension of (9;7)-complete arc

j	$Q_3^{p_j}$	T_i-distrubution	c_i-distrubution	$\mathcal{S}.$	$\mathcal{D}.$	$\mathfrak{p}$	#
1	$Q_7^{p_1}$	$(T_3,T_4,T_5,T_6,T_7,T_8)$=(12,15,16,11,4,5)	(c_1,c_2,c_3,c_4) = (12,24,13,4)	10	8	0	0
2	$Q_7^{p_2}$	$(T_3,T_4,T_5,T_6,T_7,T_8,T_9)$=(7,12,14,16,8,4,2)	(c_0,c_1,c_2)=(16,28,8)	11	9	16	2
3	$Q_7^{p_3}$	$(T_3,T_4,T_5,T_6,T_7T_8,T_9,T_{10})$=(4,10,10,15,16,5,2,1)	(c_0,c_1)=(30,21)	12	10	30	4
4	$Q_7^{p_4}$	$(T_3,T_4,T_5,T_6,T_7,T_8,T_9,T_{10},T_{11})$=(1,6,13,14,14,10,2,2,1)	(c_0,c_1)=(30,20)	13	11	30	6
5	$Q_7^{p_5}$	$(T_3,T_4,T_5,T_6,T_7T_8,T_9,T_{10},T_{11},T_{12})$=(1,2,8,18,14,9,8,1,1,1)	(c_0,c_1)=(30,19)	14	12	30	6
6	$Q_7^{p_6}$	$(T_3,T_5,T_6,T_7,T_8,T_9,T_{10},T_{11},T_{13})$=(1,5,14,11,12,6,6,1,1)	(c_0,c_1)=(30,18)	15	13	30	6
7	$Q_7^{p_7}$	$(T_4,T_5,T_6,T_7,T_8,T_9,T_{10},T_{11},T_{14})$=(1,2,10,16,14,10,5,4,1)	(c_0,c_1)=(30,17)	16	14	30	6
8	$Q_7^{p_8}$	$(T_5,T_6,T_7,T_8,T_9,T_{10},T_{11},T_{12},T_{15})$=(3,3,14,19,8,9,5,1,1)	(c_0,c_1)=(30,16)	17	15	30	8
9	$Q_7^{p_9}$	$(T_5,T_6,T_7,T_8,T_9T_{10},T_{11},T_{12},T_{16})$ =(1,2,11,15,15,9,5,4,1)	(c_0,c_1)=(30,15)	18	16	30	8
10	$Q_7^{p_{10}}$	$(T_6,T_7,T_8,T_9,T_{10}T_{11},T_{12},T_{13},T_{17})$=(3,3,15,18,9,8,5,1,1)	(c_0,c_1)=(30,14)	19	17	30	10
11	$Q_7^{p_{11}}$	$(T_6,T_7,T_8,T_9,T_{10}T_{11},T_{12},T_{13},T_{14},T_{18})$=(1,3,8,19,13,9,7,1,1,1)	(c_0,c_1)=(30,13)	20	18	30	10
12	$Q_7^{p_{12}}$	$(T_6,T_7,T_8,T_9T_{10},T_{11},T_{12},T_{13},T_{14},T_{19})$=(1,1,3,17,17,10,9,2,2,1)	(c_0,c_1)=(30,12)	21	19	30	12
13	$Q_7^{p_{13}}$	$(T_7,T_8,T_9,T_{10}T_{11},T_{12},T_{13},T_{14},T_{15}, T_{21})$=(1,2,12,14,14,13,4,1,1,1)	(c_0,c_1)=(30,11)	22	20	30	14
14	$Q_7^{p_{14}}$	$(T_8,T_9,T_{10},T_{11}T_{12},T_{13},T_{14},T_{15},T_{21})$=(2,6,14,14,14,14,9,2,1,1)	(c_0,c_1)=(30,10)	23	21	30	14
15	$Q_7^{p_{15}}$	$(T_9,T_{10},T_{11},T_{12}T_{13},T_{14},T_{15},T_{16},,T_{22})$=(5,8,15,18,11,3,1,1,1)	(c_0,c_1)=(30,9)	24	22	30	14
16	$Q_7^{p_{16}}$	$(T_9,T_{10},T_{11},T_{12}T_{13},T_{14},T_{15},T_{16},,T_{23})$=(1,9,10,15,18,7,1,1,1)	(c_0,c_1)=(30,8)	25	23	30	16
17	$Q_7^{p_{17}}$	$(T_{10},T_{11},T_{12},T_{13}T_{14},T_{15},T_{16},T_{24})$=(3,12,14,16,12,4,1,1)	(c_0,c_1)=(30,7)	26	24	30	16
18	$Q_7^{p_{18}}$	$(T_{10},T_{11},T_{12},T_{13},T_{14},T_{15},T_{16},T_{25})$=(2,4,14,19,14,7,2,1)	(c_0,c_1)=(30,6)	27	25	30	18
19	$Q_7^{p_{19}}$	$(T_{11},T_{12},T_{13},T_{14},T_{15},T_{16},T_{17},T_{26})$=(3,8,19,19,9,3,1,1)	(c_0,c_1)=(30,5)	28	26	30	20
20	$Q_7^{p_{20}}$	$(T_{11},T_{12},T_{13},T_{14},T_{15},T_{16},T_{17},T_{27})$=(1,4,13,24,14,4,2,1)	(c_0,c_1)=(30,4)	29	27	30	24
21	$Q_7^{p_{21}}$	$(T_{11},T_{12},T_{13},T_{14},T_{15},T_{16},T_{17},T_{28})$=(1,10,19,20,10,2,1)	(c_0,c_1)=(30,3)	30	28	30	24
22	$Q_7^{p_{22}}$	$(T_{13},T_{14},T_{15},T_{16},T_{17},T_{29})$=(3,16,23,16,4,1)	(c_0,c_1)=(30,2)	31	29	30	26
23	$Q_7^{p_{23}}$	$(T_{14},T_{15},T_{16},T_{17},T_{30})$=(7,24,23,8,1)	(c_0,c_1)=(30,1)	32	30	30	26
24	$Q_7^{p_{24}}$	$(T_{15},T_{16},T_{17},T_{31})$=(15,12,15,1)	c_0 =30	33	31	30	30

Table 4: Extension of (7;7)-complete arc.

j	$Q_9^{p_j}$	T_i-distrubution	c_i-distrubution	$\mathcal{S}.$	$\mathcal{D}.$	p	#
1	$Q_9^{p_1}$	(T_3,T_4,T_7,T_8)=(28,28,4,3)	(c_1,c_3) =(48,7)	8	8	0	0
2	$Q_9^{p_2}$	$(T_3,T_4,T_5,T_7,T_8,T_9)$=(14,28,14,2,4,1)	(c_0,c_1)=(32,22)	9	9	32	6
3	$Q_9^{p_3}$	$(T_3,T_4,T_5,T_6,T_7T_8,T_9,T_{10})$=(6,22,22,6,2,2,2,1)	(c_0,c_1)=(32,21)	10	10	32	4
4	$Q_9^{p_4}$	$(T_3,T_4,T_5,T_6,T_7,T_9,T_{11})$=(2,16,20,16,4,4,1)	(c_0,c_1)=(32,20)	11	11	32	4
5	$Q_9^{p_5}$	$(T_4,T_5,T_6,T_7T_9,T_{10},T_{12})$=(10,18,18,12,2,2,1)	(c_0,c_1)=(32,19)	12	12	32	4
6	$Q_9^{p_6}$	$(T_4,T_5,T_6,T_7,T_8,T_9,T_{11},T_{13})=$ (4,12,24,14,4,2,2,1)	(c_0,c_1)=(32,18)	13	13	32	4
7	$Q_9^{p_7}$	$(T_4,T_5,T_6,T_7,T_8,T_9,T_{10},T_{11},T_{14})$=(2,6,20,22,6,2,2,2,1)	(c_0,c_1)=(32,17)	14	14	32	10
8	$Q_9^{p_8}$	$(T_5,T_6,T_7,T_8,T_9,T_{11},T_{15})$=(4,16,18,16,4,4,1)	(c_0,c_1)=(32,16)	15	15	32	12
9	$Q_9^{p_9}$	$(T_6,T_7,T_8,T_9,T_{11},T_{12},T_{16})$ =(12,18,16,12,2,2,1)	(c_0,c_1)=(32,15)	16	16	32	10
10	$Q_9^{p_{10}}$	$(T_6,T_7,T_8,T_9,T_{10}T_{11},T_{12},T_{17})$=(4,18,18,10,8,2,2,1)	(c_0,c_1)=(32,14)	17	17	32	12
11	$Q_9^{p_{11}}$	$(T_7,T_8,T_9,T_{10}T_{11},T_{12},T_{18})$=(12,20,16,6,4,4,1)	(c_0,c_1)=(32,13)	18	18	32	14
12	$Q_9^{p_{12}}$	$(T_7,T_8,T_9T_{10},T_{11},T_{12},T_{19})$=(6,14,20,12,4,6,1)	(c_0,c_1)=(32,12)	19	19	32	14
13	$Q_9^{p_{13}}$	$(T_7,T_8,T_9,T_{10}T_{11},T_{12},T_{13},T_{20})$=(2,8,22,16,6,6,2,1)	(c_0,c_1)=(32,11)	20	20	32	16
14	$Q_9^{p_{14}}$	$(T_8,T_9,T_{10},T_{11}T_{12},T_{14},T_{21})$=(4,18,18,12,8,2,1)	(c_0,c_1)=(32,10)	21	21	32	16
15	$Q_9^{p_{15}}$	$(T_9,T_{10},T_{11},T_{12}T_{13},T_{14},T_{22})$=(12,20,16,8,4,2,1)	(c_0,c_1)=(32,9)	22	22	32	18
16	$Q_9^{p_{16}}$	$(T_9,T_{10},T_{11},T_{12}T_{13},T_{14},T_{23})$=(4,18,18,12,8,2,1)	(c_0,c_1)=(32,8)	23	23	32	20
17	$Q_9^{p_{17}}$	$(T_9,T_{10},T_{11},T_{12}T_{13},T_{14},T_{24})$=(2,8,20,18,10,4,1)	(c_0,c_1)=(32,7)	24	24	32	20
18	$Q_9^{p_{18}}$	$(T_{10},T_{11},T_{12},T_{13},T_{14},T_{15},T_{16},T_{25})$=(6,12,20,18,6,1)	(c_0,c_1)=(30,6)	25	25	32	22
19	$Q_9^{p_{19}}$	$(T_{10},T_{11},T_{12},T_{13},T_{14},T_{26})$=(2,8,16,24,12,1)	(c_0,c_1)=(30,5)	26	26	32	22
20	$Q_9^{p_{20}}$	$(T_{11},T_{12},T_{13},T_{14},T_{15},T_{27})$=(2,16,24,16,4,1)	(c_0,c_1)=(32,4)	27	27	32	24
21	$Q_9^{p_{21}}$	$(T_{12},T_{13},T_{14},T_{15},T_{28})$=(6,24,24,8,1)	(c_0,c_1)=(32,3)	28	28	32	28
22	$Q_9^{p_{22}}$	$(T_{13},T_{14},T_{15},T_{29})$=(14,32,15,1)	(c_0,c_1)=(32,2)	29	29	32	26
23	$Q_9^{p_{23}}$	(T_{14},T_{15},T_{30})=(30,32,1)	(c_0,c_1)=(32,1)	30	30	32	28
24	$Q_9^{p_{24}}$	(T_{15},T_{31})=(62,1)	c_0=32	31	31	32	32

Table 5: Extension of (3;3)-complete arc.

j	$Q_9^{p_j}$	T_i-distrubution	c_i-distrubution	$\mathcal{S}.$	$\mathcal{D}.$	$\mathfrak{p}$	#
1	$Q_{21}^{p_1}$	(T_1,T_2,T_3,T_4)=(24,24,8,7)	(c_3,c_7) =(56,3)	4	4	0	0
2	$Q_{21}^{p_2}$	(T_1,T_2,T_3,T_4,T_5)=(12,24,16,8,3)	(c_1,c_3)=(48,10)	5	5	0	0
3	$Q_{21}^{p_3}$	$(T_1,T_2,T_3,T_4,T_5,T_6)$=(6,18,20,12,6,1)	(c_0,c_1)=(32,25)	6	6	32	2
4	$Q_{21}^{p_4}$	$(T_1,T_2,T_3,T_4,T_5,T_6,T_7)$=(2,12,22,16,6,4,1)	(c_0,c_1)=(32,24)	7	7	32	4
5	$Q_{21}^{p_5}$	(T_2,T_4,T_6,T_8)=(14,38,10,1)	(c_0,c_1)=(32,23)	8	8	32	2
6	$Q_{21}^{p_6}$	$(T_2,T_3,T_4,T_5,T_6,T_7,T_9)$= (6,8,20,18,6,4,1)	(c_0,c_1)=(32,22)	9	9	32	6
7	$Q_{21}^{p_7}$	$(T_2,T_3,T_4,T_5,T_6,T_7,T_{10})$=(2,8,16,16,12,8, 1)	(c_0,c_1)=(32,21)	10	10	32	6
8	$Q_{21}^{p_8}$	$(T_2,T_3,T_4,T_5,T_6,T_7,T_8,T_{11})$=(2,2,14,16,14 ,12,2,1)	(c_0,c_1)=(32,20)	11	11	32	6
9	$Q_{21}^{p_9}$	$(T_3,T_4,T_5,T_6,T_7,T_8,T_9,T_{12})$ =(2,10,14,16, 14,4,2,1)	(c_0,c_1)=(32,19)	12	12	32	6
10	$Q_{21}^{p_{10}}$	(T_3,T_5,T_7,T_9,T_{13})=(2,24,30,6,1)	(c_0,c_1)=(32,18)	13	13	32	8
11	$Q_{21}^{p_{11}}$	$(T_4,T_5,T_6,T_7,T_8,T_9,T_{10},T_{14})$=(2,10,14,20, 10,2,4,1)	(c_0,c_1)=(32,17)	14	14	32	8
12	$Q_{21}^{p_{12}}$	$(T_4,T_5,T_6T_7,T_8,T_9,T_{10},T_{11},T_{15})$=(2,2,14,2 0,14,6,2,2,1)	(c_0,c_1)=(32,16)	15	15	32	8
13	$Q_{21}^{p_{13}}$	$(T_5,T_6,T_7,T_8,T_9,T_{10},T_{11},T_{16})$=(2,8,22,14, 6,8,2,1)	(c_0,c_1)=(32,15)	16	16	32	12
14	$Q_{21}^{p_{14}}$	$(T_6,T_7,T_8,T_9T_{10},T_{11},T_{12},T_{17})$=(4,18,18,10 ,8,2,2,1)	(c_0,c_1)=(32,14)	17	17	32	12
15	$Q_{21}^{p_{15}}$	$(T_6,T_8,T_{10},T_{12}T_{18})$=(4,36,18,4,1)	(c_0,c_1)=(32,13)	18	18	32	12
16	$Q_{21}^{p_{16}}$	$(T_6,T_7,T_8,T_9T_{10},T_{11},T_{12},T_{19})$=(2,2,16,20, 10,8,4,1)	(c_0,c_1)=(32,12)	19	19	32	14
17	$Q_{21}^{p_{17}}$	$(T_6,T_8,T_9,T_{10}T_{11},T_{12},T_{13},T_{20})$=(2,8,18,12 ,2,2,1)	(c_0,c_1)=(32,11)	20	20	32	14
18	$Q_{21}^{p_{18}}$	$(T_7,T_8,T_9,T_{10},T_{11},T_{12},T_{13},T_{21})$=(2,4,10,2 4,14,4,4,1)	(c_0,c_1)=(30,10)	21	21	32	18
19	$Q_{21}^{p_{19}}$	$(T_8,T_9,T_{10},T_{11},T_{12},T_{13},T_{22})$=(4,6,18,20,8, 6,1)	(c_0,c_1)=(30,9)	22	22	32	16
20	$Q_{21}^{p_{20}}$	$(T_9,T_{10},T_{11},T_{12},T_{13},T_{23})$=(6,16,14,16,10, 1)	(c_0,c_1)=(32,8)	23	23	32	18
21	$Q_{21}^{p_{21}}$	$(T_9,T_{10},T_{11},T_{12},T_{13},T_{14},T_{24})$=(2,8,20,18, 10,4,1)	(c_0,c_1)=(32,7)	24	24	32	20
22	$Q_{21}^{p_{22}}$	$(T_{10},T_{11},T_{12},T_{13},T_{14},T_{25})$=(6,12,20,18,6, 1)	(c_0,c_1)=(32,6)	25	25	32	22
23	$Q_{21}^{p_{23}}$	$(T_{11},T_{12},T_{13},T_{14},T_{15},T_{26})$=(10,20,20,10,2 ,1)	(c_0,c_1)=(32,5)	26	26	32	22
24	$Q_{21}^{p_{24}}$	$(T_{11},T_{12},T_{13},T_{14},T_{15},T_{27})$=(2,16,24,16,4, 1)	(c_0,c_1)=(32,4)	27	27	32	24
25	$Q_{21}^{p_{25}}$	$(T_{12},T_{13},T_{14},T_{15},T_{28})$=(6,24,24,8,1)	(c_0,c_1)=(32,3)	28	28	32	28
26	$Q_{21}^{p_{26}}$	$(T_{13},T_{14},T_{15},T_{29})$=(14,32,16,1)	(c_0,c_1)=(32,2)	29	29	32	26
27	$Q_{21}^{p_{27}}$	(T_{14},T_{15},T_{30})=(30,32,1)	(c_0,c_1)=(32,1)	30	30	32	28
28	$Q_{21}^{p_{28}}$	(T_{15},T_{31})=(62,1)	c_0=32	31	31	32	32

4. Conclusion

From Tables 1 to Table 5 the following are concludes:
$m_{30}(5;2) \geq 58$, $m_{29}(5;2) \geq 57$, $m_{28}(5;2) \geq 56$, $m_{27}(5;2) \geq 53$, $m_{26}(5;2) \geq 48$, $m_{25}(5;2) \geq 47$, $m_{24}(5;2) \geq 44$, $m_{23}(5;2) \geq 43$, $m_{22}(5;2) \geq 40$, $m_{21}(5;2) \geq 37$, $m_{20}(5;2) \geq 36$, $m_{19}(5;2) \geq 33$, $m_{18}(5;2) \geq 32$, $m_{17}(5;2) \geq 29$, $m_{16}(5;2) \geq 28$ $m_{15}(5;2) \geq 15$, $m_{14}(5;2) \geq 24$, $m_{13}(5;2) \geq 23$, $m_{12}(5;2) \geq 20$, $m_{11}(5;2) \geq 19$, $m_{10}(5;2) \geq 18$, $m_9(5;2) \geq 13$, $m_8(5;2) \geq 12$, $m_7(5;2) \geq 11$, $m_6(5;2) \geq 10$, $m_5(5;2) \geq 5$, $m_4(5;2) \geq 4$.
The next step is to improve the above constraints, and study all getting arcs in this research in view of coding theory.

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References

[1] A. J. Al-Rikabi, *Construction of Some Types of Sets in Three Dimensional Projective Space over Finite Field of order eight, Ph.D.Thesis, Mustansiriyah University, Baghdad, Iraq*, 2022.

[2] E. B. AL-Zangana and E. AS. Shehab, "Classification of k-sets in PG $(1,25)$, for $\mathrm{k} = 4, \ldots, 13$," *Iraqi Journal of Science,* vol. 59, no. 1B, pp. 360-368.

[3] D. Bartoli, A. A. Davydov, S. Marcugini, F. Pambianco., "On Planes Through Points off the Twisted Cubic in $\mathrm{PG}(3,\mathrm{q})$ and Multiple Covering Codes," *Finite Fields and Their Applications,* vol. 67, p. 101710, 2020.

[4] D. Baratoli, M. Giulietti and M. Montanucci., "Linear Codes From Denniston Maximal Arcs," *Des. Codes Cryptogr.,* vol. 87, pp. 795-809, 2019.

[5] A. A. Davydov, S. Marcugini and F. Pambianco, "On the Weight Distribution of the Cosets of MDS Codes," *Advances in Mathematics of Communications,* vol. 17, pp. 1115-1138, 2023.

[6] A. M. Khalaf and N. Y. K. Yahya, "New Examples in Coding Theory for Construction Optimal Linear Codes Related with Weight Distribution," *AIP Conference Proceedings,* vol. 2845, no. 1, pp. 050043(1-5), 2023.

[7] V. Starodub, R. V. Skuratovski, and S. S. Podpriatov, "Triangle Conics and Possible Applications in Cryptography," *Mathematic and Statistic,* vol. 9, pp. 749-759, 2021.

[8] A. H. Ali, Classification of Arcs in the Galois Plane of Order Thirteen, Ph. D. Thesis, University of Sussex, UK, 1993.

[9] A. L. Yasin, Cubic Arcs in the Projective Plane of Order Eight, Ph. D. Thesis, University of Sussex, UK, 1986.

[10] A. M. Sulaimaan and N. Y. K. Yahya, "The Reverse Construction of Complete $(\mathrm{k};\mathrm{n})$-Arcs in Three-Dimensional Projective Space $\mathrm{PG}(3,4)$," *AIP Conference Proceedings.,* vol. 1591, pp. 012078(1-12), 2020.

[11] R. N. Daskalov, "New Large $(\mathrm{n};\mathrm{r})$-Arcs in $\mathrm{PG}(2,\mathrm{q})$," *Iranian Journal of Mathematical Sciences and Informatics,* vol. 17, no. 1, pp. 125-133, 2022.

[12] N. A. M. Al-Seraji and R. I. K. Al-Humaidi, "On the Embedding of an Arc into a Cubic Curve in a FInite Projective Plane of Order Seven," *Al-Mustansiriyah Journal of Science,* vol. 3, pp. 158-168, 2019.

[13] Z. S. Hamed and J. W. P. Hirschfeld, "A Complete $(48,4)$-Arc in the Projective Plane Over the Field of Order Seventeen," *Baghdad Science Journal,* vol. 18, no. 4, pp. 1238-1248, 2021.

[14] V. Napolitano, "On Baer Cones in $\mathrm{PG}(3,\mathrm{q})$," *Journal of Geometry,* vol. 112, p. 37, 2021.

[15] S. Innamorati and F. Zuanni, "A Combinatorial Characterization of the Baer and the Unital Cone in PG(3,q2)," *Journal of Geometry,* vol. 111, p. 45, 2020.

[16] A. Aguglia and F. Pavese, "On Non-Singular Hermitian Varieties of $\mathrm{PG}(4, \mathrm{q}^2)$," *Discrete Mathematics,* vol. 343, no. 1, p. 111634, 2020.

[17] N. Durante and G. G. Grimaldi, "Absolute Points of Correlations of $\mathrm{PG}(4, \mathrm{q}^{\mathrm{n}})$," *Journal of Algebraic Combinatorics,* vol. 56, pp. 873-887, 2022.

[18] N. Duraante and G. G. Grimaldi, "Absolute Points of Correlations of $\mathrm{PG}(5, \mathrm{q}^{\mathrm{n}})$," *Discrete Mathematics,* vol. 346, no. 9, p. 113485, 2023.

[19] A. Aguglia, L. Giuzzi and M. Homma, "On Hermitian Varieties in $\mathrm{PG}(6, \mathrm{q}^2)$," *Ars Mathematica Cntemporanea,* vol. 21, no. 1, p. 1.04, 2021.

[20] J. W. P. Hirschfeld, *Finite Projective Space of Three Dimensions:, Oxford University Press,* 1985.

[21] J. W. P. Hirschfeld, *Projective Geometries over Finite Field, 2nd ed., Oxford Mathematical Monographs, The Clarendon press, Oxford Unversity Press,* 1998, pp. 31-131.

[22] G. Kiss and T. Szonyi, *Finite Geometries,1st ed., Chapman and Hall CRC Press,* New York,2020,10-60.