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DUAL (S.)PURE RICKART MODULES

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Abstract

This paper focuses on the study dual pure Rickart modules (or, $d.p.r.$ modules for short). We introduce the concept dual S.pure Rickart module (or, $d.s.p.r$-module for short) as a generalization of dual pure Rickart module. Suppose S is a semiradical property and let M, N be two modules. We say that M is an N- $s.d.p.r$-module if for every homomorphism $f: M \to N$, Imf is S.pure submodule of N. G. Ahmed studied $d.p.r.$ modules. In this paper aims to develop the properties of $d.p.r.$ modules and their relationship to other classes of modules such as free, projective, regular and flat. We prove that, when M and N are two modules. Then M is an N-$d.p.r.$ module if and only if for every $M_1 \leq_{\oplus} M$ and every $N_1 \leq N$, M_1 is N_1-$d.p.r.$ module. Also, we prove that a ring R is a $d.p.r.$ ring if and only if R is a regular ring.

Key word: Dual pure Rickart modules, dual S.pure Rickart module, pure submodules, S.pure submodule, flat modules.

مقاسات ريكارتيه النقية الرديفة (من النمط-S)

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الخلاصة:

سنركز في هذا البحث على دراسة المقاسات الريكارتيه النقية (أو مقاسات $d.p.r.$ للاختصار). قدمنا مقاسات ريكارتيه نقيه رديفه من النمط– S كتعميم للمقاسات الريكارتيه نقية الرديفة. لنفرض S خاصية شبه جذرية ونفرض M ، N مقاسين معرفين على حلقة R. نقول أن M هو مقاس ريكارتيه نقي رديف من النمط S–نسبة الى المقاس N (أو مقاس $d.s.p.r$ للاختصار) إذا كان لكل $f: M \to N$ ، $Im\,f$ هو مقاس جزئي نقي من النمط S– في N. درس الباحث غالب احمد المقاسات $d.p.r.$. الهدف الرئيسي من هذا البحث هو تطوير خواص المقاسات الريكارتيه النقية الرديفة وعلاقتها بفئات المقاسات الأخرى مثل الحرة والإسقاطية والمنتظمة والمسطحة. برهنا لكل M و N مقاسين, M is $N-d.p.r.$ module اذا وفقط اذا لكل $M_1 \leq_{\oplus} M$ و لكل $N_1 \leq N$, M_1 is $N_1-d.p.r.$ module . ايضا برهنا لكل حلقة R, R is a $d.p.r.$ ring اذا وفقط اذا R حلقة منتظمة.

1. Introduction

The study $d.p.r$-modules led us to develop many important properties and their applications in module theory. In addition to study and discuss the relationship of this type of class modules

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with rings and their applications in various fields. Rizvi and Roman in [1-3]. G. Ahmed in [4], [5] introduced the concepts of pure (dual pure) Rickart modules. A module M is called a pure (dual pure) Rickart module if for every $f \in End(M)$, $\ker f$ $(Im\, f)$ is a pure submodule of M. According to this, our definition is deferent to the concept of purely Rickart modules, that is introduced in [6].

N. Hamad and B. AL-Hashmi in [7] introduced semiradical property as follows: a property S is called a semiradical property if:

1- For each module M, there exists a submodule (briefly $S(M)$) such that:

a- $A \leq S(M)$ for every submodule A of M such that $S(A) = A$.

b- $S\big(S(M)\big) = S(M)$.

2- If $f: M \to N$ is an epimorphism and $S(M) = M$, then $S(N) = N$.

A semiradical property S is said to be radical property if $S\left(\frac{M}{S(M)}\right) = 0$, for each module M.

The following two properties is a radical property.

1- $S = Snr$. For a module M, let $S(M) = Snr(M) = \sum_{\substack{A \leq M \\ J(A)=A}} A$, where $J(A)$ is the Jacobson radical of A, see [8]. It is known that Snr is a radical property, see [7].

2- $S = Sa$. Let M be a module, then M is called a semiartinian module (denoted by Sa-module), if for every proper submodule A of M, $Soc(\frac{M}{A}) \neq 0$, see [8]. It is clear that every artinian module is semiartinian. For a module M, let $S(M) = Sa(M) = \sum_{\substack{A \leq M \\ A \text{ is } Sa}} A$, $Sa(M)$ is called the semiartinian submodule of M. Clearly, Sa is a radical property, see [7].

While the following three properties are semiradical property, see [7].

1- $S = Z$. For a module M, let $S(M) = Z(M)$ the singular submodule of M, then Z is a semiradical property, see [7].

2- $S = Soc$. For a module M, let $S(M) = Soc(M)$, the socle submodule of M. Soc is a semiradical property, see [7].

3- $S = \mathcal{M}$. For a module N, let $S(M) = \mathcal{M}(N) = \sum_{\substack{A \leq N \\ A \text{ is regular}}} A$, $\mathcal{M}(N)$ is called semi Broun-McCoy radical, see [9], then $\mathcal{M}$ is a semiradical property, see [7].

E. Al-Dhahari and B. Al-Bahrani in [10] introduced the concept S.pure submodule, let K be submodule of M. Then K is $s.p.$closed submodule (S.pure submodule) of M if and only if $S\left(\frac{M}{K}\right) = 0$. This observation leads us to introduce the concepts of dual S.pure Rickart modules. This paper gives some result on dual pure Rickart modules that analogue to results in [5]. As well as we introduce the concept of dual S.pure Rickart module (or, $d.s.p.r$-module) as a generalization of dual pure Rickart module. Let S be a semiradical property and let M, N be two modules. We say that M is an N- $d.s.p.r$-module if for every homomorphism $f: M \to N$, Imf is an S.pure submodule of N.

This paper consists of four sections. In Section 2, we investigate and describe relation between $d.p.r$-module and flat module. For instance, we prove for a module M_1 and a flat module M_2, then M_1 is M_2-$d.p.r.$ module if and only if for every homomorphism $f: M_1 \to M_2$, $\frac{M_2}{Im\, f}$ is a flat, see Proposition 2.1. In addition, where M be a free (projective) module, then M is a $d.p.r.$ module if and only if for all $f \in Hom(M, M)$, $\frac{M}{Im\, f}$ is flat, see Corollary 2.2. A flat module need not be a $d.p.r$-module, also the converse is not true, see Remark 2.3. Also, we prove that when R be a pure simple ring, and if R is a $d.p.r.$ ring, then R is an integral domain, see Proposition 2.4. Moreover, we prove that if R is a Bezout domain and M_1, M_2 are two modules such that for all $f \in Hom(M_1, M_2)$, $\frac{M_2}{Im\, f}$ is a nonsingular. Then M_1 is M_2-$d.p.r.$ module, see Proposition 2.5. Also, we prove that when M be a module, then M is

$d.p.r.$ module if and only if for every $f: M \to M$ be a homomorphism, $C_M + T_f \leq_p M \oplus M$, see Proposition 2.7.

In Section 3, we study the direct summand of $d.p.r.$ module, and we prove when M and N are two modules such that $M = M_1 \oplus M_2$. If M is an N-$d.p.r.$ module, then M_1 is N-$d.p.r.$ module, see Proposition 3.1. We show that, $M = \underset{j\in J}{\oplus} M_j$ be a direct sum of fully invariant submodules M_j of $M,$ for all $j \in J$. Then M is a $d.p.r$-module if and only if M_j is a $d.p.r.$ module, for all $j \in J$, see Proposition 3.2. We show that M is an N-$d.p.r.$ module if and only if for every $M_1 \leq_\oplus M$ and every $N_1 \leq N$, M_1 is an N_1-$d.p.r.$ module, see Theorem 3.3. In addition, we give a characterization for the $d.p.r$-module by means a direct summand, see Corollary 3.4. We demonstrate that, M be a module and N be a pure simple module. If M is an N-$d.p.r.$ module, then either $Hom(M,N) = 0$ or every $Hom(M,N) \neq 0$ is an epimorphism, see Proposition 3.5. In addition, when $Hom(M,N) \neq 0$. If M is an N-$d.p.r.$ module, then N is a Co-Quasi Dedekind module. In particular if M is pure simple and $d.p.r.$ module, then M is Co-Quasi Dedekind, see Corollary 3.6. We prove that, M be a pure simple and faithful module. If M is a $d.p.r.$ module, then M is divisible, see Proposition 3.7. We prove that M be a finitely generated, faithful and pure simple R-module, where R is not a field. Then M is not $d.p.r.$ module, see Proposition 3.8. We show that, if $\oplus_I R$ is M $d.p.r.$ module, for every index set I, then M is a regular module, see Proposition 3.9. Also, we give a characterization for the $d.p.r.$ ring, see Proposition 3.10. We prove that R is a regular ring if and only if every (projective, free) R-module is a $d.p.r.$ module, see Theorem 3.11.

In Section 4, we introduce the concept of $d.s.p.r$-module. We illustrate some examples and provides properties. For instance, we show that every Co-Quasi Dedekind module is a $d.s.p.r$-module Remark 4.2-4. We prove that, M and N be two modules. Let K be submodule of N. If M is N- $s.d.p.r$-module, then M is K- $s.d.p.r$-module Proposition 4.3. We prove that, M and N be modules such that $S(N) = 0$ and N has no non-trivial S.pure submodule. If M is $N - d.s.p.r$-module, then either $Hom(M,N) = 0$ or every $Hom(M,N) \neq 0$ is an epimorphism, see Proposition 4.4. In addition, M and N be modules such that $S(N) = 0$ and N has no non-trivial S.pure submodule such that $Hom(M,N) \neq 0$. If M is $N - d.s.p.r$-module, then N is Co-Quasi Dedekind. In particular, if N is a $d.s.p.r$-module, then N is Co-Quasi Dedekind, see Corollary 4.5. Also, we give a characterization for the $d.s.p.r$-module, see Theorem 4.6.

Recall that a submodule K of a module M is said to be a pure submodule if $K \cap IM = IK$, for each finitely generated ideal I of R, see [11]. Recall that a module M is said to be a regular module if each submodule of M is pure, see [9]. A ring R is a pure simple if 0 and R are the only pure ideals of R, see [12]. A module M is called a flat module, if for every short exact sequence of R-module: $0 \to A \to B \to C \to 0$, the sequence $0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0$ is also exact, see [13]. Recall that a ring R is said to be a flat ring if each finitely generated ideal in R is flat, equivalently, each ideal in R is flat, see [14]. A ring R is called a Bezout ring if every finitely generated ideal is principal, see [15]. For a module M, the singular submodule of a module M define as $Z(M) = \{m \in M : ann(m) \leq_e R\}$. A module M is called a singular module if $Z(M) = M$ and M is called a nonsingular module $Z(M) = 0$, see [16]. Recall that a submodule N of a module M is said to be a fully invariant submodule if for every $f \in End(M), f(N) \subseteq N$, see [17]. Recall that a module M is said to be a Quasi Dedekind module if for each $0 \neq f \in End(M),$ is a monomorphisem, see [18]. Recall that a module M is said to be a Co-Quasi Dedekind module if for each $0 \neq f \in End(M)$, $Im\, f = M$, see [19]. If $ann(M) = 0$, then M is called a faithful module where $ann(M) = \{r \in R \mid rm = 0, \forall m \in M\}$, see [15].

For a left module M, $End(M)$ means the endomorphism ring of M. The observes $K \leq M$, $K \leq_p M$, $K \leq_{\oplus} M$ and $K \leq_{s.p} M$ mean that K is a submodule, a pure submodule, direct summands and S.pure submodule of M, respectively. In this paper, S will be a semiradical property unless otherwise stated, R be an associative ring with identity and M be a until left R-module.

2. Dual Pure Rickart Modules and Flat Modules

In this section we study the relation between $d.p.r.$ modules and flat modules.

Proposition 2.1: Let $M_1$and M_2 be two modules such that $\forall f \in Hom(M_1, M_2)$, $\frac{M_2}{Im\, f}$ is a flat module, then M_1 is M_2-$d.p.r.$ module. The opposite is true if M_2 is a flat module.
Proof: Let $f: M_1 \to M_2$ be a homomorphism. Consider the following short exact sequence

$$0 \to Im\, f \xrightarrow{i} M_2 \xrightarrow{\pi} \frac{M_2}{Im\, f} \to 0.$$

Where i is the inclusion map and π is the natural epimorphism. Since $\frac{M_2}{Im\, f}$ is flat, then $Im\, f \leq_p M_2$, by [[13], Proposition 3.67, p. 147]. Thus M_1 is M_2-$d.p.r.$ module.
For the converse, let M_1be M_2-$d.p.r.$ module, M_2 be a flat module and let $f: M_1 \to M_2$ be a homomorphism. Consider the following short exact sequence

$$0 \to Im\, f \xrightarrow{i} M_2 \xrightarrow{\pi} \frac{M_2}{Im\, f} \to 0.$$

Where i is the inclusion map and π is the natural epimorphism. Since M_1 is M_2-$d.p.r.$ module, then $Im\, f \leq_p M_2$. But M_2 is flat, therefore $\frac{M_2}{Im\, f}$ is flat, by [[13], Proposition 3.60, p.139].
Corollary 2.2: Let N be a free (projective) module. Then N is $d.p.r.$ module if and only if $\forall f \in Hom(N, N)$, $\frac{M}{Im\, f}$ is flat.
Proof: Clear by [8] and Proposition 2.1.

Remark 2.3: A flat module need not be a $d.p.r.$ module as the following example show. Consider the module Z_4 as Z_4-module. Since Z_4 is free, then Z_4 is flat, by [8]. Let $f: Z_4 \to Z_4$ be a map defined by $f(\bar{x}) = 2\bar{x}$, for all $\bar{x} \in Z_4$. On can easily show that $Im\, f = \{\bar{0}, \bar{2}\}$ is not pure in Z_4, therefore Z_4 is not a $d.p.r.$ module.
The converse is also not true, for example the module Z_6 as Z-module. Since $6Z_6 = 0$, then Z_6 is not torsion free. Hence, Z_6 is not flat. But Z_6 is semisimple, therefore Z_6 is regular. Thus Z_6 $d.p.r.$ module.

Proposition 2.4: Let R be a pure simple ring. If R is $d.p.r.$ ring, then R is an integral domain.
Proof: Let $a \in R$. Consider the following short exact sequence

$$0 \to Ra \xrightarrow{i} R \xrightarrow{\pi} \frac{R}{Ra} \to 0.$$

Where i is the inclusion map and π is the natural epimorphism. Since R is $d.p.r.$ ring, then $\frac{R}{Ra}$ is flat, by Corollary 2.4. Hence $Ra \leq_p R$, by [[13], Proposition 3.67, p. 147]. So R is PF-ring. But R is pure simple, therefore R is an (ID) integral domain by [16].

Proposition 2.5: Let R be a Bezout domain and M_1, M_2 be two modules such that $\forall f \in Hom(M_1, M_2)$, $\frac{M_2}{Im\, f}$ is nonsingular. Then M_1 is M_2-$d.p.r.$ module.
Proof: Let $f: M_1 \to M_2$ be a homomorphism. Consider the following short exact sequence $0 \to Im\, f \xrightarrow{i} M_2 \xrightarrow{\pi} \frac{M_2}{Im\, f} \to 0.$

Where i is the inclusion map and π is the natural epimorphism. Since R is an integral domain, then $T\left(\frac{M_2}{Im\,f}\right) = Z\left(\frac{M_2}{Im\,f}\right) = 0$ and hence $\frac{M_2}{Im\,f}$ is torsion free. But R is Bezout domain, therefore $\frac{M_2}{Im\,f}$ is flat, by [[13], Corollary 2.2. 3.1, p.23]. So $Im\,f \leq_p M_2$. Thus M_1 is M_2-$d.p.r.$ module.

Remark 2.6: Let M be a module and $f: M \to M$ be a homomorphism.
Let $C_M = M\oplus 0, D_M = 0\oplus M$ and $\bar{f}: C_M \to D_M$ be a map define by $\bar{f}(m, 0) = (0, f(m))$, for every $m \in M$. It is clear that $M\oplus M = C_M \oplus D_M$, $\bar{f}$ is an R-homomorphism and $Imf = 0\oplus Im\bar{f}$. Let $T_f = \{x + \bar{f}(x), x \in C_M\}$. Clearly that $T_f \leq M\oplus M$ and $C_M + T_f = C_M \oplus Im\bar{f}$.

Proposition 2.7: Let M be a module. Then M is $d.p.r.$ module if and only if for every $f: M \to M$ be a homomorphism, $C_M + T_f \leq_p M\oplus M$.
Proof: Let $f: M \to M$ be a homomorphism. Since M is $d.p.r.$ module, then $Im\,f \leq_p M$ and hence $0\oplus Im\,f \leq_p 0\oplus M$. Therefore $Im\,\bar{f} \leq_p 0\oplus M$. So $C_M \oplus Im\,\bar{f} \leq_p C_M \oplus M$. Claim that $C_M + T_f = C_M \oplus Im\bar{f}$. To show that: let $x_1 + \bar{f}(x_2) \in C_M \oplus Im\bar{f}$, where $x_1, x_2 \in C_M$. Hence, $x_1 + \bar{f}(x_2) = x_1 - x_2 + x_1 + \bar{f}(x_2) \in C_M \oplus Im\,\bar{f}$. Therefore, $C_M \oplus Im\,\bar{f} \subseteq C_M + T_f$. It is clear that $C_M + T_f \subseteq C_M \oplus Im\,\bar{f}$. Thus $C_M + T_f \leq_p M\oplus M$.

For the converse, let $f: M \to M$ be a homomorphism. Since $C_M + T_f \leq_p M\oplus M$ and $C_M + T_f = C_M \oplus Im\bar{f}$, then $C_M \oplus Im\bar{f} \leq_p M\oplus M$. Hence, $0\oplus Im\bar{f} \leq_p 0\oplus M$. But $Imf = 0\oplus Im\bar{f}$, therefore $Imf \leq_p M$. Thus M is $d.p.r.$ module.

3. Direct Summands of Dual Pure Rickart Modules

In this section we study the direct summand of $d.p.r.$ modules and give a characterization of it by means of the direct summands. Also, we show that a ring R is a $d.p.r.$ ring if and only if R is a regular ring.

Proposition 3.1: Let M and N be two modules such that $M = M_1 \oplus M_2$. If M is N-$d.p.r.$ module, then M_1 is N-$d.p.r.$ module.
Proof: Suppose that M is N-$d.p.r.$ module and let $f: M_1 \to N$ be a homomorphism. Let $P: M \to M_1$ be the projection map. Consider the map $fop: M \to N$. Since M is N-$d.p.r.$ module, then $Im\,fop \leq_p N$. But

$Im\,fop = \{fop(m); m \in M\}$
$= \{f(p(m));\ m \in M\}$
$= \{f(p(a + b)); a \in M_1, b \in M_2\}$
$= \{f(a); a \in M_1\} = Im\,f$.

Therefore, $Im\,f \leq_p N$. Thus M_1 is N-$d.p.r.$ module.

Proposition 3.2: Let $M = \overset{\oplus}{_{j\in J}} M_j$ be a direct sum of fully invariant submodules M_j of M, for all $j \in J$. Then M is $d.p.r$-module if and only if M_j is $d.p.r.$ module, for all $j \in J$.
Proof: ⇒) Clear.
Conversely, let $f \in End(M)$. Since M_j be a fully invariant submodule, for all $j \in J$, then we can consider $f|_{Mj}: M_j \to M_j$. Clearly, $Im\,f|_{Mj} \subseteq Im\,f \cap M_j$. Let $f(x) \in Im\,f \cap M_j$, then $x = \sum_{i\in J} x_i$, for all $i \in J$ and $x_i \neq 0$ for at most a finite number of $i \in J$. But $f(x) = f\left(\sum_{i\in J} x_i\right) = \sum_{i\in J} f(x_i) \in M_j$, therefore $f(x_i) = 0$, for all $i \neq j$ and hence $f(x) = f(x_J) \in Im\,f|_{Mj}$. Thus $Im\,f|_{Mj} = Im\,f \cap M_j$.

Claim that $Im\, f = \oplus_{j\in J}\,(Im\, f|_{Mj})$. To show that, let $f(x) \in Im\, f$ and let $x = \sum_{j\in J} x_j$, where $x_j \in M_j$, for each $j \in J$ and $x_j \neq 0$ for at most a finite number of $j \in J$. Hence $f(x) = f\left(\sum_{j\in J} x_j\right) = \sum_{j\in J} f\left(x_j\right)$. Since M_j be a fully invariant submodule, for all $j \in J$, therefore $f\left(x_j\right) \in Im\, f \cap M_j$, for all $j \in J$. So, $f(x) \in \oplus_{j\in J}\left(Im\, f \cap M_j\right) = \oplus_{j\in J}\, Im\,(f|_{Mj})$.Thus $Im\, f \subseteq \oplus_{j\in J}\,(Im\, f|_{Mj})$. Clearly that $\oplus_{j\in J}\,(Im\, f|_{Mj}) \subseteq Im\, f$. Hence $Im\, f = \oplus_{j\in J}\,(Im\, f|_{Mj})$. But M_j is a $d.p.r.$ module, $\forall j \in J$, therefore $Im\, f|_{Mj} \leq_p M_j$. So $Im\, f = \oplus_{j\in J}\,(Im\, f|_{Mj}) \leq_P M$, by [[9], Remark 7-2, p.17]. Thus M is $d.p.r.$ module.

Theorem 3.3: Let M and N be two modules. Then M is N-$d.p.r.$ module if and only if for every $M_1 \leq_\oplus M$ and every $N_1 \leq N$, M_1 is N_1-$d.p.r.$ module.

Proof: Let $M_1 \leq_\oplus M$ and $N_1 \leq N$. Then $M = M_1 \oplus M_2$, for some submodule M_2 of M. Let $f: M_1 \to N_1$ be a homomorphism, $i: N_1 \to N$ be the inclusion map and $P: M \to M_1$ be the projection map. Consider the map $iofop: M \to N$. Since M is N-$d.p.r.$ module, then $Im\; iofop \leq_p N$.

But $Im\; iofop = \{iofop(m); m \in M\}$

$$= \{i\left(f\left(p(m_1+m_2)\right)\right);\ m_1 \in M_1, m_2 \in M_2\}$$
$$= \{i\left(f(m_1)\right);\ m_1 \in M_1\}$$
$$= \{f(m_1);\ m_1 \in M_1\} = Im\, f.$$

Therefore, $Im\, f \leq_p N$. But $Im\, f \leq N_1$, therefore $Im\, f \leq_p N_1$, by [[9], Remark 7-2, p.16] Thus M_1 is N_1-$d.p.r.$ module.

The converse is clear, take $M = M_1$ and $N = N_1$.

The following corollary are characterizations for the $d.p.r$-module.

Corollary 3.4: The following statements are equivalent for a module M:

1- M is a $d.p.r.$ module;

2- For every $N \leq M$, every $K \leq_\oplus M$ is N-$d.p.r.$ module;

3- For every pair K, $N \leq_\oplus M$ and every $f \in Hom(M,N)$, the image of the restricted map $f|_K \leq_p N$.

Proof: 1⇔2 Clear by Theorem 3.3.

2 ⇒) 3 Let K, $N \leq_\oplus M$ and let $f: M \to N$ be a homomorphism. Since $f|_K: K \to N$, then by our assumption K is N-$d.p.r.$ module. Thus $Im\, f|_K \leq_p N$.

3⇒)1 Clear (by taking $K = N = M$).

Proposition 3.5: Let M be a module and N be a pure simple module. If M is N-$d.p.r.$ module, then either

1- $Hom(M,N) = 0$ or,

2- Every $Hom(M,N) \neq 0$ is an epimorphism.

Proof: Suppose that $Hom(M,N) \neq 0$ and let $f: M \to N$ be a non-zero R-homomorphism. Since M is N-$d.p.r.$ module, then $Im\, f \leq_p N$. But N is pure simple, therefore $Im\, f = N$. Thus f is an epimorphism.

Corollary 3.6: Let M be a module and let N be a pure simple module such that $Hom(M,N) \neq 0$. If M is N-$d.p.r.$ module, then N is a Co-Quasi Dedekind module. In particular, if M is a pure simple and $d.p.r.$ module, then M is Co-Quasi Dedekind.

Proof: By Proposition 3.5, there is an epimorphism $f: M \to N$. Assume N is not Co-Quasi Dedekind R-module. Then there exists $0 \neq g \in End(N)$ such that $Im\, g \neq N$. Since f is an epimorphism, then $Im\, gof = Im\, g \neq N$. Since M is an N-$d.p.r.$ module, then $Im\, gof \leq_p N$.

But N is a pure simple, therefore $Im\ g = N$, which is a contradiction. Thus, N is a Co-Quasi Dedekind.

Proposition 3.7: Let M be a pure simple and faithful module. If M is a $d.p.r.$ module, then M is divisible.
Proof: Assume that M is a pure simple, faithful and $d.p.r.$ module. Let $0 \neq r \in R$ and let $f: M \to M$ be a map defined by $f(r) = rm,$ for all $r \in R$. Clearly, f is a homomorphism. So, $Im\ f = rM \leq_p M$. But M is pure simple and faithful module, then $rM \neq 0$, therefore $0 \neq rM = M$. Thus, M is divisible.

Proposition 3.8: Let N be a finitely generated, faithful and pure simple R-module, where R is not a field. Then N is not $d.p.r.$ module.
Proof: Suppose that N is a $d.p.r.$ module. Let $0 \neq r \in R$ such that $R \neq (r)$. Let $f: N \to N$ be a map defined by $f(r) = rn, \forall r \in R$. Clearly, f is a homomorphism. Since N is a $d.p.r.$ module, then $Im\ f = rN \leq_p N$. Since N is a faithful module, then $rN \neq 0$. But N is a pure simple, therefore $rN = N$. Since N is finitely generated and faithful, then N is $1/2$ cancellation, by [20]. Therefore, $R = (r)$ which is a contradiction. Thus N is not a $d.p.r.$ module.

Proposition 3.9: Let M be a module. If $\oplus_I R$ is $M\ d.p.r.$ module, for each index set I, then M is a regular module.
Proof: Let N be a submodule of M and let $\{n_\alpha;\ \alpha \in \Lambda\}$ be a set of generators of N. Let $f_\alpha: R \to Rn_\alpha$be a map defined by $f_\alpha(r) = rn_\alpha,$ for all $r \in R$ and $\alpha \in \Lambda$. Now define $f: \oplus_I R \to N$ by $f((r_\alpha)_{\alpha\in\Lambda}) = \sum r_\alpha n_\alpha$. Clearly that f is an epimorphism. Let $i: N \to M$ be the inclusion map. Consider $iof: \oplus_I R \to M$. Since $\oplus_I R$ is $M\ d.p.r.$ module, then $Im\ iof = N \leq_p M$. Thus M is a regular module.

Now, we give a characterization of $d.p.r.$ ring.

Proposition 3.10: Let R be a ring. Then R is a $d.p.r.$ ring if and only if R is a regular ring.
Proof: Suppose that R is a $d.p.r.$ ring. Let Rx be cyclic ideal in R and let $f: R \to Rx$ be a map defined by $f(x) = rx,$ for all $r \in R$. Clearly, f is an epimorphism. Let $i: Rx \to R$ be the inclusion map. Consider the map $iof: R \to R$. Since R is a $d.p.r.$ ring, then $Im\ iof \leq_p R$. But $Im\ iof = Im\ f = Rx$, so $Rx \leq_p R$. Thus R is a regular ring.

Conversely, assume that R is a regular ring. Let $f: R \to R$ be a homomorphism. Since R is a regular ring, then $Im\ f \leq_p R$. Thus, R is a $d.p.r.$ ring.

Theorem 3.11: Let R be a ring. The following statements are equivalent:
1- R is a regular ring;
2- Every R-module is $d.p.r.$ module;
3- Every projective R-module is $d.p.r.$ module;
4- Every free R-module is $d.p.r.$ module.

Proof: 1⇒2⇒3⇒4 Clear.
4⇒)1 Let I be an ideal of R, then there exists a free module F and an epimorphism $f: F \to I$. Let $i: I \to R$ be the inclusion map. Consider the map $iof: F \to R$. Since $F \oplus R$ is free, then $F \oplus R$ is $d.p.r.$ module. Therefore, F is $R\ d.p.r.$ module, by Corollary 3.4, and hence $Im\ iof = Im\ f = I \leq_p R$. Thus R is a regular ring.

4- Dual S.pure Rickart module

In this section we introduce the concept of dual S.pure Rickart module. We illustrate it some examples. We also give some basic properties. We start by the following definition:

Definition 4.1: Let M and N be modules. We say that M is $N-$dual S.pure Rickart module (or, $d.s.p.r$-module) if for each homomorphism $f: M \to N$, $Im\, f \leq_{s.p} N$. In particular, if M is $M - d.s.p.r$-module, then we say that M is a $d.s.p.r$-module.

Remarks and Examples 4.2:

1- Let $S = Z$. Consider the module $Z_2 \oplus Z_2$ as Z_2-module. Let $0 \neq f: Z_2 \oplus Z_2 \to Z_2$ be a homomorphism. Since $Im\, f = Z_2$ and $S\left(\frac{Z_2}{Z_2}\right) = S(0) = 0$, then $Im\, f \leq_{s.p} Z_2$. Thus $Z_2 \oplus Z_2$ is $Z_2 - d.s.p.r$-module.

2- Let $S = Soc$. Consider the modules Z_8 as Z-modules. Claim that Z_8 is not a $d.s.p.r$-module. To show that, let $f: Z_8 \to Z_8$ be a homomorphism define by $f(n) = 4n, \forall n \in Z_8$. Clearly $Im\, f = \{\bar{0}, \bar{4}\}$. Since $\frac{Z_8}{\{\bar{0},\bar{4}\}} \cong Z_4$, then $S\left(\frac{Z_8}{\{\bar{0},\bar{4}\}}\right) = S(Z_4) = \{\bar{0}, \bar{2}\} \neq 0$. Therefore, $\ker f$ is not S.pure submodule of Z_8. Thus Z_8 is not a $d.s.p.r$-module.

3- $s.p.r$-module need not be $d.s.p.r$-module. For example, let $S = \mathcal{M}$. Consider the module Z as Z-module. Claim that Z is not $d.s.p.r$-module. To show that, let $0 \neq f: Z \to Z$ be a homomorphism defined by $f(n) = 2n$, for all $n \in Z$. Since $Im\, f = 2Z$ and $S\left(\frac{Z}{2Z}\right) \cong S(Z_2) = Z_2 \neq 0$, then $Im\, f$ is not s.pure submodule of Z. Thus Z is not $d.s.p.r$-module. One can easily show that Z is $s.p.r$-module.

4- Every Co-Quasi Dedekind module is $d.s.p.r$-module. To show that, let $0 \neq f: M \to M$ be a homomorphism. Since M is Co-Quasi Dedekind, then $Im\, f = M$. Therefore, $S\left(\frac{M}{Im\, f}\right) = S(0) = 0$ and hence $Im\, f \leq_{s.p} M$. Thus M is $d.s.p.r$-module.

The converse is not true in general. For example, let $S = Snr$. The module Z_6 as Z-module is $d.s.p.r$-module. To show that, let $0 \neq f: Z_6 \to Z_6$ be a homomorphism. Since Z_6 is noetherian, then $\frac{Z_6}{Im\, f}$ noetherian, by [8]. Therefore, $S\left(\frac{Z_6}{Im\, f}\right) = 0$. So, $Im\, f \leq_{s.p} Z_6$. Thus Z_6 is $d.s.p.r$-module. One the other hand let $f: Z_6 \to Z_6$ be a homomorphism define by $f(n) = 3n, \ \forall n \in Z_6$. Clearly that $Im\, f = \{\bar{0}, \bar{3}\}$. Thus Z_6 is not a Co-Quasi Dedekind.

Proposition 4.3: Let M_1 and M_2 be modules and let K be submodule of M_2. If M_1 is $M_2 - d.s.p.r$-module, then M_1 is $K - d.s.p.r$-module.

Proof: Assume that M_1 is $M_2 - d.s.p.r$-module. Let $0 \neq f: M_1 \to K$ be a homomorphism and let $i: K \to M_2$ be the inclusion map. Consider the map $iof: M_1 \to M_2$. Since $M_1$is $M_2 - d.s.p.r$-module, then $Im\, f = Im\, iof \leq_{s.p} M_2$ and hence $S\left(\frac{M_2}{Im\, f}\right) = 0$. But $\frac{K}{Im\, f} \leq \frac{M_2}{Im\, f}$, therefore $S\left(\frac{K}{Im\, f}\right) = 0$ and hence $Im\, f \leq_{s.p} K$. Thus M_1 is $K - d.s.p.r$-module.

Proposition 4.4: Let M and N be modules such that $S(N) = 0$ and N has no non-trivial S.pure submodule. If M is an $N - d.s.p.r$-module, then either

1- $Hom(M, N) = 0$ or,

2- Every $Hom(M, N) \neq 0$ is an epimorphism.

Proof: Suppose that $Hom(M, N) \neq 0$. Let $f: M \to N$ be a non-zero homomorphism. Since M is an $N - d.s.p.r$-module, then $Im\, f \leq_{s.p} N$. But N has no non-trivial S.pure submodule, therefore $Im\, f = N$. Thus f is an epimorphism.

Corollary 4.5: Let M and N be modules such that $S(N) = 0$ and N has no non-trivial S.pure submodule such that $Hom(M, N) \neq 0$. If M is an $N - d.s.p.r$-module, then N is Co-Quasi Dedekind. In particular, if N is a $d.s.p.r$-module, then N is Co-Quasi Dedekind.

Proof: Assume that $Hom(M, N) \neq 0$. Hence there is an epimorphism $f: M \to N$, by Proposition 3.4 and $Im\, f = N$. Let $0 \neq g: N \to N$ be a homomorphism. Consider the map $gof: M \to N$. Since M is an $N - d.s.p.r$-module by assumption, then $Im\, gof \leq_{s.p} N$. But f is an epimorphism, therefore $Im\, gof = Im\, g \leq_{s.p} N$. Since N has no non-trivial S.pure submodule, then $Im\, g = N$. Thus N is Co-Quasi Dedekind.

The following theorems are characterizations for the $d.s.p.r$-module.

Theorem 4.6: Let M be an R-module. Then the following statements are equivalent:

1- M is $d.s.p.r$-module;

2- For every $N \leq M$, every $K \leq_{\oplus} M$ is $N - d.s.p.r$-module;

3- For every pair $K, L \leq_{\oplus} M$ and every $f \in Hom(M, L)$, the image of the restricted map $Im\, f|_K \leq_{s.p} L$.

Proof: 1⇒) 2 Assume that M is a $d.s.p.r$-module and let $N \leq M$, $K \leq_{\oplus} M$. We want to show that K is an $N - d.s.p.r$-module. Let $f: K \to N$ be a homomorphism and $M = K \oplus K_1$, for some submodule K_1 of M. Define $g: M \to M$ be a map defined by

$$g(x) = \begin{cases} f(x) & if\ x \in K; \\ 0 & if\ x \in K_1. \end{cases}$$

It is clear g is a homomorphism. But M is a $d.s.p.r$-module, therefore $Im\, g \leq_{s.p} M$. Since

$$\begin{aligned} Im\, g &= \{ g(a + b), a \in K, b \in K_1 \} \\ &= \{ f(a), a \in K \} \\ &= Im\, f. \end{aligned}$$

Therefore, $Im\, f \leq_{s.p} M$ and hence $S\left(\frac{M}{Im\, f}\right) = 0$. But $\frac{N}{Im\, f} \leq \frac{M}{Im\, f}$, so $S\left(\frac{N}{Im\, f}\right) = 0$ and hence $Im\, f \leq_{s.p} N$. Thus K is an $N - d.s.p.r$-module.

2⇒)3 Let $K, L \leq_{\oplus} M$ and let $f: M \to L$ be a homomorphism. Consider we have the map $f|_K: K \to L$. But K is an $L - d.s.p.r$-module, therefore $Im\, f|_K \leq_{s.p} L$.

3⇒)1 Let $f: M \to M$ be a homomorphism. Since $f|_K: K \to L$, then $Im f|_K \leq_{s.p} L$ by assumpation. Therefore, K is $L - d.s.p.r$-module. Take $K = L = M$. Thus M is $d.s.p.r$-module.

5. Conclusions:

In this work, the class of Pure Rickart and Dual Pure Rickart modules have been generalized to new concepts called $S.$pure Rickart modules which was presented in a research under publication and dual $S.$pure Rickart modules which was presented in this research. Several characteristics of this type of modules have been studied. In addition, we see relations between dual pure Rickart modules and flat modules.

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