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On Supplement -Small Submodules

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Abstract.
This paper studied a supplement-small submodule, which is generalization of the small submodule. We present a characterization for the supplement-small submodule. We give an important property of this kind of sub module, also the concept of supplement-small radical of R module is investigated.

Keywords: small submodules, supplement submodules, supplement-small sub modules, supplement-small radical of an R −module.

حول المقاسات الجزئية الصغيرة المكملة

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الخلاصة.
هذه الورقة تدرس المقاسات الجزئية الصغيرة المكملة كتعميم على المقاس الجزئي الصغير. قدمنا توصيف للمقاس الجزئي الصغير المكمل وقمنا بدراسة خصائص مهمة لهذا النوع من المقاسات الجزئية كما تم دراسة مفهوم جذر جاكوبسن للمقاس الجزئي الصغير المكمل .

1. Introduction

Consider a module D is untial left R −module and R be commutative ring with identity, and consider D be a module. A proper submodule A of a module D is called small ($A \ll D$), if there is sub module T of module D so that $D = T + A$, implies that $T = D$ [1] , [2] . A submodule L of D is a supplement in D, if and only if $D = A + L$ and $A \cap L \ll L$, $A \leq D$ [3] , [4] . Different types of small submodules were developed and investigated as a generalization by the numerous authors in [5-17] . The definition we present in this work is as follows: Assuming D is R module, a proper submodule A is mentioned a supplement - small sub module if $A + L = D$, where L submodule of D then L is a supplement submodule in D. We introduce this definition as generalization of small submodules. Several properties of this type of submodules are studies. In [18] the author gives the concept of large −small radical of D, while in our article we define the supplement−small radical of D as follows: the sum of all supplement-small submodule of R module D known as the supplement−small radical of D, it refers as $\mathrm{Rad}_{sp}(D)$.

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2.Supplement-small submodule

Definition 2.1

Assume that D is R-module. A proper sub module A of D is named supplement-small (sp−small) sub module of D, and represented as $(A \ll_{sp} M)$, if $D = A + L$, where $L \leq D$, then L is supplement submodule of D.

Examples and remarks 2.2

1. Each small sub module is sp-small sub module. Thus, in Z −module Z_4, $\{\bar{0}, \bar{2}\} \ll Z_4$ and $\{\bar{0}, \bar{2}\} \ll_{sp} Z_4$ given that, $\{\bar{0}, \bar{2}\} + Z_4 = Z_4$ and Z_4 is supplement in Z_4.

2. The following example told us the opposite of (1) is not true to show that: Z −module Z_6- module, $\{\bar{0}, \bar{3}\}$ is a supplement-small sub module in Z_6, since the only sub module L such as, $\{\bar{0}, \bar{3}\} + L = Z_6$ are $\{\bar{0}, \bar{2}, \bar{4}\}$ and Z_6 as well $\{\bar{0}, \bar{2}, \bar{4}\}$ are supplement .Then $\{\bar{0}, \bar{3}\} \ll_{sp} Z_6$, in similar way $\{\bar{0}, \bar{2}, \bar{4}\} \ll_{sp} Z_6$.

3. Since every direct summand sub module is a supplement sub module then if D is semi-simple module and U is a proper sub module of D, then U is supplement-small sub module in D.

Following propositions give properties of supplement-small sub module.

Proposition 2.3

Assume D is R module and T, Y are sub modules of D so that $T \leq Y \leq D$, if $Y \ll_{sp} D$ then $T \ll_{sp} D$.

Proof:

Suppose L is a sub module of D so that $T + L = D$. Since $T \leq V$, then $Y + L = D$ and as $Y \ll_{sp} D$ therefore, L is a supplement submodule in D and hence $T \ll_{sp} D$.

Corollary 2.4

Assume D is an R −module such that $K_1 \ll_{sp} D$ or, $K_2 \ll_{sp} D$ then $K_1 \cap K_2 \ll_{sp} D$.

Proof: If $K_1 \ll_{sp} D$, and since $K_1 \cap K_2 \leq K_1 \leq D$ then from Proposition 2.3, we have $K_1 \cap K_2 \ll_{sp} D$.

Remark 2.5

An opposite of corollary 2.4 is untrue this is because for instance, in Z_{18} as Z – module, let $K_1 = 2Z_{18}$, $K_2 = 9Z_{18}$ such that $K_1 \cap K_2 = 0 \ll_{sp} Z_{18}$. But $2Z_{18} + 3Z_{18} = Z_{18}$ and $3Z_{18}$ is not supplement submodule in Z_{18}. So, K_1 not supplement – small submodule in Z_{18}.

Corollary 2.6:

Assume D is an R −module and K_1, K_2 are submodules of D, if $K_1 + K_2 \ll_{sp} D$ then $K_1 \ll_{sp} D$ and $K_2 \ll_{sp} D$.

Proof:

Since $K_1 \leq K_1 + K_2 \leq D$ and $K_1 + K_2 \ll_{sp} D$, therefore from Proposition 2.3, we have $K_1 \ll_{sp} D$. Similarly, we have $K_2 \ll_{sp} D$.

The symbols (SUSP) refers to the summation for two supplement sub modules of R module D is again supplement, and then the module D has the supplement sum property.

Remark 2.7

Assume D is an R −module with (SUSP), and W_1, W_2 are submodules of D, if $W_1 \ll_{sp} D$ and $W_2 \ll_{sp} D$, then $W_1 + W_2 \ll_{sp} D$.

Proof:

suppose L be sub module of D so that $(W_1 + W_2) + T = D$, implies $W_1 + (W_2 + T) = D$, and $W_1 \ll_{sp} D$, then $(W_2 + T) \leq_{sp} D$. Since $T \leq W_2 + T \leq_{sp} M$, but $T \leq_{sp} T$ and $\{0\} \leq_{sp} W_2$ hence $T = T + \{0\} \leq_{sp} W_2 + T$ by (SUSP), and so $T \leq_{sp} W_2 + T \leq_{sp} D$. Then $T \leq_{sp} D$ according to [19] Proposition 8.2.7, P. 173], we obtain $W_1 + W_2 \ll_{sp} D$.

Proposition 2.8

Assume D be R module has (SUSP), and let V, L be sub modules of D so that $V \leq L \leq D$, if $V \ll_{sp} D$, then $V \ll_{sp} L$, where L be direct summand of D.

Proof:

Suppose T is sub module of L so that $V + T = L$. Because L be direct summand of D, then $D = L \oplus W$ where W a submodule of D, implies $L + W = D$, so $(V + T) + W = D$, and hence $V + (T + W) = D$, since $V \ll_{sp} D$, then $T + W \leq_{sp} D$. Thus, $T \leq T + W \leq_{sp} D$, since $T \leq_{sp} T$ and $\{0\} \leq_{sp} W$, $T = T + \{0\} \leq_{sp} T + W$, therefore $T \leq_{sp} T + W \leq_{sp} D$, as a result $T \leq_{sp} D$ according to [19] ,Proposition 8.2.7, P. 173]. As well as since $T \leq L$ and from [19] ,Proposition 8.2.7, P. 173] we have, $T \leq_{sp} L$ and hence $V \ll_{sp} L$.

Proposition 2.9

Suppose D is R module, with B, V are sub modules of D so that $Y \leq V \leq D$. If $V \ll_{sp} D$, then $\frac{V}{Y} \ll_{sp} \frac{D}{Y}$.

Proof:

To prove $\frac{V}{Y} \ll_{sp} \frac{D}{Y}$, suppose $\frac{U}{Y}$ be sub module of $\frac{D}{Y}$ so that $\frac{V}{Y} + \frac{U}{Y} = \frac{D}{Y}$, therefore, $\frac{V+U}{Y} = \frac{D}{Y}$ and hence $V + U = D$, as $V \ll_{sp} D$, we get $U \leq_{sp} D$ and from [19], Proposition 8.2.6, p.172] we obtain $\frac{U}{Y} \leq_{sp} \frac{D}{Y}$, and then we have $\frac{V}{Y} \ll_{sp} \frac{D}{Y}$.

Proposition 2.10

Suppose D is R module, with H, V are sub modules of D and D has (SUSP) , H is a supplement in D such that $H \leq V \leq D$, if $\frac{V}{H} \ll_{sp} \frac{D}{H}$, then $V \ll_{sp} D$.

Proof:

Let $V + U = D$,so $\frac{V}{H} + \frac{U+H}{H} = \frac{D}{H}$ since $\frac{V}{H} \ll_{sp} \frac{D}{H}$ then $\frac{U+H}{H} \leq_{sp} \frac{D}{H}$ and from [20, Proposition 8.2.6, p.172], $U + H \leq_{sp} D$, as $U \leq_{sp} U$ and $\{0\} \leq_{sp} H$, so $U \leq_{sp} U + H$, but $U \leq U + H \leq_{sp} D$, hence $U \leq_{sp} D$ from [19] , Proposition 8.2.7, p.173], and then $U \leq_{sp} M$ this implies $V \ll_{sp} D$.

Proposition 2.11

Assuming $f: D_1 \to D_2$ an isomorphism with D_1 and D_2 be R modules, if $K \ll_{sp} D_1$, then $f(K) \ll_{sp} D_2$.

Proof:

Suppose that $W \leq D_2$ such that $f(K) + W = D_2$, since f is an epimorphism we have $f^{-1}(f(K) + W) = f^{-1}(D_2)$, then $K + f^{-1}(W) = D_1$, and because $K \ll_{sp} D_1$, we obtain $f^{-1}(W) \leq_{sp} D_1$, so there is a submodule H in D_1. From that we have $f^{-1}(W) + H = D_1$ as well as, $f^{-1}(W) \cap H \ll f^{-1}(W)$. Since f is monomorphism we get $f(f^{-1}(W) + H) = f(D_1)$ hence $W + f(H) = D_2$ and $f(f^{-1}(W) \cap H) \ll f(f^{-1}(W))$, then $W \cap f(H) \ll W$, and so W is supplement in D_2 and hence $f(K) \ll_{sp} D_2$.

The symbols (SUIP) refers to the intersection of two supplement submodules of module D is again supplements.

The following proposition describes that the direct sum of sp-small submodule is also sp-small submodule.

Proposition 2.12

Assume D is an R −module with (SUIP), and K_1, K_2 are submodules of D, then $K_1 \ll_{sp} D$ and $K_2 \ll_{sp} D$, if and only if $K_1 \oplus K_2 \ll_{sp} D$.

Proof:

(⇒) Suppose U be sub module of D, so that $K_1 + K_2 + U = D$. Because of $K_1 \ll_{sp} D$ and $K_2 \ll_{sp} D$, then $K_2 + U \leq_{sp} D$, as well as $K_1 + U \leq_{sp} D$. Since D is (SUIP), we

get $(K_1 + U) \cap (K_2 + U) \leq_{sp} D$ and hence $U = (K_1 + U) \cap (K_2 + U) \leq_{sp} D$, so $U \leq_{sp} D$. Thus, $K_1 \oplus K_2 \ll_{sp} D$.
(⇐) Since $K_1 \leq K_1 \oplus K_2 \leq D$, as $K_1 \oplus K_2 \ll_{sp} D$. Therefore, by Proposition 2.3, we get $K_1 \ll_{sp} D$. Similarly, we have $K_2 \ll_{sp} D$.

Theorem 2.13

Assume W, Y and U are sub modules of R module D with (SUSP) , $W \leq Y \leq U \leq D$,where W , Y are supplement submodules in D, then $\frac{U}{W} \ll_{sp} \frac{D}{W}$ if and only if $\frac{U}{Y} \ll_{sp} \frac{D}{Y}$ and $\frac{Y}{W} \ll_{sp} \frac{D}{W}$.

Proof:
(⇒) Let $\frac{U}{W} \ll_{sp} \frac{D}{W}$. We have to demonstrate $\frac{U}{Y} \ll_{sp} \frac{D}{Y}$. Let $\frac{V}{Y}$ be sub module of $\frac{D}{Y}$ so that $\frac{U}{Y} + \frac{V}{Y} = \frac{D}{Y}$ then $\frac{U+V}{Y} = \frac{D}{Y}$, hence $U + V = D$, implies $\frac{U+V}{W} = \frac{D}{W}$ then $\frac{U}{W} + \frac{V}{W} = \frac{D}{W}$, since $\frac{U}{W} \ll_{sp} \frac{D}{W}$, then $\frac{V}{W} \leq_{sp} \frac{D}{W}$ as W is supplement in D, hence $V \leq_{sp} D$, by [19], Proposition 8.2.6 ,p.172] and since $Y \leq V \leq D$ then $\frac{V}{Y} \leq_{sp} \frac{D}{Y}$ by [19], Proposition 8.2.6, p.172] and $\frac{U}{Y} \ll_{sp} \frac{D}{Y}$. Now, to prove $\frac{Y}{W} \ll_{sp} \frac{D}{W}$, as $\frac{U}{W} \ll_{sp} \frac{D}{W}$ then $U \ll_{sp} D$ by Proposition 2.9, and so $Y \ll_{sp} D$ by Propositions 2.3 and $\frac{U}{Y} \ll_{sp} \frac{D}{Y}$ from Proposition 2.10.
Conversely, let $\frac{U}{Y} \ll_{sp} \frac{D}{Y}$, we have proved $\frac{U}{W} \ll_{sp} \frac{D}{W}$. Let $\frac{H}{W}$ be sub module of $\frac{D}{W}$ such that $\frac{U}{W} + \frac{H}{W} = \frac{D}{W}$ therefore, $\frac{U+H}{W} = \frac{D}{W}$ from that we have, $U + H = D$, implies $\frac{U+H}{Y} = \frac{D}{Y}$, as well $\frac{U}{Y} + \frac{H}{Y} = \frac{D}{Y}$ as, $\frac{U}{Y} \ll_{sp} \frac{D}{Y}$, then $\frac{H}{Y} \leq_{sp} \frac{D}{Y}$. Since Y is supplement, hence $H \leq_{sp} D$ by [19] , Proposition 8.2.6, p.172] and since $W \leq H \leq D$, then $\frac{H}{W} \leq_{sp} \frac{D}{W}$ from [19], Proposition 8.2.6, p.172], hence $\frac{U}{W} \ll_{sp} \frac{D}{W}$.

3. Supplement-small radical submodule

Rad (D) : the sum of all small sub module of D, see [4].
The following definition is a generalization of the above concept:

Definition 3.1

Sum of every supplement−small sub module of R module D is called the supplement-small radical of D, its denoted $\mathrm{Rad}_{sp}(D)$.

$$\mathrm{Rad}_{sp}(D) = \sum \{ U \leq D \mid U \ll_{sp} D\}.$$

Remarks and examples 3.2

1. $\mathrm{Rad}_{sp}(Z_4) = \{\bar{0}, \bar{2}\}$.
2. $\mathrm{Rad}_{sp}(Z_6) = Z_6$.
3. Each radical of D is the supplement−small radical of D.
4. The converse of (3) is not true, since $\mathrm{Rad}_{sp}(Z_6) = Z_6$, but $\mathrm{Rad}(Z_6) = \{\bar{0}\}$.
5. In Z_{24} as Z- module $\mathrm{Rad}_{sp}(Z_{24}) = 6Z_{24}$.
6. If D is a semisimple, then $\mathrm{Rad}_{sp}(D) = D$, as the example Z_6 as Z- module.

Proposition 3.3

Assume D is an $R-$ module. If $f: D \to N$ be an isomorphism then $f(\mathrm{Rad}_{sp}(D)) \leq \mathrm{Rad}_{sp}(N)$.

Proof:
Let $L \ll_{sp} D$, $f(L) \ll_{sp} N$ by Proposition 2.11 hence $f(L) \leq Rad_{sp}(N)$. Thus, $f\left(Rad_{sp}(D)\right) = f\sum_{L \ll_{sp}}(L) = \sum_{L \ll_{sp}} f(L) \leq Rad_{sp}(N)$. So, $f\left(Rad_{sp}(D)\right) \leq Rad_{sp}(N)$.

Proposition 3.4

Assume D is R module, as well N is sub module of D, so that U supplement sub module of D ($U \leq_{sp} D$), then $Rad_{sp}(U) \leq Rad_{sp}(D)$.

Proof:
Suppose K be sub module of $Rad_{sp}(U)$ so that $K \ll_{sp} U$ because $U \leq_{sp} D$, as well $K \leq U \leq D$, so from [19] , proposition 8.2.7, p.173], we have $K \leq_{sp} D$ therefore K is a submodule of $Rad_{sp}(D)$, and so $Rad_{sp}(U) \leq Rad_{sp}(D)$.

Proposition 3.5

The supplement-small sub module of R module D is an arbitrary sum of these sub modules if and only if $Rad_{sp}(D) \ll_{sp} D$.

Proof:
(⇒) Since $Rad_{sp}(D)$ = sum of every supplement-small submodules then by Definition 3.1 we have $Rad_{sp}(D) \ll_{sp} D$.
(⇐) Suppose $Rad_{sp}(D) \ll_{sp} D$. Let $\{K_\alpha\}_{\alpha\in\wedge}$ be a family of sp-small submodules of D. As $\sum_{\alpha\in\wedge} K_\alpha \leq Rad_{sp}(D) \ll_{sp} D$. Therefore, $\sum_{\alpha\in\wedge} K_\alpha \ll_{sp} D$, by Proposition 2.3.

4. Conclusions

In this article, several of supplement-small submodules are presented as a generalization of small-submodules. For example, where $f: D_1 \to D_2$ be an isomorphism such that D_1 and D_2 are R −modules, if $K \ll_{sp} D_1$, then $f(K) \ll_{sp} D_2$. Also, the direct sum of sp-small submodule is also sp-small submodule are provided. Lastly, the concept of supplement-small radical of R module D is studied.

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