DOI: 10.24996/ijs.2020.61.2.21

ISSN: 0067-2904

Topologized Soft Fundamental Group of Soft Topological Group

Hiyam Hassan Kadhem[1*], Noor Abdul Moneem Jawad[2]
[1]Department of Mathematics, Faculty of Education, University of Kufa, Najaf, Iraq
[2]Department of Mathematics, Faculty of Education for Girls, University of Kufa, Najaf, Iraq

Received: 13/6/ 2019 Accepted: 28/ 8/2019

Abstract
In this paper, we show that each soft topological group is a strong small soft loop transfer space at the identity element. This indicates that the soft quasitopological fundamental group of a soft connected and locally soft path connected space, is a soft topological group.

Keywords: fundamental group, locally path connected space, soft topological group, quasitopological fundamental group.

تبولوجي الزمرة الاساسية الناعمة للزمرة التبولوجية الناعمة

هيام حسن كاظم[1*]، نور عبد المنعم جواد[2]
[1]قسم الرياضيات ، كلية التربية ، جامعة الكوفة ، النجف ، العراق
[2]قسم الرياضيات ، كلية التربية للبنات ، جامعة الكوفة ، النجف ، العراق

الخلاصة
في هذا البحث ، نوضح أن كل مجموعة تبولوجية ناعمة هي مسار صغير قوي لتحويل الفضاء عند العنصر المحايد. يشير هذا إلى أن المجموعة الأساسية شبه النقطية الناعمة لزمرة تبولوجية متصلة بمسارات ناعمة متصلة محليًا هي زمرة تبولوجية ناعمة.

Introduction

The fundamental group awarded with the quotient topology is tempted by the natural surjective map $q : \Omega(X,x_0) \longrightarrow \pi_1(X,x_0)$, where $\Omega(X,x_0)$ is the loop space of (X,x_0) with compact-open topology, signified by $\pi_1^{qtop}(X,x_0)$ and becomes a quasitopo-logical group [1,2]. Torabi et al. [3] showed that the quasitopological fundamental group of a connected locally path connected, semi locally small generated space, is a topological group. Spanier [4] presented a different topology on the fundamental group which was called the whisker topology by Brodskiy et al. [5] and signified by $\pi_1^{wh}(X,x_0)$.

Hamed Torabi [6] presented that the quasitopological fundamental group of a connected locally path connected topological group is a topological group.

In this paper we will demonstrate that the soft quasitopological fundamental group of a soft connected locally soft path connected topological group is a soft topological group.

Definition1.1:

Let $(X,\mathcal{T})$ is a soft topology and let V is an equivalent relationship on X when $q(x) = [x]$ then X/V is all soft equivalent classes, when $q: X \longrightarrow X/V$ is a soft quotient function then $\grave{\mathcal{T}} = \{A \subseteq X/V : q^{-1}(A) \in \mathcal{T}\}$ is soft quotient topology and $(X/V, \grave{\mathcal{T}})$ is termed soft quotient topological space.

*Email: hiyamh.kadhim@uokufa.edu.iq

the soft fundamental group $\pi_1(X,x_0)$ with the soft quotient topology X/V is signified by $\pi_1^{sqtop}(X,x_0)$.

Theorem 1.2.

Let (X,x) be a soft locally path connected soft pointed space and $V=\{V_\alpha \mid \alpha \in P(X,x)\}$ be a soft path open cover of (X,x). Then $\tilde{\pi}(V,x)$ is a soft open subgroup of $\pi_1^{sqtop}(X,x_0)$.

Definition 1.3:

Let $\tilde{X}$ is the soft space of soft homotopy classes of based soft paths in X. For any soft pointed topological space (X,x_0) the soft whisker topology on $\tilde{X}$ is defined by the soft basis $B([\alpha],U) = \{[\alpha * \beta]\}$, where α is a soft path in X from x_0 to x_1 , U is a soft neighborhood of x_1 in X , and β is a soft path in U originating at x_1, the soft path β is said to be a U − soft whisker. We represent $\tilde{X}$ with the soft whisker topology by $\tilde{X}_{swh}$.

The soft fundamental group $\pi_1(X,x_0)$ with the soft subspace topology congenital from $\tilde{X}_{swh}$ is signified by $\pi_1^{swh}(X,x_0)$.

Definition 1.4:

A soft topological space X is called a small soft loop transfer (SSLT for short) space at x if for each soft path α in X with $\alpha(0) = x_0$ and for each soft neighborhood U of x_0 there is a soft neighborhood V of $\alpha(1) = x$ such that for each soft loop β in V based at x there is a soft loop γ in U based at x_0 which is soft homotopic to $\alpha * \beta * \bar{\alpha}$ corresponding to I. The soft space X is called an $SSLT$ space if X is $SSLT$ at x_0 for each $x_0 \in X$.

Definition 1.5:

Let (X,x_0) be a soft locally path connected space and let $\mathcal{V} = \{V_\alpha | \alpha \in P(X,x_0)\}$ be a soft path open cover of X by the soft neighborhoods V_α excluding $\alpha(1)$. Define $\tilde{\pi}(\mathcal{V},x_0)$ as the subgroup of $\pi_1(X,x_0)$ involving of the soft homotopy classes of soft loops that can be represented by a product of the following form

$$\prod_{j=1}^{n} \alpha_j \beta_j \alpha_j^{-1}$$

where α_j are arbitrary soft path starting at x_0 and each β is a soft loop inside the soft open set $V_{\alpha j}$ for all $j \in \{1,2,\dots,n\}$. We call $\tilde{\pi}(V,x_0)$ the soft path Spanier group of $\pi_1(X,x_0)$ with admiration to $\mathcal{V}$.

Proposition 1.6:

Let X is a soft topological space, then we say that X is SSLT at x_0 if and only if for each soft open neighborhood $U \subseteq X$ enclosing x_0 is found a soft path open cover $\mathcal{V}$ of X at x such that $\tilde{\pi}(\mathcal{V},x_0) \leq i_*\pi_1(U,x_0)$.

Proof :

Let U be a soft open neighborhood of x_0. As X is SSLT at x_0 , for each soft path α from x_0 to $\alpha(1)$ there is a soft open neighborhood V_α of $\alpha(1)$ such that for each soft loop β in V_α based at $\alpha(1)$ we have $[\alpha * \beta * \alpha^{-1}] \in i_*\pi\,(U,x_0)$, where $i_* : \pi_1(U,x_0) \longrightarrow \pi_1(X,x_0)$ is the homomorphism convinced by the soft inclusion map $i : U \longrightarrow X$. Consider $\mathcal{V} = \{V_\alpha | \alpha \in P\,(X,x_0)\}$. Hence each generator of $\tilde{\pi}(\mathcal{V},x_0)$ goes to $i_*\pi_1(U,x_0)$ which indicates that $\tilde{\pi}(\mathcal{V},x_0) \leq i_*\pi_1(U,x_0)$.

On the other hand, let α be a soft path from x_0 to $\alpha(1)$ and U be a soft open neighborhood enclosing x_0 . By the definition of the soft path Spanier group, there is a $V_\alpha \in \mathcal{V}$ such that $[\alpha * \beta * \alpha^{-1}] \in \tilde{\pi}(V,x_0)$ for each soft loop β in V_α based at $\alpha(1)$. Thus, by assumption, $[\alpha * \beta * \alpha^{-1}] \in i_*\pi_1(U,x_0)$ which indicates that α is an SSLT path. Therefore X is an SSLT space at x_0.

Corollary 1.7:

A soft topological space X is SSLT at x_0 if and only if for each soft open neighborhood $U \subseteq X$ enclosing x_0, $i_*\pi_1(U,x_0)$ is an open subgroup of $\pi_1^{sqtop}(X,x_0)$.

Theorem 1.8:

Let X be a soft connected locally path soft connected space, then X is SSLT at x_0 if and only if $\pi_1^{swh}(X, x_0) = \pi_1^{sqtop}(X, x_0)$.

Proof:

Let X be SSLT at x_0. It is ample to expression that $\pi_1^{swh}(X, x_0)$ is bristlier than $\pi_1^{sqtop}(X, x_0)$. Let the assortment $\{[\alpha]\ i_*\pi_1(U, x_0) \mid [\alpha] \in \pi_1(X, x_0)\}$ customs a basis for the soft whisker topology on $\pi_1(X, x_0)$. Thus, it be enough to verify that $[\alpha]i_*\pi_1(U, x_0)$ is a soft open subset of $\pi_1^{sqtop}(X, x_0)$, where U is a soft open neighborhood of x_0. Using Proposition 1.6, there is a soft path open cover V of X such that $\tilde{\pi}(\mathcal{V}, x_0) \leq i_*\pi_1(U, x_0)$. Since $\tilde{\pi}(\mathcal{V}, x_0)$ is soft open in $\pi_1^{sqtop}(X, x_0)$ (Theorem 1.2) and $\pi_1^{sqtop}(X, x_0)$ is a soft quasitopological group, we imply that $[\alpha]i_*\pi_1(U, x_0)$ is a soft open subset of $\pi_1^{sqtop}(X, x_0)$.

On the other hand, assume $\pi_1^{swh}(X, x_0) = \pi_1^{sqtop}(X, x_0)$. The subset $i_*\pi_1(U, x_0)$ is a soft open basis in $\pi_1^{swh}(X, x_0)$. Then, the subset $i_*\pi_1(U, x_0)$ is soft open in $\pi_1^{sqtop}(X, x_0)$. Therefore Corollary 2.4 indicates that X is SSLT at x_0.

Corollary 1.9:

For a soft connected and soft locally path soft connected space X, if X is SSLT at x_0, then $\pi_1^{sqtop}(X, x_0)$ and $\pi_1^{swh}(X, x_0)$ are soft topological groups.

Note 1.10: [7]

if $\{f_s : (X_s, \mathcal{T}_s, E) \longrightarrow (Y_s, \acute{\mathcal{T}}_s, E)\}$ is a family of soft continuous functions, then the soft function $\prod_{s\in S} f_s : (\prod_{s\in S} X_s, \mathcal{T}, E) \longrightarrow \prod_{s\in S} Y_s, \acute{\mathcal{T}}, E)$ is soft continuous.

2.Topologized soft fundamental group of soft topological group

Let G be a soft topological group and α be a soft path in G, then we denote the soft homotopy class α by $[\alpha]$ and the inverse of α by $\bar{\alpha}$ where $\bar{\alpha} : I \longrightarrow G$ by $\bar{\alpha}(t) = \alpha(1-t)$. Also we define $\alpha^{-1} : I \longrightarrow G$ by $\alpha^{-1}(t) = \left(\alpha(t)\right)^{-1}$ and denote the constant soft path $\alpha : I \longrightarrow G$ at $a \in G$ by C_a.

Definition 2.1:

Let G be a soft topological group with the multiplication soft function $m : G \times G \longrightarrow G$, given by $(x, y) \longrightarrow xy$. Let α, β be two soft paths in G. We define the soft path $\alpha.\beta : I \longrightarrow G$ by $\alpha.\beta(t) = m(\alpha(t), \beta(t))$. Since the multiplication soft function and α, β are soft continuous $\alpha.\beta : I \longrightarrow G$ is soft continuous (by Note 1.10) .

Let f be a soft path in G and $a \in G$. We denote the soft path $C_a.f$ and $f.C_a$ by ${}^a f$ and f^a respectively.

Lemma 2.2:

If G is a soft topological group and λ, γ be two soft loops in $\acute{G}$ based at $a \in \acute{G}$ and $b \in G$ respectively, then $[\lambda.\gamma] = [\lambda^a][{}^a\gamma]$. In particular, if λ, γ be two soft loops in G based at the soft identity element e_G, then $[\lambda.\gamma] = [\lambda][\gamma]$.

Proof:

Consider the soft continuous multiplication function $m : G \times G \longrightarrow G$, given by $(x, y) \longrightarrow xy$. Let $\theta : \pi_1(G, a) \times \pi_1(G, b) \longrightarrow \pi_1\left(G \times G, (a, b)\right)$ be the soft isomorphism defined by $([\lambda], [\gamma]) \longrightarrow [(\lambda, \gamma)]$. Since $m_*\theta : \pi_1(G, a) \times \pi_1(G, b) \longrightarrow \pi_1(G, ab)$ is a soft homomorphism and $([\lambda], [\gamma]) = ([\lambda], [C])([C], [\gamma])$, we have

$$m_*\theta([\lambda], [\gamma]) = m_*\theta\big(([\lambda], [C_b])([C_a], [\gamma])\big) = m_*\theta(([\lambda], [C_b]))m_*\theta(([C_a], [\gamma])) = [\lambda^b][{}^a\gamma].$$

On the other hand $m_*\theta([\lambda], [\gamma]) = [\lambda.\gamma]$, which indicates that $[\lambda^b][{}^a\gamma] = [\lambda.\gamma]$.

Definition 2.3:

A soft topological space X is said to be a strong small soft loop transfer (strong SSLT) space at x_0 if for each $x \in X$ and for each soft neighborhood U of x_0 there is a soft neighborhood V of x such that for each soft path α in X with $\alpha(0) = x_0, \alpha(1) = x$ and for each soft loop β in V based at x

there is a soft loop γ in U based at x_0 which is soft homotopic to $\alpha * \beta * \bar{\alpha}$ relative to I . The soft space X is said to be a strong SSLT space if X is strong SSLT at x_0 for each $x_0 \in X$.

Theorem 2.4:

A soft topological group G is a strong SSLT space at the identity element e_G.

Proof:

Let U be a soft neighborhood of e_G in G and $x \in G$. We show that for each soft loop β based at x in the soft neighborhood $xU = \{xu | u \in U\}$ of x and each soft path α in G with $\alpha(0) = e_G$, $\alpha(1) = x$, there is a soft loop ξ in U based at e_G which is soft homotopic to $\alpha * \beta * \bar{\alpha}$ relative to I . For this let λ be a soft loop in G based at e_G such that

$$\lambda(t) = \begin{cases} \alpha(3t) & 0 \le t \le 1/3 \\ x & 1/3 \le t \le 2/3 \\ \bar{\alpha}(3t-2) & 2/3 \le t \le 1 \end{cases}$$

Also let γ be a soft loop in G based at e_G such that

$$\gamma(t) = \begin{cases} e_G & 0 \le t \le 1/3 \\ x^{-1}\beta(3t-1) & 1/3 \le t \le 2/3 \\ e_G & 2/3 \le t \le 1 \end{cases}$$

Therefore by Lemma 2.2 we have $[\lambda][\gamma] = [\lambda.\gamma]$. Note that

$$(\lambda.\gamma)(t) = \begin{cases} \alpha(3t) & 0 \le t \le 1/3 \\ \beta(3t-1) & 1/3 \le t \le 2/3 \\ \bar{\alpha}(3t-2) & 2/3 \le t \le 1 \end{cases}$$

If ξ is a soft loop in U based at e_G such that for each $t \in I$, $\gamma(t) = x^{-1}\beta(t)$ then we have $[\xi] = [\alpha * \bar{\alpha}][\xi] = [\lambda][\gamma] = [\lambda.\gamma] = [\alpha * \beta * \bar{\alpha}]$

Hence G is a strong SSLT space at e_G.

Corollary 2.5:

Let G be a soft topological group. Then G is an SSLT space at the identity element e_G .

Corollary 2.6:

Let G be a soft connected and soft locally soft path connected soft topological group, then $\pi^{\text{sqtop}}(G, e_G) = \pi^{\text{swh}}(G, e_G)$ is a soft topological group.

Proof:

By corollary 2.5, G is an SSLT space at e_G . Therefore $\pi_1^{sqtop}(G, e_G) = \pi_1^{swh}(G, e_G)$ by Theorem 1.8, Hence $\pi_1^{sqtop}(G, e_G)$ and $\pi_1^{swh}(G, e_G)$ are soft topological group by Corollary 1.9.

Proposition 2.7:

Let G connected and locally soft path connected soft topological group and $H \le \pi_1(G, e_G)$. Then the next statements are equivalent.

(i) H is a soft open subgroup of $\pi_1^{sqtop}(G, e_G)$.

(ii) H is a soft open subgroup of $\pi_1^{swh}(G, e_G)$.

(iii) There is a soft neighborhood U of e_G s.t. $i * \pi_1(U, e_G) \le H$.

Proof. (i) ⇔ (ii) deduce from Corollary 2.6.

(ii) ⇒ (iii) : Let H be a soft open subgroup of $\pi_1^{swh}(G, e_G)$. Since $i * \pi_1(V, e_G)$ is a soft open basis in $\pi_1^{swh}(G, e_G)$, then there is a soft neighborhood U of e s.t. $i_*\pi_1(U, e_G) \le H$.

(iii) ⇒ (ii) : Let there is a soft neighborhood U of e_G s.t. $i_*\pi_1(U, e_G) \le H$. Since $i_*\pi_1(U, e_G) \le H$ is a soft open set in $\pi_1^{swh}(G, e_G)$ and $i_*\pi_1(U, e_G) \le H$ and $\pi_1^{swh}(G, e_G)$ is a soft topological group, Hence H is a soft open subgroup of $\pi_1^{swh}(G, e_G)$.

Definition 2.8:

Let $H \le \pi_1(X, x_0)$. A soft topological space X is called an H-small soft loop transfer (H-SSLT) space at x_0 if for each soft path α in X with $\alpha(0) = x_0$ and for each soft neighborhood U of x_0 there is a soft neighborhood V of $\alpha(1) = x$ such that for each soft loop β in V based at x there is a soft loop γ in U based at x_0 such that $[\alpha * \beta * \bar{\alpha} * \bar{\gamma}] \in H$.

It is simple to see that each SSLT space at x is an H -SSLT space at x , for any soft subgroup H of $\pi_1(X, x_0)$, so each soft topological group G is a H-SSLT space at e_G, for any soft subgroup H of $\pi_1(G, e)$.

Theorem 2.9:

Let $H \trianglelefteq \pi_1(X, x_0)$ and X be a H-SSLT at x_0. So X is soft homotopically path Hausdorff for H iff X is soft homotopically Hausdorff for H.

Lemma 2.10:

Let C is a subset of $\pi_1(X, x_0)$ and C $\neq \pi_1(X, x_0)$. W say that X is soft homotopically path-Hausdorff for C if C is closed in $\pi_1^{sqtop}(X, x_0)$, and we say that C is closed in $\pi_1^{sqtop}(X, x_0)$, if X is soft homotopically path-Hausdorff for C and soft locally soft path connected.

Proposition 2.11:

Let G soft connected and soft locally soft path connected soft topological group and $H \leq \pi_1(G, e)$. So the next statements are equivalent.

(i) H is a soft closed subgroup of $\pi_1^{sqtop}(G, e)$.

(ii) H is a soft closed subgroup of $\pi_1^{swh}(G, e)$.

(iii) G is soft homotopically Hausdorff for H .

(iv) G is soft homotopically soft path Hausdorff for H .

Proof:

(i) $\Leftrightarrow$ (ii) deduce from corollary 2.6.

(iii) $\Leftrightarrow$ (iv) deduce from Theorem 2.9, since G is an H-SSLT at e_G and $\pi_1(G, e_G)$ is abelian, so H is a soft normal subgroup of $\pi_1(G, e_G)$.

(iv) $\Leftrightarrow$ (i) deduce from Lemma 2.10.

Corollary 2.12:

A soft connected locally soft path connected soft topological group G is a soft homotopically Hausdorff if and only if $\pi_1^{sqtop}(G, e_G)$ is a soft Hausdorff space.

Proof:

Assume that G is a soft homotopically Hausdorff. So G is a soft homotopically Hausdorff relative to the soft trivial subgroup $H = \{1\}$. Hence by Proposition 2.11 $\{e_G\}$ is closed in $\pi_1^{sqtop}(G, e_G)$. Therefore for each $g \in G$, $\{g\}$ is closed in $\pi_1^{sqtop}(G, e_G)$ since $\pi_1^{sqtop}(G, e_G)$ is a soft quasitopological group. Hence $\pi_1^{sqtop}(G, e_G)$ is $\mathcal{T}_0$, which indicates that it is a soft Hausdorff space since $\pi_1^{sqtop}(G, e_G)$ is a soft topological group. The converse is trivial. ■

Theorem 2.13:

A soft topological group G is a strong SSLT space if G is an abelian group or a soft path connected space.

Proof:

Let G be an abelian soft topological group and $a \in G$. We show that G is a strong SSLT space at a. For this let U be a soft neighborhood of a in G and $b \in G$. We show that for each soft loop β based at b in the soft neighborhood $ba^{-1}U = \{ba^{-1} u | u \in U\}$ of b and each soft path α in G with $\alpha(0) = a, \alpha(1) = b$, there is a soft loop γ in U based at a which is soft homotopic to $\alpha * \beta * \bar{\alpha}$ relative to I . Let λ be a soft loop in G based at a such that

$$\lambda(t) = \begin{cases} \alpha(3t) & 0 \leq t \leq 1/3 \\ b & 1/3 \leq t \leq 2/3 \\ \bar{\alpha}(3t-2) & 2/3 \leq t \leq 1 \end{cases}$$

Also let γ be a soft loop in G based at a such that

$$\gamma(t) = \begin{cases} e_G & 0 \leq t \leq 1/3 \\ ab^{-1}\beta(3t-1) & 1/3 \leq t \leq 2/3 \\ e_G & 2/3 \leq t \leq 1 \end{cases}$$

Therefore by Lemma 2.2 we have $[\lambda^a][^a\gamma] = [\lambda.\gamma]$. Note that

$$(\lambda.\gamma)(t) = \begin{cases} \alpha^a(3t) & 0 \leq t \leq 1/3 \\ bab^{-1}\,\beta(3t-1) & 1/3 \leq t \leq 2/3 \\ \bar{\alpha}^a(3t-2) & 2/3 \leq t \leq 1 \end{cases}$$

Since G is abelian, hence $bab^{-1} = a, \alpha^a =^a \alpha$ and $\bar{\alpha}^a =^a \bar{\alpha}$. Therefore
$\lambda.\gamma = (^a\alpha) * (^a\beta) * (^a\bar{\alpha}) =^a (\alpha * \beta * \bar{\alpha})$.
Since G is abelian, so $[^a\lambda] = [\lambda^a]$. Hence
$[^a(\lambda * \gamma)] = [(^a\lambda) * (^a\gamma)] = [^a\lambda][^a\gamma] = [\lambda^a][^a\gamma] = [\lambda.\gamma] = [^a(\alpha * \beta * \bar{\alpha})]$.
Therefore $[\lambda * \gamma] = [(\alpha * \beta * \bar{\alpha})]$. If $\gamma =^{b^{-1}} \beta$, then γ is a soft loop in U based at a since β is a soft loop in $ba^{-1}U$ based at b. Since $[\lambda] = [\alpha * \bar{\alpha}]$, we have
$[f] = [C_a][f] = [\alpha * \bar{\alpha}][f] = [\lambda][\gamma] = [(\alpha * \beta * \bar{\alpha})]$
Hence G is a strong SSLT space at a.

Now let G be a soft path connected soft topological group and $a \in G$. We show that G is a strong SSLT space at a. For this let U be a soft neighborhood of a in G and $b \in G$. We show that for each soft loop β based at b in the soft neighborhood $ba^{-1}U = \{ba^{-1}u | u \in U\}$ of b and each soft path α in G with $\alpha(0) = a, \alpha(1) = b$, there is a soft loop f in U based at a which is soft homotopic to $\alpha * \beta * \bar{\alpha}$ relative to I . Since G is soft path connected so there is a soft path g in G from e_G to a. By proof of Theorem 2.4, we have

$$[g * \alpha * \beta * \bar{\alpha} * \bar{g}] = [(g * \alpha) * \beta * \overline{(g * \alpha)}] = [(g * \alpha) * \overline{(g * \alpha)}][^{b^{-1}}\beta] = [C_{e_G}][^{b^{-1}}\beta] = [^{b^{-1}}\beta]$$

Also
$[g * (^{ab^{-1}}\beta) * g] = [g * \bar{g}][^{b^{-1}}\beta] = [C_{e_G}][^{b^{-1}}\beta] = [^{b^{-1}}\beta]$.
Therefore $[g * \alpha * \beta * \bar{\alpha} * \bar{g}] = [g * (^{ab^{-1}}\beta) * \bar{g}]$, which indicates that $[\alpha * \beta * \bar{\alpha}] = [^{ab^{-1}}\beta]$. If $f = {}^{b^{-1}}\beta$, then f is a soft loop in U based at a and $[\alpha * \beta * \bar{\alpha}] = [f]$. Hence G is a strong SSLT space at a.■

Corollary 2.14:

A soft topological group G is an SSLT space if G is an abelian group or a soft path connected space.

Corollary 2.15:

Let G be a soft path connected topological group, then $\pi_1^{swh}(G, e_G)$ is a soft topological group.

Conclusion:

The study has reached that the soft quasitopological fundamental group of a soft connected and locally soft path connected space is a soft topological group.

References

1. Dugundji, J.A. **1950.** Topological Fundamental Group, *Proc. Nat. Acad. Sci.* USA, **36**: 141-143.
2. Brazas, J. **2013.** The fundamental group as a topological group, *Topology Appl*. **160** (1): 170-188.
3. Torabi, H., Pakdaman, A. and Mashayekhy, B. **2015**. Topological Fundamental Groups and small Generated coverings, *Math. Slovaca*, **65**: 1153-1164.
4. Spanier, E.H. **1966.** *Algebraic topology*, McGraw-Hill, New York.
5. Brodskiy, N., Dydak, J., Labuz, B. and Mitra, A. **2006.** Topological and uniform structures on universal covering spaces, arXiv:1206-0071.
6. Torabi, H. **2018.** On Topologized Fundamental Group and Covering Spaces of Topological Groups. arXiv:1803.00741
7. yasin, T. and Bayramov, S. **2017.** Topology on soft continuous function spaces. *Comput. Math. Appl*., 22-32.