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Fixed Point Theorem for Set Valued Mapping with Rational Condition

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Abstract
In this paper, we generalized the principle of Banach contractive to the relative formula and then used this formula to prove that the set valued mapping has a fixed point in a complete partial metric space. We also showed that the set-valued mapping can have a fixed point in a complete partial metric space without satisfying the contraction condition. Additionally, we justified an example for our proof.

Keywords: partial metric space, Hausdorff metric, usual metric, lower semi continuous function and Banach contraction principle.

نظرية النقطة الثابتة للدالة المتعددة مع شرط الانكماش الكسري

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الخلاصة
عممنا في بحثنا مبدأ باناخ الانكماشي الى الصيغة الكسرية ثم استخدمنا هذه الصيغة في اثبات ان الدالة المتعددة تمتلك نقطة ثابتة في الفضاء المتري الجزئي الكامل . كذلك اثبتنا انه من الممكن ان تمتلك الدالة المتعددة نقطة ثابتة في الفضاء المتري الجزئي الكامل من غير ان تحقق شرط الانكماش ثم اعطينا مثال يعزز هذا الاثبات

1-Introduction and preliminaries

Studying the set valued mappings started with the work of Nadler in 1969 [1] who proved that a set valued contractive mapping of complete metric space X into the family of closed bounded subsets of X has a fixed point. Later in the same year, several authors provided some new set-valued fixed point results by considering the δ –distance. In 2018, Sokol [2] presented a Kennan-type fixed point theorem for multivalued mapping defined on complete metric spaces. The existence and uniqueness of common fixed points of a family of self-mappings, which satisfies s the generalization of the rational contractive condition in 2-Banach spaces, was established [3]. In 1994, Mathews [4] introduced the partial metric space, which is a metric space with a non-self-distance property, and obtained a Banach-type fixed point theorem on complete partial metric space. After that, many authors studied the partial metric space and the fixed point theory in this space. Jukrapong and Suthep [5] introduced two new types of monotone set- valued mappings in a partial metric space and proved some fixed point theorems of these two types. The aim of this paper is to prove that the set-valued mapping which satisfies the generalization of the rational contractive condition can have a fixed point in complete partial metric spaces.

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A partial metric on a nonempty set X is a function p : X×X→[0,+∞) such that the following conditions hold: for all u, v, w ∈ X,
(1) p (u, u) = p(v, v) = p(u, v) if and only if u = v,
(2) p (u, u) ≤ p(u, v),
(3) p (u, v) = p (v, u),
(4) p (u, w) ≤ p(u, v) + p(v, w) – p(v, v).
The pair (X, p) is called a partial metric space.

The definitions from the theory of metric spaces, such as those of Cauchy sequences, convergent sequences, complete space and others, can be generalized to a partial metric space [6].

Dentition 1.1

A sequence $x = (x_n)$ of points in a partial metric space (X, p) is Cauchy if there exists a ≥ 0, such that for each ε > 0 there exists a positive integer k, such that for all n, m > k, $|p(x_n, x_m) - a| < \varepsilon$. That is, if $\lim_{n,m\to\infty} p(x_n, x_m) = a$.

Definition 1.2

A sequence $x = (x_n)$ of points in a partial metric space (X, p) converges to x_0 in X, If

$$\lim_{n\to\infty} p(x_n, x_0) = \lim_{n\to\infty} p(x_n, x_n) = p(x_0, x_0) .$$

Definition 1.3

A partial metric space (X, p) is complete if every Cauchy sequence converges.

Remark1.4 [7]

If p is a partial metric on X, then the function $p^s : X \times X \to [0, +\infty)$ such that $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ defines a usual metric on X.

The relation between the partial metric space (X, p) and the usual metric space (X, p^s) is given by the following lemma.

Lemma 1.5 [6]

Let (X, p) be a partial metric space. Then,
(a) A sequence (x_n) in X is Cauchy in (X, p) if and only if it is Cauchy in (X, p^s).
(b) A partial metric space (X, p) is complete if and only if the usual metric space (X, p^S) is complete.

Remark 1.6 [8]

1- *CB(X)* is the collection of all nonempty closed bounded subsets of *X*.
2- C(X) is the collection of all closed subsets of X.
3- P(x ,N) = inf {p(x, a) , a ∈ N
4- $\delta_p(M,N) = sup\{p(a,N), a \in M\}$, $\delta_p(N,M) = sup\{p(b,M), b \in N\}$
5- For $M, N \in CB(X), H_p(M,N) = max\{\delta_p(M,N), \delta_p(M,N)\}$
6- H_p is said to be partial Hausdorff metric on CB(X).
7- H_p is said to be generalized partial Hausdorff metric on C(X)

Lemma 1.7 [9]

For M , N ,W ∈ *CB(X)*, we have :
(a) $HP(M,M) \leq HP(M,N)$
(b) $HP(M,N) = HP(N,M)$

$$(c)\ HP(M,N) \leq HP(M,W) + HP(W,N) - inf\{p(w,w) : w \in W\}$$

Lemma 1.8 [10]

Let (X,P) be a partial metric space , if M,N ∈ $CB(X)$, then for any a ∈ M, there exists b = b (a) ∈N, such that p(a, b) ≤ $h\, H_p$(M,N), where $h > 1$.

2 – Banach contraction principle for the set valued mapping in partial metric spaces

The principle of Banach contraction plays an important role in the fixed point theory for a single-valued mapping in a complete metric space.
This principle is used to prove that the set-valued mapping in a complete partial metric space has a fixed point, as we see in the following main result in this section.

Theorem 2.1

Let (X,P) be a complete partial metric space and T be a set-valued mapping from X into all nonempty closed bounded subsets of X, such that :

(1) $H(Tx, Ty) \leq \frac{a\, p\,(y,Ty)\,[1+p(x,Tx)]}{1+p(x,y)} + b\, p(x,y)$,
For each $x, y \in X$, $a, b \in (0,1)$, such that $a+b < 1$
(2) For some $x_0 \in X$,the sequence $\{T^n(x_0)\}$ has a subsequence $\{T^{nk}(x_0)\}$ with $x^* = \lim_{n\to\infty} T^{nk}(x_0)$, then T has a fixed point, that is $x^* \in T(x^*)$.

Proof:

Let $x_0 \in X$ and $x_1 \in T x_0$, then by Lemma (1.8) there exists $x_2 \in T x_1$, such that $p(x_1, x_2) \leq h\, H(T x_0, T x_1)$ with $h = \frac{1}{\sqrt{a+b}}$.

Again, there exists $x_3 \in T x_2$ such that $p(x_2, x_3) \leq h\, H(T x_1, T x_2)$.
If we continue in this process, then, for each $n \geq 1$, there exists
$x_{n+1} \in T x_n$ with $p(x_n, x_{n+1}) \leq h\, H(T x_{n-1}, T x_n)$

Now, since

$$H(Tx_{n-1}, Tx_n) \leq \frac{\boldsymbol{a\, p\,(x_n, Tx_n)\,[1+p}(x_{n-1}, Tx_{n-1})]}{\boldsymbol{1+p}(x_{n-1}, x_n)} + b\, p(x_{n-1}, x_n)$$

then,

$$p(x_n, x_{n+1}) \leq h\left[\frac{\boldsymbol{a\, p\,(x_n, Tx_n)\,[1+p}(x_{n-1}, Tx_{n-1})]}{\boldsymbol{1+p}(x_{n-1}, x_n)} + b\, p(x_{n-1}, x_n)\right]$$

$$\leq h\left[\frac{\boldsymbol{a\, p\,(x_n, x_{n+1})\,[1+p}(x_{n-1}, x_n)]}{\boldsymbol{1+p}(x_{n-1}, x_n)} + b\, p(x_{n-1}, x_n)\right]$$

Hence,

$$(1-ha)\, p(x_n, x_{n+1}) \leq hb\, p(x_{n-1}, x_n)$$

$$p(x_n, x_{n+1}) \leq \frac{hb}{(1-ha)}\, p(x_{n-1}, x_n)$$

$$\leq \frac{hb}{(1-ha)}\left[\frac{hb}{(1-ha)}\, p(x_{n-2}, x_{n-1})\right]$$

$$\leq$$

$$\vdots$$

$$\leq \left(\frac{hb}{(1-ha)}\right)^n p(x_0, x_1).$$

Therefore, by choosing the two natural numbers n, m, such that $m > n$, we have

$$p(x_n, x_{n+m}) \leq \left[\left(\frac{hb}{(1-ha)}\right)^n + \left(\frac{hb}{(1-ha)}\right)^{n+1} + \cdots + \left(\frac{hb}{(1-ha)}\right)^{n+m-1}\right] p(x_0, x_1)$$

$$\leq \frac{\left(\frac{hb}{(1-ha)}\right)^n}{1-\left(\frac{hb}{(1-ha)}\right)}\, P(x_0, x_1) \to 0 \ as\ n \to \infty \text{ due to } \left(\frac{hb}{(1-ha)}\right) < 1 .$$

On the other hand,
by the definition of $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$, we get for any $m \in N$

$$p^s(x_n, x_{n+m}) = 2p(x_n, x_{n+m}) - p(x_n, x_n) - p(x_m, x_m)$$

$$\leq 2\, p(x_n, x_{n+m}) \to 0 \ as\ n \to \infty$$

This yields that (x_n) is a Cauchy sequence in (X, p^s) .

Since (X, p) is complete partial metric space, then, from Lemma 1.5, (X, p^s) is a complete metric space and hence (x_n) converges to some x_0 in X with respect to (X, p^s) , so $\lim_{n\to\infty} p^s(x_n, x_0) = 0$.
Furthermore, the sequence (x_n) converges in (X, p^s) to a point x_0 in X if and only if
$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x_0) = p(x_0, x_0)$.
That is, $p(x_0, x_0) = \lim_{n\to\infty} p(x_n, x_0) = \lim_{n\to\infty} p(x_n, x_n) = 0$.
Now, by using the fourth property of a partial metric space for each i, we have

$$p(x_{0i}, T_i x_{0i}) \leq p(x_{0i}, Tx_n) + p(Tx_n, T_i x_{0i}) - H(Tx_n, Tx_n)$$

$$\leq p(x_{0i}, Tx_n) + p(Tx_n, T_i x_{0i}) - \frac{\boldsymbol{a\, p\,(x_n, Tx_n)\,[1+p}(x_n, Tx_n)]}{\boldsymbol{1+p}(x_n, x_n)} -$$

$b\, p(x_n, x_n) \rightarrow 0$ as $n \rightarrow \infty$.

That is $x_{0i} \in T_i x_{0i}$ for each i, and this completes the proof.

Remark 2.2

In a previous study [3], Theorem 2.1 for a single valued mapping in a complete metric space was proved. Note that, if we have a sequence of set valued mapping (T_i) converging point-wise, to T such that T has a fixed point, then this fixed point represents the limit point of the sequence of fixed points of the sequence of set-valued mapping {T_i }.

We state and proof this main result in the following theorem.

Theorem 2.3

Let {T_i } be a sequence of set-valued mapping from a complete partial metric space X into the set of all closed bounded subsets of X convergence point-wise to T, and

$H(T_i x, T_i y) \leq \frac{a\, p(y, T_i y)[1+p(x, T_i x)]}{1+p(x,y)} + b\, p(x,y),$ for each $x, y \in X$, $a, b \in (0,1)$, such that $a + b < 1$, $i = 1,2, \ldots$

If each T_i has a fixed point x_{0i} and T has a fixed point x_0 , then the sequence $\{x_{0n}\}$ convergence to x_0.

Proof

Since H is a Hausdorff metric space, then, by using Lemma (1.7), we get

$$H\left(T(x_0), T_n(x_{0n})\right) \leq H\left(T(x_0), T_n(x_0)\right) + H\left(T_n(x_0), T_n(x_{0n})\right) - \inf\{p(y, y) : y \in T_n(x_0)\}$$

$$\leq H\left(T(x_0), T_n(x_0)\right) + \frac{a\, p(x_{0n}, T_n(x_{0n}))[1+p(x_0, T_n(x_0))]}{1+p(x_0, x_{0n})} + b\, p(x_0, x_{0n})$$

But, $p(x_0, x_{0n}) \leq H\left(T(x_0), T_n(x_{0n})\right)$

Hence,

$$p\,(x_0, x_{0n}) \leq H\left(T(x_0), T_n(x_0)\right) + \frac{a\, p(x_{0n}, T_n(x_{0n}))[1+p(x_0, T_n(x_0))]}{1+p(x_0, x_{0n})} + b\, p(x_0, x_{0n})$$

That is, $(1-b)\, p\,(x_0, x_{0n}) \leq H\left(T(x_0), T_n(x_0)\right) + \frac{a\, p(x_{0n}, T_n(x_{0n}))[1+p(x_0, T_n(x_0))]}{1+p(x_0, x_{0n})}$

Then, $p\,(x_0, x_{0n}) \leq \frac{H\left(T(x_0), T_n(x_0)\right)}{(1-b)} \rightarrow 0 \quad as \quad T_n(x_0) \rightarrow T(x_0)$.

Hence, $x_{0n} \rightarrow x_0$

Now, $p\,(x_{0i}, T_i x_{0i}) \leq p\,(x_{0i}, T_i x_n) + p\,(T_i x_n, T_i x_{0i})$ - $H(T_i x_n, T_i x_n)$

$$\leq p\,(x_{0i}, T_i x_n) + p\,(T_i x_n, T_i x_{0i}) - \frac{a\, p(x_n, T_i(x_n))[1+p(x_n, T_i(x_n))]}{1+p(x_n, x_n)} - b\, p(x_n, x_n)$$

$$\leq p\,(x_{0i}, x_{n+1}) + p\,(x_{n+1}, T_i x_{0i}) - \frac{a\, p(x_n\, x_{n+1})[1+p(x_n, T_i(x_n))]}{1+p(x_n, x_n)} - b\, p(x_n, x_n)$$

$$\leq p\,(x_{0i}, x_{n+1}) + p\,(x_{n+1}, T_i x_{0i}) \rightarrow 0$$

Hence, $p\,(x_{0i}, T_i x_{0i}) \rightarrow 0$

That is, $x_{0i} \in T_i x_{0i}$ for each i, and this completes the proof.

3- Fixed point theorem for the set-valued mapping in a partial metric space

Nadler [1] proved that a contraction set-valued mapping in a complete metric space has a fixed point. In another study [7], it was proved that a contractive set-valued mapping in a complete partial metric space has a fixed point.

In this section, we proved the existence of a fixed point for set-valued mapping in a complete partial metric space without using the contraction condition.

Let (X, p) be a complete partial metric space and N(X) be a set of all nonempty subsets of X. Let $T: X \rightarrow N(X)$ be set-valued mapping. A point $x_0 \in X$ is said to be fixed point of T if $x_0 \in T\, x_0$.

Define g: X → R by g(x) = p(x, T(x)). g is lower semi-continuous, if for any $(x_n) \subset X$ and $x \in X$, if $x_n \to x$ then, $g(x) \le \lim_{n\to\infty} g(x_n)$.
For a positive constant a ∈ (0, 1), define the set $I_a^x \subset X$ as:

$$I_a^x = \{ y \in T(x): a\, p(x,y) \le p(x,T(x))\}.$$

Theorem 3.1

Let (X, p) be a complete partial metric space and T: X →C(X) be a set-valued mapping. If there exists a constant k ∈ (0, 1), such that for any x ∈ X there is $y \in I_a^x$ satisfying p (y, T (y)) ≤ k p(x, y), then T has a fixed point in X, provided that k < a and f is lower semi-continuous.

Proof

Since T (x) ∈ C(X) for any x ∈ X, I_a^x is nonempty for any constant a ∈ (0, 1), then for any initial point $x_0 \in X$ there exists $x_1 \in I_a^{x_0}$, such that $p(x_1,T x_1) \le kp(x_0, x_1)$ and, for $x_1 \in X$, there is $x_2 \in I_a^{x_1}$ satisfying $p(x_2,T x_2) \le k\, p(x_1, x_2)$.
By continuing this process, we can get an iterative sequence $(x_n)_{n=0}^{\infty}$ where $x_{n+1} \in I_a^{x_n}$ and

$$p(x_{n+1},T x_{n+1}) \le k\, p(x_n, x_{n+1}) \quad n= 0,1,2,3,\ldots \tag{1}$$

Since, $x_{n+1} \in I_a^{x_n}$,

then
$$a\, p(x_n, x_{n+1}) \le p(x_n, T x_n) \tag{2}$$

By the above two inequalities, we have

$$p(x_{n+1}, x_{n+2}) \le \frac{k}{a}\, p(x_n, T x_n) \quad , n= 0, 1, 2, \ldots$$

Now, look at the following inequality

$$\begin{aligned} p(x_{n+1}, x_{n+1}) &\le \frac{k}{a} \left[\frac{k}{a}\, p(x_{n-1}, T x_{n-1})\right] \\ &\le \left(\frac{k}{a}\right)^2 \left[\frac{k}{a}\, p(x_{n-2}, T x_{n-2})\right] \\ &\vdots \\ &\le \left(\frac{k}{a}\right)^n p(x_0, T x_0) \end{aligned}$$

By using the fourth property for the partial metric space for any $n, m \in N$, we have

$$\begin{aligned} p(x_n, x_{n+m}) &\le \left[\left(\frac{k}{a}\right)^n + \left(\frac{k}{a}\right)^{n+1} + \cdots + \left(\frac{k}{a}\right)^{n+m-1}\right] p(x_0, x_1) \\ &\le \frac{\left(\frac{k}{a}\right)^n}{1-\frac{k}{a}}\, P(x_0, x_1) \to 0 \text{ as } n \to \infty \quad \text{due to } \frac{k}{a} < 1. \end{aligned}$$

By the definition of $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$, we get for any $m \in N$,

$$\begin{aligned} p^s(x_n, x_{n+m}) &= 2p(x_n, x_{n+m}) - p(x_n, x_n) - p(x_m, x_m) \\ &\le 2\, p(x_n, x_{n+m}) \to 0 \text{ as } n \to \infty \end{aligned}$$

This yields that (x_n) is a Cauchy sequence in (X, p^s).

Since (X, p) is complete partial metric space, then, from lemma 1.5, (X, p^s) is a complete metric space and hence (x_n) converges to some x_0 in X with respect to p^s, so $\lim_{n\to\infty} p^s(x_n, x_0) = 0$.
Again, the sequence (x_n) converges in (X, p^s) to a point x_0 in X if and only if

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x_0) = p(x_0, x_0)$$

That is, $p(x_0, x_0) = \lim_{n\to\infty} p(x_n, x_0) = \lim_{n\to\infty} p(x_n, x_n) = 0$.
Furthermore,
Since $\{g(x_n)\}_{n=0}^{\infty} = \{p(x_n, Tx_n)\}_{n=0}^{\infty}$ is decreasing, then its converges to zero
Since *f* is lower semi continuous, then $0 \le g(x_0) \le \lim_{n\to\infty} g(x_n) = 0$.
Hence, g *(x)* = 0 and then $p(x_0, Tx_0) = 0$.
That is, $x_0 \in T x_0$, and this completes the proof.
Now, we would like to give an example for the set-valued mapping in a complete partial metric space which has a fixed point and in the same time does not satisfy the contraction condition.

Example 3.2

Let $X = \left\{\frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{3^n}, \ldots\right\} \cup \{0, 1\}$

P(x, y) = max {x , y}, for x ,y ∈ X; then X is a complete partial metric space.
Let we define the mapping T: X → C(X) as:

$$T(x) = \begin{cases} \{\frac{1}{3^n} \ , 1\} & , x = \frac{1}{3^n}, n = 0,1,2, \ldots \\ \left\{0 \ , \frac{1}{3}\right\} & , x = 0 \end{cases}$$

Obviously, *T* is not a contractive mapping, but it has a fixed point.
To show this, let $x = \frac{1}{3^n}$, y = 0, then
p(x, y) = max {x, y} = $\frac{1}{3^n}$, H(T(x),T(y)) = max { $\frac{1}{3^n}$, 1 } = 1
That is, there is no $k \in (0,1)$, satisfying the following inequality of contraction condition

$$1 = H(Tx, Ty) \le k\ (p(x,y) = k\ (\frac{1}{3^n})$$

On the other hand, it is easy to show that T has a fixed point by using the previous theorem
f (x) = p(x , T x) = inf $\{p(x,y), y \in Tx\}$ = inf { max $(x,y), y \in Tx$ }
Then,

$$f(x) = \begin{cases} \frac{1}{3^n} & x = \frac{1}{3^n}, n = 1,2,3, \ldots \\ 0 & x = 0 \\ 1 & x = 1 \end{cases}$$

Hence, f is continuous.
Furthermore, by the definition of $I_a^x = \{ y \in T(x) : ap(x,y) \le p(x, T(x)) \}$.
There exists y ∈ $I_{0.3}^x$ such that p(y, T (y)) = $\frac{1}{3}$ $p(x,y)$.
Hence, the existence of a fixed point follows from Theorem 3.1.

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