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A Generalized Subclass of Starlike Functions Involving Jackson's $(p,q)-$ Derivative

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Abstract
In this paper, we generalize many earlier differential operators which were studied by other researchers using our differential operator. We also obtain a new subclass of starlike functions to utilize some interesting properties.

Keywords: Differential operator, starlike functions, coefficient inequality, inclusion properties, convexity.

1. Introduction

Let A represents the class of all analytic functions φ defined in the open unit disk $\Theta=\{z\in \mathrm{C}: |z|<1\}$, and normalized by the conditions $\varphi(0)=0$ and $\varphi'(0)=1$. Therefore, each $\varphi\in A$ has a Taylor-Maclaurin series extension of the form:

$$\varphi(z)=z+\sum_{h=2}^{\infty}a_h z^h,(z\in\Theta) \tag{1.1}$$

Furthermore, let S represents the class of all functions $\varphi\in A$ which are univalent in Θ. The quantum calculus (henceforth $q-$ calculus) is considered as a crucial tool that is used to explore the subclasses of analytic functions. $q-$ calculus operators were used by Kanas and Raducanu to investigate some significant classes of functions which are analytic in Θ [1]. The importance of the fractional calculus applications is obvious in many topics of mathematics, such as in the fields of $q-$ transform analysis, ordinary fractional calculus, and operator theory. Recently, researchers paid more attention to the area of $q-$ calculus and several new operators have been proposed. The application of $q-$ calculus was first founded by Jackson who developed the $q-$ integral and $q-$ derivative in a systematic way [2]. After that, through several studies on quantum groups, the geometrical interpretation of $q-$ analysis was identified. Unlike the typical calculus, this calculus has no limits notion. A good detailed work on the calculus and it's applications in operator theory is found in aprevious report [3], while more information were provided in other articles [4, 5].

The main structure of $(p,q)-$ calculus was established on only one parameter, but since then it was generalized to the post-quantum calculus (represented by $(p,q)-$ calculus). In this section, we assume that we can obtain calculus by substituting $p=1$ in calculus.

To be fulfilled, some brief notations and definitions of $(p,q)-$ calculus are provided below: For Jackson's derivative where $0<p<q\le 1$ and $\varphi\in A$, the following is provided [2]:

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$$D_{p,q}\varphi(z)=\begin{cases}\frac{\varphi(pz)-\varphi(qz)}{(p-q)z} & for \quad z\neq 0.\\ \varphi'(0) & for \quad z=0.\end{cases} \tag{1.2}$$

From (1.2), we have

$$D_{p,q}\varphi(z)=1+\sum_{h=2}^{\infty}[h]_{p,q}a_h z^{h-1} \tag{1.3}$$

Where

$$[h]_{p,q}=p^{h-1}+p^{h-2}q+p^{h-3}q^2+...+pq^{h-2}+q^{h-1}=\frac{p^h-q^h}{p-q} \tag{1.4}$$

is named $(p,q)-$ bracket. It's notable that when $p=1$, the bracket is an obvious generalization of the $q-$ number, that is

$$[h]_{1,q}=\frac{1-q^h}{1-q}=[h]_q, q\neq 1$$

For $p=1$, one can notice that the Jackson's $(p,q)-$ derivative will be reduced to the $q-$ derivative, as previously described [2]. It was clearly proved that for a function $\gamma(z)=z^h$, the $D_{p,q}\gamma(z)=D_{p,q}z^h=\frac{p^h-q^h}{p-q}z^{h-1}=[h]_{p,q}z^{h-1}$ is obtained. For $\varphi\in A$, the Sălăgean $(p,q)-$ differential operator is defined as follows [6]:

$$\begin{aligned}\Gamma^0_{p,q}\varphi(z)&=\varphi(z),\\ \Gamma^1_{p,q}\varphi(z)&=zD_{p,q}\varphi(z),\\ &\dots\\ \Gamma^k_{p,q}\varphi(z)&=\Gamma^1_{p,q}(\Gamma^{k-1}_{p,q}\varphi(z)),\\ &=z+\sum_{h=2}^{\infty}[h]^k_{p,q}a_h z^h,\quad (k\in \mathrm{N}_0=\mathrm{N}\cup\{0\}, z\in\Theta)\end{aligned} \tag{1.5}$$

It's observable when $p=1$ and $\lim_{q\to 1^-}$, the well-known Sălăgean operator is obtained [6]:

$$\Gamma^k\varphi(z)=z+\sum_{h=2}^{\infty}h^k a_h z^h,\ (z\in\Theta) \tag{1.6}$$

Now let

$$\Lambda^{0,k}_{\beta,\delta,\lambda,p,q}\varphi(z)=\Gamma^{k}_{p,q}\varphi(z),$$

$$\Lambda^{1,k}_{\beta,\delta,\lambda,p,q}\varphi(z)=(1-\beta(\delta-\lambda))\Gamma^{k}_{p,q}\varphi(z)+\beta(\delta-\lambda)z\left(\Gamma^{k}\varphi(z)\right)'$$

$$=z+\sum_{h=2}^{\infty}[h]^{k}_{p,q}[1+\beta(\delta-\lambda)(h-1)]a_h z^h. \tag{1.7}$$

$$\Lambda^{2,k}_{\beta,\delta,\lambda,p,q}\varphi(z)=(1-\beta(\delta-\lambda))\Lambda^{1,k}_{\beta,\delta,\lambda,p,q}\varphi(z)+\beta(\delta-\lambda)z\left(\Lambda^{1,k}_{\beta,\delta,\lambda,p,q}\varphi(z)\right)'$$

$$=z+\sum_{h=2}^{\infty}[h]^{k}_{p,q}[1+\beta(\delta-\lambda)(h-1)]^2 a_h z^h.$$

In general, we have

$$\Lambda^{\zeta,k}_{\beta,\delta,\lambda,p,q}\varphi(z)=(1-\beta(\delta-\lambda))\Lambda^{\zeta-1,k}_{\beta,\delta,\lambda,p,q}\varphi(z)+\beta(\delta-\lambda)z\left(\Lambda^{\zeta-1,k}_{\beta,\delta,\lambda,p,q}\varphi(z)\right)'$$

$$=z+\sum_{h=2}^{\infty}[h]^{k}_{p,q}[1+\beta(\delta-\lambda)(h-1)]^{\zeta}a_h z^h \tag{1.8}$$

Where $\beta\geq 0,\lambda\geq 0,\delta\geq 0$ *and* $\zeta\in\mathrm{N}_0$.

It is observable that we have $\Lambda^{0,0}_{\beta,\delta,\lambda,p,q}\varphi(z)=\varphi(z),$ and $\Lambda^{1,0}_{\beta,\delta,\lambda,p,q}\varphi(z)=z\varphi'(z).$ It is noticeable that when $p=1$, the differential operator $\Lambda^{\zeta,k}_{\beta,q}\varphi(z)$ that was defined and studied by Frasin and Murugusundaramoorthy is obtained [7]. Also, it is noticeable that when $p=1$ and $\lim_{q\to 1}$, the following differential operator is obtained:

$$\Lambda^{\zeta,k}_{\beta}\varphi(z)=z+\sum_{h=2}^{\infty}h^{k}[1+\beta(\delta-\lambda)(h-1)]^{\zeta}a_h z^h$$

It is noticeable that when $\delta=1$ and $\lambda=0$, we find the differential operator $\Lambda^{\zeta,k}_{\beta,p,q}\varphi(z)$ that was defined and studied by Feras Yousef [8]. Furthermore, when $k=0$ we find the differential operator $\Lambda^{\zeta}_{\beta,\delta,\lambda}$ that was defined and studied by Ibrahim and Darus [9, 10], and when $\delta=1$, $\lambda=0$ and $k=0$ we identify the differential operator Λ^{ζ}_{β} defined and studied by Al-Oboudi [10], while if $\zeta=0$, we identify Sălăgean differential operator Λ^{ζ} [6].

By using the differential operator $\Lambda^{\zeta,k}_{\beta,\delta,\lambda,p,q}\varphi(z)$, we say that a function $\varphi(z)$ belonging to A is in the class $Q^{\zeta,k}_{p,q}(\beta,\delta,\lambda,\mu,b)$ if and only if

$$\left|\frac{(1-\beta(\delta-\lambda))z\left(\Lambda^{\zeta,k+1}_{\beta,\delta,\lambda,p,q}\varphi(z)\right)'+\beta(\delta-\lambda)z\left(\Lambda^{\zeta,k+1}_{\beta,\delta,\lambda,p,q}\varphi(z)\right)'}{(1-\beta(\delta-\lambda))\Lambda^{\zeta,k}_{\beta,\delta,\lambda,p,q}\varphi(z)+\beta(\delta-\lambda)\Lambda^{\zeta,k}_{\beta,\delta,\lambda,p,q}\varphi(z)}-1\right|$$

$$<\mu\left|\frac{(1-\beta(\delta-\lambda))z\left(\Lambda^{\zeta,k+1}_{\beta,\delta,\lambda,p,q}\varphi(z)\right)'+\beta(\delta-\lambda)z\left(\Lambda^{\zeta,k+1}_{\beta,\delta,\lambda,p,q}\varphi(z)\right)'}{(1-\beta(\delta-\lambda))\Lambda^{\zeta,k}_{\beta,\delta,\lambda,p,q}\varphi(z)+\beta(\delta-\lambda)\Lambda^{\zeta,k}_{\beta,\delta,\lambda,p,q}\varphi(z)}-b\right|,\quad (k,\zeta\in\mathrm{N}_0)$$

(1.9)

for some $\mu(0\leq\mu<1)$, $\beta,\delta,\lambda\geq 0,$ and $0\leq b<1$ for all $z\in\Theta.$

Let T denotes the subclass of A consisting of functions of the form

$$\varphi(z)=z-\sum_{h=2}^{\infty}a_h z^h \quad (a_h\geq 0, z\in\Theta) \tag{1.10}$$

Further, we define the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$ by

$P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)=Q_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)\cap T$

The main target of this paper is to provide a systematic investigation of some important features and characteristics of the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$. Some interesting corollaries and natural consequences of the main findings are also considered. Some important techniques used earlier by many researchers were applied in this work (see Al-Hawary et al. [11, 12], Aouf and Srivastava [13], and Frasin et al. [14- 19]).

2. Coefficient inequality

In this section, we find the coefficient inequality for the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$.

Theorem 2.1. Let the function $\varphi(z)$ be defined by (1.10). Then $\varphi(z)\in P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$ if and only if

$$\sum_{h=2}^{\infty}[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}a_h\le\mu(1-b) \tag{2.1}$$

The result is sharp.

$$f(z)=z-\frac{\mu(1-b)}{[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}}z^h \tag{2.2}$$

Proof. Suppose that the inequality (2.1) holds. Then we have for $z\in\Theta$ and $|z|<1$:

$$\Big|(1-\beta(\delta-\lambda))z\left(\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k+1}\varphi(z)\right)'+\beta(\delta-\lambda)z\left(\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k+1}\varphi(z)\right)'-$$

$$(1-\beta(\delta-\lambda))\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k}\varphi(z)-\beta(\delta-\lambda)\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k}\varphi(z)\Big|-$$

$$\mu\Big|(1-\beta(\delta-\lambda))z\left(\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k+1}\varphi(z)\right)'+\beta(\delta-\lambda)z\left(\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k+1}\varphi(z)\right)'-$$

$$b(1-\beta(\delta-\lambda))\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k}\varphi(z)-b\beta(\delta-\lambda)\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k}\varphi(z)\Big|$$

$$=\left|\sum_{h=2}^{\infty}[h]_{p,q}^{k}(h[h]_{p,q}-1)[1+(h-1)\beta(\delta-\lambda)]^{\zeta}a_hz^h\right|-$$

$$\mu\left|z(1-b)-\sum_{v=2}^{\infty}[h]_{p,q}^{k}(h[h]_{p,q}-b)[1+(h-1)\beta(\delta-\lambda)]^{\zeta}a_hz^h\right|$$

$$\le\sum_{h=2}^{\infty}[h]_{p,q}^{k}\left\{(1+\mu)h[h]_{p,q}-1-b\mu\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}a_hz^h-\mu(1-b)$$

$$\le 0$$

where $\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k}\varphi(z)$ is given by (1.8).

This implies

$$\sum_{h=2}^{\infty}[h]_{p,q}^{k}\left\{(1+\mu)h[h]_{p,q}-1-b\mu\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}a_hz^h\le\mu(1-b)$$

which shows that $\varphi(z)\in P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$. For the converse, assume that

$$\frac{\left|\frac{(1-\beta(\delta-\lambda))z\left(\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k+1}\varphi(z)\right)'+\beta(\delta-\lambda)z\left(\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k+1}\varphi(z)\right)'}{(1-\beta(\delta-\lambda))\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k}\varphi(z)+\beta(\delta-\lambda)\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k}\varphi(z)}-1\right|}{\mu\left|\frac{(1-\beta(\delta-\lambda))z\left(\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k+1}\varphi(z)\right)'+\beta(\delta-\lambda)z\left(\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k+1}\varphi(z)\right)'}{(1-\beta(\delta-\lambda))\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k}\varphi(z)+\beta(\delta-\lambda)\Lambda_{\beta,\delta,\lambda,p,q}^{\zeta,k}\varphi(z)}-b\right|}$$

$$\frac{\left|-\sum_{v=2}^{\infty}[h]_{p,q}^{k}(h[h]_{p,q}-1)[1+(h-1)\beta(\delta-\lambda)]^{\zeta}a_h z^h\right|}{\mu\left|z(1-b)-\sum_{h=2}^{\infty}[h]_{p,q}^{k}(h[h]_{p,q}-b)[1+(h-1)\beta(\delta-\lambda)]^{\zeta}a_h z^h\right|} \tag{2.3}$$

$$<1$$

Since the $\operatorname{Re}(z)\le|z|$ for all z, it follows from (2.3) that

$$\operatorname{Re}\left\{\frac{\sum_{h=2}^{\infty}[h]_{p,q}^{k}(h[h]_{p,q}-1)[1+(h-1)\beta(\delta-\lambda)]^{\zeta}a_h z^h}{z(1-b)\mu-\mu\sum_{h=2}^{\infty}[h]_{p,q}^{k}(h[h]_{p,q}-b)[1+(h-1)\beta(\delta-\lambda)]^{\zeta}a_h z^h}\right\}<1. \tag{2.4}$$

By choosing values of z on the real axis and letting $|z|\to 1^-$ through the real values, we obtain

$$\sum_{h=2}^{\infty}[h]_{p,q}^{k}(h[h]_{p,q}-1)[1+(h-1)\beta(\delta-\lambda)]^{\zeta}a_h$$

$$\le(1-b)\mu-\mu\sum_{h=2}^{\infty}[h]_{p,q}^{k}(h[h]_{p,q}-b)[1+(h-1)\beta(\delta-\lambda)]^{\zeta}a_h$$

This gives the required condition.

Corollary 2.2. Let the function $\varphi(z)$, defined by (1.10), be in the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b).$ Then

$$a_h\le\frac{\mu(1-b)}{[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}},\quad(h\ge 2). \tag{2.5}$$

The inequality in (2.1) is obtained for the function $\varphi(z)$ given by (2.2).

3. Growth and Distortion Theorems

Theorem 3.1. Let the function $\varphi(z),$ defined by (1.10), be in the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b).$ Then for $|z|=r<1,$

$$\left|\Lambda_{\beta,\delta,\lambda,p,q}^{i,j}\varphi(z)\right|\ge r-\frac{\mu(1-b)}{[2]_{p,q}^{k-j}\left\{2[2]_{p,q}(1+\mu)-\mu b-1\right\}[1+\beta(\delta-\lambda)]^{\zeta-i}}r^2 \tag{3.1}$$

and

$$\left|\Lambda_{\beta,\delta,\lambda,p,q}^{i,j}\varphi(z)\right|\le r+\frac{\mu(1-b)}{[2]_{p,q}^{k-j}\left\{2[2]_{p,q}(1+\mu)-\mu b-1\right\}[1+\beta(\delta-\lambda)]^{\zeta-i}}r^2, \tag{3.2}$$

$$(0\le i\le\zeta,0\le j\le k,z\in\Theta)$$

The inequalities in (3.1) and (3.2) are obtained for $\varphi(z)$ given by

$$\varphi(z)=z-\frac{\mu(1-b)}{[2]_{p,q}^{k}\left\{2[2]_{p,q}(1+\mu)-\mu b-1\right\}[1+(\beta(\delta-\lambda)]^{\zeta}} \tag{3.3}$$

Proof. Note that the function $\varphi(z)\in P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$ if and only if

$\Lambda_{\beta,\delta,\lambda,p,q}^{i,j}\varphi(z)\in P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$

and that

$$\Lambda_{\beta,\delta,\lambda,p,q}^{i,j}\varphi(z)=z-\sum_{h=2}^{\infty}[h]_{p,q}^{j}[1+(h-1)\beta(\delta-\lambda)]^{i}a_h z^h \tag{3.4}$$

By the theorem 2.1

$$[2]^k_{p,q}\left\{2[2]_{p,q}(1+\mu)-\mu b-1\right\}[1+(\beta(\delta-\lambda)]^{\zeta}\sum_{h=2}^{\infty}[h]^j_{p,q}(1+\beta(\delta-\lambda))^i a_h$$
$$\leq \sum_{h=2}^{\infty}[h]^k_{p,q}\left\{2[2]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta} a_h \leq \mu(1-b) \tag{3.5}$$

Which implies,

$$\sum_{h=2}^{\infty}[h]^j_{p,q}[1+\beta(\delta-\lambda)]^i a_h z^h \leq \frac{\mu(1-b)}{[2]^k_{p,q}\left\{2[2]_{p,q}(1+\mu)-\mu b-1\right\}[1+(\beta(\delta-\lambda)]^{\zeta-i}} \tag{3.6}$$

The assertions (3.1) and (3.2) of Theorem4.1 would now follow readily from (3.4) and (3.6). Finally, we note that the equalities (3.1) and (3.2) are achieved for the function $\varphi(z)$, defined by

$$\Lambda^{i,j}_{\beta,\delta,\lambda,p,q}\varphi(z)=z-\frac{\mu(1-b)}{[2]^k_{p,q}\left\{2[2]_{p,q}(1+\mu)-\mu b-1\right\}[1+(\beta(\delta-\lambda)]^{\zeta-i}}z^2 \tag{3.7}$$

Hence, the proof has been completed.

Taking $i=j=0$ in Theorem 2.1, we obtain this corollary.

Corollary 3.2.

Let $\varphi(z)$, defined by (1.10), be in the class $P^{\zeta,k}_{p,q}(\beta,\delta,\lambda,\mu,b)$. Then, for $|z|=r<1$,

$$|\varphi(z)|\geq r-\frac{\mu(1-b)}{[2]^k_{p,q}\left\{2[2]_{p,q}(1+\mu)-\mu b-1\right\}[1+\beta(\delta-\lambda)]^{\zeta}}r^2 \tag{3.8}$$

and

$$|\varphi(z)|\leq r+\frac{\mu(1-b)}{[2]^k_{p,q}\left\{2[2]_{p,q}(1+\mu)-\mu b-1\right\}[1+\beta(\delta-\lambda)]^{\zeta}}r^2 \tag{3.9}$$

The equalities in (3.8) and (3.9) are achieved for the function $\varphi(z)$ given by (4.3).

4. Inclusion properties

We begin this section by showing the following inclusion relation.

Theorem 4.1. Let the hypotheses of theorem th1 be satisfied. Then

$$P^{\zeta,k}_{p,q}(\beta_1,\delta,\lambda,\mu_1,b)\supseteq P^{\zeta,k}_{p,q}(\beta_2,\delta,\lambda,\mu,b)$$
$$P^{\zeta,k}_{p,q}(\beta,\delta_1,\lambda,\mu,b)\supseteq P^{\zeta,k}_{p,q}(\beta,\delta_1,\lambda,\mu,b)$$
$$P^{\zeta,k}_{p,q}(\beta,\delta,\lambda_1,\mu,b)\supseteq P^{\zeta,k}_{p,q}(\beta,\delta,\lambda_2,\mu,b)$$
$$P^{\zeta,k}_{p,q}(\beta,\delta,\lambda,\mu_1,b)\supseteq P^{\zeta,k}_{p,q}(\beta,\delta,\lambda,\mu_2,b)$$

Proof. Let the function $\varphi(z)$, defined by (1.10), be in the class $P^{\zeta,k}_{p,q}(\beta,\delta,\lambda,\mu,b)$ and let $\beta_1\geq\beta_2$. Then, by theorem 2.1, we have

$$\sum_{h=2}^{\infty}[h]^k_{p,q}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta_1(\delta-\lambda)]^{\zeta}a_h$$
$$\leq \sum_{h=2}^{\infty}[h]^k_{p,q}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta_1(\delta-\lambda)]^{\zeta}a_h$$
$$\leq \mu(1-b)$$

Hence, $P^{\zeta,k}_{p,q}(\beta_1,\delta,\lambda,\mu_1,b)\supseteq P^{\zeta,k}_{p,q}(\beta_2,\delta,\lambda,\mu,b).$

and

$$\sum_{h=2}^{\infty}[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta_1-\lambda)]^{\zeta}a_h$$
$$\leq\sum_{h=2}^{\infty}[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta_2-\lambda)]^{\zeta}a_h$$
$$\leq\mu(1-b)$$

Hence, $P_{p,q}^{\zeta,k}(\beta,\delta_1,\lambda,\mu,b)\supseteq P_{p,q}^{\zeta,k}(\beta,\delta_2,\lambda,\mu,b)$.

$$\sum_{h=2}^{\infty}[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda_1)]^{\zeta}a_h$$
$$\leq\sum_{h=2}^{\infty}[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda_1)]^{\zeta}a_h$$
$$\leq 1-\mu(1-b)$$

Hence, $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda_1,\mu,b)\supseteq P_{p,q}^{\zeta,k}(\beta,\delta,\lambda_2,\mu,b)$.

Employing a similar procedure, we can prove that $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu_1,b)\supseteq P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu_2,b)$.

5. Closure Theorems

This section has begun with proving that the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$ is closed under convex linear combinations.

Theorem 5.1. The class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$ is a convex set.

Proof. Let the functions

$$\varphi_{\varepsilon}(z)=z-\sum_{h=2}^{\infty}a_{\varepsilon,h}z^h \quad (a_{\varepsilon,h}\geq 0;\varepsilon=1,2;z\in\Theta) \tag{5.1}$$

be in the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$. It is sufficient to show that the function $\gamma(z)$ defined by

$$\gamma(z)=\xi\varphi_1(z)+(1-\xi)\varphi_2(z) \tag{5.2}$$

is also in the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$. Since, for $0\leq\xi\leq 1$,

$$\gamma(z)=z-\sum_{h=2}^{\infty}\left\{\xi a_{1,h}+(1-\xi)a_{2,h}\right\}z^c, \tag{5.3}$$

by using theorem 2.1 , we have

$$\sum_{h=2}^{\infty}[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}\left\{\xi a_{1,h}+(1-\xi)a_{2,h}\right\}\leq\mu(1-b) \tag{5.4}$$

which means that $\gamma(z)\in P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$. Hence $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$ is a convex set.

Theorem 5.2. Let $\varphi_1(z)=z$ and

$$\varphi_h(z)=z-\frac{\mu(1-b)}{[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}}z^h,\quad (h\geq 2;k,\zeta\in\mathrm{N}_0)$$

(5.5)

for $0\leq\mu<1$ and $0\leq\beta(\delta-\lambda)\leq 1$. Then $\varphi(z)$ is in the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$ if and only if it could be expressed in the form:

$$\varphi(z)=\sum_{h=1}^{\infty}\omega_h\varphi_h(z), \tag{5.6}$$

where

$$\omega_h\geq 0\ (h\geq 1)\ \ and\ \ \sum_{h=1}^{\infty}\omega_h=1 \tag{5.7}$$

Proof. Assume that

$$\varphi(z)=\sum_{h=1}^{\infty}\omega_h\varphi_h(z)$$

$$=z-\sum_{h=2}^{\infty}\frac{\mu(1-b)}{[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}}\omega_h z^h$$

Then it follows that

$$\sum_{h=2}^{\infty}\frac{[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}}{\mu(1-b)}$$

$$\frac{\mu(1-b)}{[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}}\omega_h=\sum_{h=2}^{\infty}\omega_h=1-\omega_1$$

Thus, by Theorem 2.1, $\varphi(z)\in P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$.

Conversely, suppose that $\varphi(z)$, defined by (1.10), is in the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$. Then

$$a_h\le\frac{\mu(1-b)}{[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}},\quad (h\ge 2;k,\zeta\in\mathrm{N}_0).$$

considering

$$\omega_h=\frac{[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(v-1)\beta(\delta-\lambda)]^{\zeta}}{\mu(1-b)}a_h\quad (h\ge 2;k,\zeta\in\mathrm{N}_0)$$

and

$$\omega_1=1-\sum_{h=2}^{\infty}\omega_h$$

It's observable that $\varphi(z)$ can be expressed in (5.6). Which completes the proof.

6. Radii of close-to-convexity, starlikenss, and convexity

In this section, we shall determine the radii of close-to-convexity, starlikeness, and convexity for the functions belonging to the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$.

Theorem 6.1. Let the function $\varphi(z)$, defined by (1.10), be in the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$. Then $\varphi(z)$ is close-to-convex of order $\sigma(0\le\sigma<1)$ in $|z|<r_1$, where

$$r_1=\inf\left\{\frac{(1-\sigma)h^{-1}[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}}{\mu(1-b)}\right\}^{\frac{1}{(h-1)}},(h\ge 2)\quad (6.1)$$

The result is sharp, with the extremal function $\varphi(z)$ given by (2.2).

Proof. We need to show that

$$\left|\varphi'(z)-1\right|\le 1-\sigma\ \text{for}\ |z|<r_1$$

where r_1 is given by (6.1). Then we yield from definition (1.10)

$$\left|\varphi'(z)-1\right|\le\sum_{h=2}^{\infty}h a_h|z|^{h-1}.$$

Thus,

$$\left|\varphi'(z)-1\right|\le 1-\sigma$$

if

$$\sum_{h=2}^{\infty}\left(\frac{h}{1-\sigma}\right)a_h|z|^{h-1}\leq 1 \quad (6.2)$$

But, by Theorem 6.1, (6.2) holds true if

$$\left(\frac{h}{1-\sigma}\right)|z|^{h-1}\leq\frac{[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}}{\mu(1-b)}$$

that is, if

$$|z|\leq\left(\frac{(1-\sigma)h^{-1}[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}}{\mu(1-b)}\right)^{\frac{1}{(h-1)}} \quad (h\geq 2) \quad (6.3)$$

Theorem 6.1 follows readily form (6.4).

Theorem 6.2. Let $\varphi(z)$, defined by (1.10), be in the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$. Then $\varphi(z)$ is a starlike of order $\sigma(0\leq\sigma<1)$ in $|z|<r_2$, where

$$r_2=\inf\left\{\frac{(1-\sigma)[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}}{\mu(h-\sigma)(1-b)}\right\}^{\frac{1}{(h-1)}},(h\geq 2) \quad (6.4)$$

The result is sharp, with the extremal function $\varphi(z)$ given by (2.2).

Proof. We need to show that

$$\left|\frac{z\,\varphi'(z)}{\varphi(z)}-1\right|\leq 1-\sigma \text{ for } |z|<r_2$$

where r_2 is given by (6.4). Indeed, definition (1.10) implies that

$$\left|\frac{z\,\varphi'(z)}{\varphi(z)}-1\right|\leq\frac{\sum_{h=2}^{\infty}(h-1)a_h|z|^{h-1}}{1-\sum_{h=2}^{\infty}a_h|z|^{h-1}}$$

Thus,

$$\left|\frac{z\,\varphi'(z)}{\varphi(z)}-1\right|\leq 1-\sigma$$

if

$$\sum_{h=2}^{\infty}\left(\frac{h-\sigma}{1-\sigma}\right)a_h|z|^{h-1}\leq 1 \quad (6.5)$$

But, by Theorem 2.1, (6.5) holds true if

$$\left(\frac{h-\sigma}{1-\sigma}\right)|z|^{h-1}\leq\frac{[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}}{\mu(h-\sigma)(1-b)} \quad (6.6)$$

that is, if

$$|z|\leq\left(\frac{(1-\sigma)[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}}{\mu(h-\sigma)(1-b)}\right)^{\frac{1}{(h-1)}} \quad (h\geq 2) \quad (6.7)$$

Theorem 6.2 follows readily form (6.7).

Corollary 6.3. Let the function $\varphi(z)$, defined by (1.10), be in the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$. Then $\varphi(z)$ is a convex of order $\sigma(0\leq\sigma<1)$ in $|z|<r_2$, where

$$r_3=\inf\left\{\frac{(1-\sigma)h^{-1}[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}}{\mu(h-\sigma)(1-b)}\right\}^{\frac{1}{(h-1)}},(h\geq 2) \tag{6.8}$$

The result is sharp, with the extremal function $\varphi(z)$ given by (2.2).

7. Integral means inequality

For any two functions, φ and Γ, analytic in Θ, φ is said to be subordinate to Γ in Θ, written as $\varphi(z)\prec\Gamma(z)$, if there exists a Schwarz function $\omega(z)$, analytic in Θ, with

$\omega(0)=0$ and $|\omega(z)|<1$ for all $z\in\Theta$,

such that $\varphi(z)=\Gamma(\omega(z))$ for all $z\in\Theta$. Furthermore, if the function Γ is univalent in Θ, then we have the following equivalence [10]:

$\varphi(z)\prec\gamma(z)\Leftrightarrow\varphi(0)=\gamma(0)$ and $\varphi(\Theta)\subset\gamma(\Theta)$.

To prove the integral means inequality for functions belonging to the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$, we need the following subordination result found by Littlewood [16].

Lemma 7.1. If the functions φ and Γ are analytic in Θ with $\varphi(z)\prec\Gamma(z)$, then for $\eta>0$ and $z=re^{i\theta}$ $(0<r<1)$,

$$\int_0^{2\pi}|\varphi(z)|^{\eta}\,d\theta\leq\int_0^{2\pi}|\gamma(z)|^{\eta}\,d\theta \tag{7.1}$$

By applying Theorem 2.1 with the extremal function and Lemma 7.1, we achieve the following theorem.

Theorem 7.2. Let $\left\{[h]_{p,q}^{k}\left\{h[h]_{p,q}(1+\mu)-\mu b-1\right\}[1+(h-1)\beta(\delta-\lambda)]^{\zeta}\right\}_{h=2}^{\infty}$ be a non-decreasing sequence. If $\varphi(z)\in P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$, then

$$\int_0^{2\pi}\left|\varphi(re^{i\theta})\right|^{\eta}d\theta\leq\int_0^{2\pi}\left|\gamma(re^{i\theta})\right|^{\eta}d\theta\quad(0<r<1;\ \eta>0), \tag{7.2}$$

where

$$\varphi_*(z)=z-\frac{\mu(1-b)}{[2]_{p,q}^{k}\left\{2[2]_{p,q}(1+\mu)-\mu b-1\right\}[1+(\beta(\delta-\lambda)]^{\zeta}}z^2 \tag{7.3}$$

Proof. Let the function $\varphi(z)$, defined by (1.10), be in the class $P_{p,q}^{\zeta,k}(\beta,\delta,\lambda,\mu,b)$. Then we need to show that

$$\int_0^{2\pi}\left|1-\sum_{h=2}^{\infty}a_{h,h}z^{h-1}\right|^{\eta}d\theta\leq\int_0^{2\pi}\left|1-\frac{\mu(1-b)}{[2]_{p,q}^{k}\left\{2[2]_{p,q}(1+\mu)-\mu b-1\right\}[1+(\beta(\delta-\lambda)]^{\zeta}}z\right|^{\eta}d\theta \tag{7.4}$$

Thus, by applying Lemma 7.1, it would suffice to show that

$$1-\sum_{h=2}^{\infty}a_h z^{h-1}\prec 1-\frac{\mu(1-b)}{[2]_{p,q}^{k}\left\{2[2]_{p,q}(1+\mu)-\mu b-1\right\}[1+(\beta(\delta-\lambda)]^{\zeta}}z \tag{7.5}$$

If the subordination (7.5) holds true, then there exists an analytic function ω with $\omega(0)=0$ and $\omega(z)|<1$ such that

$$1-\sum_{h=2}^{\infty}a_{h,h}z^{h-1}=1-\frac{\mu(1-b)}{[2]_{p,q}^{k}\left\{2[2]_{p,q}(1+\mu)-\mu b-1\right\}[1+(\beta(\delta-\lambda)]^{\zeta}}\omega(z). \tag{7.6}$$

Using Theorem 2.1, we have

$$|\omega(z)| = \left| \sum_{h=2}^{\infty} \frac{[2]_{p,q}^{k} \left\{ 2[2]_{p,q}(1+\mu) - \mu b - 1 \right\} [1+(\beta(\delta-\lambda)]^{\zeta}}{\mu(1-b)} a_h z^{h-1} \right|$$

$$\leq |z| \sum_{h=2}^{\infty} \frac{[2]_{p,q}^{k} \left\{ 2[2]_{p,q}(1+\mu) - \mu b - 1 \right\} [1+(\beta(\delta-\lambda)]^{\zeta}}{\mu(1-b)} a_h \leq |z| < 1,$$

which proves the subordination (7.5). So the proof is completed.

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