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Analytical Study of Seven-Dimensional Projective Space

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Abstract

Analytical studies of finite projective spaces of interest make understanding space objects much easier. In this paper, an analytic study is given of the finite projective space of dimension seven. Formulas of some combinatorial properties and details are found like, number of i-spaces through j-spaces, $i > j$, spaces resulting from the intersection of two spaces, i-space with j-space $i \neq j$, and the relation between lines and skew lines. Also, the basic conditions to spread the space by lines were also found. All obtained results were implemented on the finite projective space of dimension seven over the finite field of order two as an example.

Keywords: Finite field, Finite projective space, Spared, Vector space.

دراسة تحليلية للفضاء الإسقاطي السباعي الأبعاد

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الخلاصة

الدراسات التحليلية للمساحات الإسقاطية المنتهية ذات الاهتمام تجعل فهم الأجسام الفضائية أسهل بكثير. في هذا البحث تم تقديم دراسة تحليلية للفضاء الإسقاطي المنتهي للبعد السابع. تم ايجاد صيغ لبعض الخصائص والتفاصيل العددية مثل عدد الفضاءات ذات البعد i من خلال الفضاء ذات البعد j حيث , $i > j$ والفضاءات الناتجة عن تقاطع فضائين، فضاء من البعد i مع فضاء من البعد j ، $i \neq j$، والعلاقة بين الخطوط و خطوط الانحراف. كما تم وضع الشروط الأساسية لتجزئة الفضاء بالخطوط. تم تطبيق جميع النتائج التي تم الحصول عليها على الفضاء الإسقاطي المنتهي للبعد السابع على الحقل المنتهي من الرتبة الثانية كمثال.

1. Introduction

Given a finite-dimensional vector space $V = V(n+1, d)$ over a *Galois field* $GF(d) = F_d$, where d prime power. A projective space of dimension n over F_d is the set of all subspaces of the vector space V and denoted by $P_d(n)$, whose elements: points, lines, planes, ..., primes (hyperplanes) are the subspace of $V$of dimension $1,2,3,\dots,n$. A subspace of dimension m or m-space written Π_m of $P_d(n)$ is a set of points all of whose representing vectors form, together with the zero a subspace of dimension $m+1$ of V. A subspace $\Pi_{-1}$of dimension -1 is the empty set, a subspace Π_0 of dimension zero has already been called a

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point; a subspace Π_1 of dimension one, Π_2 of dimension two are respectively a line and a plane. A subspace Π_{n-1} of dimension $n-1$ is a hyperplane, see [1-2].

Motivation: In recent years, many studies have been presented on finite projective spaces, some analytically and some others using computers, especially studies on geometric concepts such as cap, arc, and spread properties. The increased interest of researchers in these studies comes due to the connection between these concepts and many purely applied concepts, such as coding theory (construct constant dimension codes [3-6], and non-constant dimension codes [7-10]), cryptography (related to linear cipher)[11-13], and network systems [14-16].

Regarding the projective plane, it has been studied extensively by many researchers [17-19], and many books have been published about it, the most important of which is as [1]. As for finite projective spaces with a dimension greater than 2, they are considered more difficult to study compared to the plane. Therefore, many studies work on specific spaces in terms of dimension and the finite field. For example, in [20], an analytical study of the projective space of dimension three is given in addition, to studies over a certain finite field. The projective space $P_5(3)$ was studied in [21], where an analytical study of the space was presented in addition to a study of cap, arc. Also, in [22], they presented their research on the third-dimensional projective space over the finite field of order 8, where they studied the terms cap and arc using the idea of group action. See also for fixed field [23-25], and non-fixed field [26-29]. A number of new studies have appeared about finite projective space of higher dimension regarding to arc and cap as for the fourth, see [30-33], fifth, see [34], sixth, see [35-36], and seventh dimensions, see [37]. As for the spread, it has been studied theoretically and with the help of computers, where the classification and the maximum size of spread was found, as shown in [38-40] for dimension five, [41] for dimension seven, and [42] for general finite field.

Objectives: The objectives of the research are to present an analytical study on the 7-dimensional projective space over the finite field F_q, as well as to present the necessary conditions for the spread the space $P_d(7)$ into lines. All results are performing on the space $P_2(7)$ as an example. The introduction section is the most important part of the format and must be included in any type of papers.

2. Some Symbols in $P_d(n)$

$\theta_d^n =$ the number of points of $P_d(n)$,

$\varphi_d^n(r) =$ the number of Π_r,

$\Phi_d^n(s,r) =$ the number of Π_r through Π_s of $P_d(n)$,

$\varkappa_d^n(t,s,r) =$ the number of Π_r meeting a fixed Π_s in a fixed Π_t,

$\aleph_d^n(t,s,r) =$ the number of Π_r meeting a fixed Π_s in some Π_t,

$\varepsilon_d^n(t,s,r) =$ the number of order pairs of (Π_s,Π_r) meeting in some Π_t,

$[r,s]_- = \begin{cases} \prod_{i=r}^{i=s}(d^i-1) & if\ s \geq r \\ 1 & if\ s < r \end{cases}$,

$\begin{bmatrix} n \\ m \end{bmatrix}_d = \frac{(d^n-1)(d^n-d)\dots(d^n-d^{m-1})}{(d^m-1)(d^m-d)\dots(d^m-d^{m-1})}$, Gaussian coefficient,

$\gcd(m,n) =$ greatest common divisor.

3. Some Properties of $P_d(7)$

This item is dedicated to finding important equations that are considered mathematical examines when searching in finite projective spaces.

Theorem 3.1[**1, Ch.** 3]**:** In $P_d(7)$, the following are holds:

(i) $\theta_d^n = \frac{d^{n+1}-1}{d-1}$; (ii) $\varphi_d^n(r) = \frac{[n-r+1,n+1]_-}{[1,r+1]_-}$;

(iii) $\Phi_d^n(s,r) = \frac{[r-s+1,n-s]_-}{[1,n-r]_-}$ $, r > s$;
(iv) $\varkappa_d^n(t,s,r) = d^{(r-t)(s-t)} \frac{[n-r-s+t+1,n-s]_-}{[1,r-t]_-}$;
(v) $\aleph_d^n(t,s,r) = d^{(r-t)(s-t)} \frac{[n-r-s+t+1,n-s]_-[s-t+1,s+1]_-}{[1,r-t]_-[1,t+1]_-} = \varphi_d^n(r)\varkappa_d^n(t,s,r)$;
(vi) $\varepsilon_d^n(t,s,r) = d^{(r-t)(s-t)} \frac{[n-r-s+t+1,n-s]_-[s-t+1,n-t]_-}{[1,r-t]_-[1,n-s]_-} = \Phi_d^n(s,r)\varkappa_d^n(t,s,r)$.

Theorem 3.2: In $P_d(7)$, the following numbers are satisfied:
(i) $\varphi_d^7(0) = (d+1)(d^2+1)(d^4+1) = \theta_d^7 = \begin{bmatrix}7\\0\end{bmatrix}_d$, number of points in $P_d(7)$.
(ii) $\varphi_d^7(1) = \frac{(d^2+1)(d^4+1)(d^7-1)}{(d-1)} = \begin{bmatrix}7\\1\end{bmatrix}_d$, number of lines in $P_d(7)$.
(iii) $\varphi_d^7(2) = \frac{(d^2+1)(d^4+1)(d^6-1)(d^7-1)}{(d-1)(d^3-1)} = \begin{bmatrix}7\\2\end{bmatrix}_d$, number of planes in $P_d(7)$.
(iv) $\varphi_d^7(3) = \frac{(d^4+1)(d^5-1)(d^6-1)(d^7-1)}{(d-1)(d^2-1)(d^3-1)} = \begin{bmatrix}7\\3\end{bmatrix}_d$, number of Π_3 in $P_d(7)$.
(v) $\varphi_d^7(4) = \frac{(d^2+1)(d^4+1)(d^6-1)(d^7-1)}{(d-1)(d^3-1)} = \begin{bmatrix}7\\4\end{bmatrix}_d$, number of Π_4 in $P_d(7)$.
(vi) $\varphi_d^7(5) = \frac{(d^2+1)(d^4+1)(d^7-1)}{(d-1)} = \begin{bmatrix}7\\5\end{bmatrix}_d$, number of Π_5 in $P_d(7)$.
(vii) $\varphi_d^7(6) = \varphi_d^7(0) = \begin{bmatrix}7\\6\end{bmatrix}_d$, number of hyperplanes in $P_d(7)$.

Theorem3.3: In $P_d(7)$, the following numbers are satisfied:
(i) $\aleph_d^7(0,i,0) = \frac{d^{i+1}-1}{d-1} = \theta_d^i, i = 1,2,3,4,5,6,7$, number of points in Π_i of $P_d(7)$;
(ii) $\aleph_d^7(1,i,1) = \frac{(d^{i+1}-1)(d^i-1)}{(d-1)(d^2-1)}, i = 2,3,4,5,6,7$, number of lines in Π_i of $P_d(7)$;
(iii) $\aleph_d^7(2,i,2) = \frac{(d^{i-1}-1)(d^i-1)(d^{i+1}-1)}{(d-1)(d^2-1)(d^{3`}-1)}, i = 3,4,5,6,7$, number of planes in Π_i of $P_d(7)$;
(iv) $\aleph_d^7(3,i,3) = \frac{(d^{i-2}-1)(d^{i-1}-1)(d^i-1)(d^{i+1}-1)}{(d-1)(d^2-1)(q^3-1)(d^4-1)}, i = 4,5,6,7$, number of Π_3 in Π_i of $P_d(7)$;
(v) $\aleph_d^7(4,i,4) = \frac{(d^{i-3}-1)(d^{i-2}-1)(d^{i-1}-1)(d^i-1)(d^{i+1}-1)}{(d-1)(d^2-1)(d^3-1)(d^4-1)(d^5-1)}, i = 5,6,7$, number of Π_4 in Π_i of $P_d(7)$;
(vii) $\aleph_d^7(5,i,5) = \frac{(d^{i-4}-1)(d^{i-3}-1)(d^{i-2}-1)(d^{i-1}-1)(d^i)(d^{i+1}-1)}{(d-1)(d^2-1)(d^3-1)(d^4-1)(d^5-1)(d^6-1)}, i = 6,7$, number of Π_5 in Π_i of $P_d(7)$;
(viii) $\aleph_d^n(6,7,6) = \varphi_d^7(6)$, number of hyperplanes in $P_d(7)$.

Lemma 3.4: In $P_d(7)$,
(i) $\Phi_d^7(0,1) = \Phi_d^7(0,6) = \frac{(d^7-1)}{d-1}$, number of lines through a fixed point;
(ii) $\Phi_d^7(0,2) = \Phi_d^7(0,5) = \frac{(d^6-1)(d^7-1)}{(d-1)(d^2-1)}$, number of planes through a fixed point;
(iii) $\Phi_d^7(0,3) = \Phi_d^7(0,4) = \frac{(d^5-1)(d^6-1)(d^7-1)}{(d-1)(d^2-1)(d^3-1)}$, number of Π_3 through a fixed point; ;
(iv) $\Phi_d^7(1,2) = \Phi_d^7(1,6) = \frac{(d^6-1)}{(d-1)}$, number of planes and hyperplanes through a fixed line;
(v) $\Phi_d^7(1,3) = \Phi_d^7(1,5) = \frac{(d^5-1)(d^6-1)}{(d-1)(d^2-1)}$, number of Π_3 through a fixed line;
(vi) $\Phi_d^7(1,4) = \frac{(d^2+1)(d^5-1)(d^6-1)}{(d-1)(d^3-1)}$, number of Π_4 through a fixed line;
(vii) $\Phi_d^7(2,3) = \Phi_d^7(2,6) = \frac{(d^5-1)}{(d-1)}$, number of Π_3 through a fixed plane;
(viii) $\Phi_d^7(2,4) = \Phi_d^7(2,5) = \frac{(d^2+1)(d^5-1)}{(d-1)}$, number of Π_4 through a fixed plane;

(ix) $\Phi_d^7(3,4) = \Phi_d^7(2,6) = (d+1)(d^2+1)$, number of Π_4 through a fixed Π_3 ;
(x) $\Phi_d^7(3,5) = \frac{(d^2+1)(d^3-1)}{d-1}$, number of Π_5 through a fixed Π_3 ;
(xi) $\Phi_d^7(4,5) = \Phi_d^7(4,6) = d^2+d+1$, number of Π_5 through a fixed Π_4 ;
(xii) $\Phi_d^7(5,6) = d+1$, the number of hyperplanes through a fixed Π_5 .

Corollary 3.5: In Theorem 1(iv), if $t = r$ then
(i) $\varkappa_d^7(0,0,1) = \varkappa_d^7(0,0,6) = \Phi_d^7(0,1) = \Phi_d^7(0,6) = \frac{(d^7-1)}{d-1}$;
(ii) $\varkappa_d^7(0,0,2) = \varkappa_d^7(0,0,5) = \Phi_d^7(0,2) = \Phi_d^7(0,5) = \frac{(d^6-1)(d^7-1)}{(d-1)(d^2-1)}$;
(iii) $\varkappa_d^7(0,0,3) = \varkappa_d^7(0,0,4) = \Phi_d^7(0,3) = \Phi_d^7(0,4) = \frac{(d^5-1)(d^6-1)(d^7-1)}{(d-1)(d^2-1)(d^3-1)}$;
(iv) $\varkappa_d^7(1,1,2) = \varkappa_d^7(1,1,6) = \Phi_d^7(1,2) = \Phi_d^7(1,6) = \frac{(d^6-1)}{d-1}$;
(v) $\varkappa_d^7(1,1,3) = \varkappa_d^7(1,1,5) = \Phi_d^7(1,3) = \Phi_d^7(1,5) = \frac{(d^5-1)(d^6-1)}{(d-1)(d^2-1)}$;
(vi) $\varkappa_d^7(1,1,4) = \Phi_d^7(1,4) = \frac{(d^2+1)(d^5-1)(d^6-1)}{(d-1)(d^3-1)}$;
(vii) $\varkappa_d^7(2,2,3) = \varkappa_d^7(2,2,6) = \Phi_d^7(2,3) = \Phi_d^7(2,6) = \frac{(d^5-1)}{d-1}$;
(viii) $\varkappa_d^7(2,2,4) = \varkappa_d^7(2,2,5) = \Phi_d^7(2,4) = \Phi_d^7(2,5) = \frac{(d^2+1)(d^5-1)}{d-1}$;
(ix) $\varkappa_d^7(3,3,4) = \varkappa_d^7(3,3,6) = \Phi_d^7(3,4) = \Phi_d^7(2,6) = (d+1)(d^2+1)$;
(x) $\varkappa_d^7(3,3,5) = \Phi_d^7(3,5) = (d^2+1)(d^2+d+1)$;
(xi) $\varkappa_d^7(4,4,5) = \varkappa_d^7(4,4,6) = \Phi_d^7(4,5) = \Phi_d^7(4,6) = d^2+d+1$;
(xii) $\varkappa_d^7(5,5,6) = \Phi_d^7(5,6) = d+1$.

Theorem 3.6[1, Ch. 3]: If $\mathcal{L}$ is a set of points in $P_d(n)$ that has a non-empty intersection with every Π_r then $|\mathcal{L}| \geq \theta_d^{n-r}$. Also $|\mathcal{L}| = \theta_d^{n-r}$ if and only if $\mathcal{L}$ is an Π_{n-r}.

Theorem 3.7: In $P_d(7)$.
(i) A subset is the line if and only if meets every hyperplane.
(ii) A subset is a plane if and only if meet Π_5 .
(iii) A subset is Π_3 if and only if meet Π_4.
(iv) A subset is a hyperplane if and only meets every line.

Theorem 3.8: $P_d(7)$, the number of lines
(i) skew to a given line is $\frac{d^{13}+d^{11}+d^9-d^8-d^6-d^4}{d-1}$.
(ii) Meeting two skew lines is $(d+1)^2$.
(iii) Skew to each of two skew lines is $\frac{d^{13}+d^{11}+d^9-2d^8-d^7-d^6-d^4+d^3+2d^2-d}{d-1}$.

Proof:
(i) Every point in $P_d(7)$ there exist $\frac{(d^7-1)}{(d-1)}$ lines through from it, So the number of lines meeting a fixed line is $(d+1)\left(\frac{(d^7-1)}{(d-1)}-1\right)$, Therefor the number of lines skew to a given line are $\frac{(d^7-1)(d^8-1)}{(d-1)(d^2-1)} - (d+1)\left(\frac{(d^7-1)}{(d-1)}-1\right) - 1 = \frac{(d^7-1)(d^8-1)}{(d-1)(d^2-1)} - \frac{d^8+d^7-d^2-d}{d-1} - 1 = \frac{d^{13}+d^{11}+d^9-d^8-d^6-d^4}{d-1}$.
(ii) Let l_1, l_2 be two skew lines and let r any point on l_1 then there exist $(d+1)$ lines joining r with l_2, and since l_1 has $(d+1)$ points, so the number of transversals of l_1 and l_2 are $(d+1)^2$.

(iii) Let l_1, l_2 be two skew lines, for any point r on l_1 there exist $\Phi_d^7(0,1) - 1$ lines through from it ; by (ii) there exist $(d+1)$ lines joining r with l_2, so the number of lines through r and not meeting l_2 is $\frac{(d^7-1)}{(d-1)} - 1 - (d+1)$, and since l_1 has $(d+1)$ points, so the number of lines meeting l_1 and not meeting l_2 is $(d+1)\left[\frac{(d^7-1)}{(d-1)} - 1 - (d+1)\right] = (d+1)\left[\frac{d^7-d^2-d+1}{d-1}\right] = \frac{d^8+d^7-d^3-2d^2+1}{d-1}$, the number of lines meeting l_2 and not meeting l_1 have the same value.
by (ii) the number of transversals of l_1 and l_2 are $(d+1)^2$, therefore the number of lines meeting l_1 or l_2 are $2\left[\frac{d^8+d^7-d^3-2d^2+1}{d-1}\right] + (d+1)^2 + 2 = \frac{2d^8+2d^7-d^3-3d^2+d-1}{d-1}$; now the number of lines skews to l_1 and l_2 are $\varphi_d^7(1) - \frac{2d^8+2d^7-d^3-3d^2+d-1}{d-1} = \frac{(d^7-1)(d^4+1)(d^2+1)}{(d-1)} - \frac{2d^8+2d^7-d^3-3d^2+d-1}{d-1} = \frac{d^{13}+d^{11}+d^9-2d^8-d^7-d^6-d^4+d^3+2d^2-d}{d-1}$. ■

Theorem 3.9: In $P_d(7)$,
(i) Every eight points that lie on a hyperplane are linearly dependent, dually every eight hyperplanes intersect in a point iff the set of coefficients these hyperplanes are linearly dependent;
(ii) a point P on the hyperplane can be determined by other seven distinct points P_i, $i = 1, \dots, 7$;
(iii) any eight distinct points determined the equation of a hyperplane passes through these points;
(iv) a subset of $P_d(7)$ with order more than seven will be linearly dependent.

Theorem 3.10: In $P_d(7)$
(i) Intersection of any two distinct Π_6 accurse a Π_5.
(ii) Intersection of any two Π_5 accurse a Π_3 or Π_4.
(iii) Intersection of any two Π_4 accurse a line or plane or Π_3.
(iv) Intersection of any two Π_3 accurse a point, ine or plane.
(v) Intersection of any two planes accurse empty set or point or line.
Proof:
Usually, intersection of any two distinct liner spaces of dimension m is a liner space of dimension less than m.
(i) From the parameter $\aleph_d^n(t,s,r)$ = the number of Π_r meeting a fixed Π_s in some Π_t,
$\aleph_d^7(0,6,6) = \aleph_d^7(1,6,6) = \aleph_d^7(2,6,6) = \aleph_d^7(3,6,6) = \aleph_d^7(4,6,6) = 0$, but $\aleph_d^7(5,6,6) \neq 0$, therefore, meeting of two hyperplanes will give a Π_5.
(ii) $\aleph_d^7(0,5,5) = \aleph_d^7(1,5,5) = \aleph_d^7(2,5,5) = 0$, but $\aleph_d^7(3,5,5) \neq 0$, $\aleph_d^7(4,5,5) \neq 0$. Therefore, a meeting of two Π_5 will give either Π_3 or Π_4 .
(iii) Since $\aleph_d^7(0,4,4) = 0$,and $\aleph_d^7(1,4,4) \neq 0$, $\aleph_d^7(2,4,4) \neq 0$, $\aleph_d^7(3,4,4) \neq 0$, then a meeting of two Π_4 will give either line or plane or Π_3 ,
(iv) $\aleph_d^7(0,3,3) \neq 0, \aleph_d^7(1,3,3) \neq 0$, $\aleph_d^7(2,3,3) \neq 0$, so a meeting of two Π_3 will give either point, line or plane.
(v) $\aleph_d^7(0,2,2) \neq 0, \aleph_d^7(1,2,2) \neq 0$, then the meeting of any two planes will give a point or line. ■

4. Partition of $P_d(7)$

In this section, we will address the subject of segmenting the projective space ending in the seventh dimension theoretically.

Definition 4.1[1, Ch. 4]: A spread $\mathcal{F}$ of $P_d(n)$ is a set of Π_m which is a partition of the points of $P_d(n)$,by Π_m ; such that every point of $P_d(n)$ lies in exactly one Π_m of the partition $\mathcal{F}$. That is, any two elements of $\mathcal{F}$ are disjoint.

Corollary 4.2[1, Ch. 4]: A spread in $P_d(n)$ exists if and only if $m+1$ divides $n+1$.

Theorem 4.3 [1, Ch. 4]: If $\gcd(m+1, n+1) > 1$, then

(i) There exists an Π_m of cycle N such that, $\theta_d^n = rN$, $r = \theta_d^l$, r divides gcd (θ_d^m, θ_d^n), and $l+1$ divides gcd (θ_d^m, θ_d^n).

(ii) The points represented by $0, N, \dots (r-1)N$ lie in an Π_l .

Theorem 4.4: Let $\mathrm{M}_1 = \frac{\varphi_d^7(1)-N}{\theta_d^7}$, the number of cycles of lines of length θ_d^7, then

(i) There is a line of cycles N less than θ_d^7,

(ii) the parameters in theorem15(i) have the following values: $l=1$,$r=d+1$, $N = d^6 + d^4 + d^2 + 1$,

(iii) the number of partitions of lines of length θ_d^7, $\mathrm{M}_1 = \frac{d^{16}-d^{15}-d^{10}+d^9-d^8+d^7+d^2-d}{d^{11}-d^{10}-d^9+d^8-d^3+d^2+d-1}$,

Proof:

(i) From Theorem 4,3(i) the points of $P_d(7)$ can be spread to lines of cycle N, Since $\gcd(2,8) = 2$, and $l+1/\gcd(2,8)$, so $l=1$, and $r = \theta_d^1 = d+1$

Therefore, $N = \frac{\theta_d^7}{r} = \frac{\frac{d^8-1}{d-1}}{d+1} = \frac{d^8-1}{(d-1)^2} = d^6 + d^4 + d^2 + 1.$

(ii) $\mathrm{M}_1 = \frac{\varphi_d^7(1)-N}{\theta_d^7} = \frac{\frac{(d^7-1)(d^8-1)}{(d-1)(d^2-1)}-(d^6+d^4+d^2+1)}{\frac{d^8-1}{d-1}} = \frac{\frac{d^{15}-d^8-d^7+1}{d^3-d^2-d+1}-(d^6+d^4+d^2+1)}{\frac{d^8-1}{d-1}}$

$\mathrm{M}_1 = \frac{(d-1)(d^{15}-d^9-d^7+d)}{(d^8-1)(d^3-d^2-d+1)} = \frac{d^{16}-d^{15}-d^{10}+d^9-d^8+d^7+d^2-d}{d^{11}-d^{10}-d^9+d^8-d^3+d^2+d-1}$. ∎

5. Spread of $P_2(7)$ by Lines

In this section, the statistics details and spared of $P_2(7)$ by lines are given. The GAP program (https://www.gap-system.org.) is used to perform all designed algorithms in this work.

Let $\mathcal{M}$ be $8*8$ non-singular matrix over F_2, where the coefficient of the eight rows are scalar of a non-singular polynomial over F_2

$$\mathcal{M} = \begin{pmatrix} 0 & & & & & & & \\ \vdots & & & I_7 & & & & \\ 0 & & & & & & & \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

such that $|\mathcal{M}| \neq 0$. The $\mathcal{M}$ matrix is usually called the companion matrix of order 255. This matrix is used to find the points and hyperplanes of this space. Let $\mathrm{P}_0 = [1,0,0,0,0,0,0,0]$, then the point of $P_2(7)$ can be found by the equation $\mathrm{P}_i = \mathrm{P}_0\mathcal{M}^i, i = 0,1,\dots,254$.

Let H_0 be the hyperplane with eight coordinates is zero, so H_0 will contain the following points $H_0[i]$, $i = 1,\dots,255$: 1, 2, 3, 4, 5, 6, 7, 14, 15, 17, 19, 20, 22, 25, 27, 33, 34, 35, 37, 40, 41, 43, 44, 45, 46, 48, 50, 51, 52, 54, 55, 57, 58, 60, 62, 64, 66, 67, 70, 71, 73, 77, 79, 80, 81, 85, 88, 89, 95, 98, 100, 101, 102, 103, 104, 105, 107, 108, 109, 110, 111, 113, 118, 122, 127, 128, 129, 130, 132, 133, 137, 139, 142, 143, 144, 149, 150, 153, 156, 165, 167, 169, 173, 174, 178, 179, 181, 182, 185, 187, 189, 190, 191, 192, 195, 199, 200, 201, 202, 203, 206, 207, 208, 209, 213, 214, 215, 218, 219, 222, 223, 224, 226, 227, 228, 231, 233, 234, 239, 241, 242, 244, 245, 246, 248, 250, 253.

Then by using H_0 can be constructed for all other 254 hyperplanes, the cyclic formula $H_i = H_0\mathcal{M}^i, i = 0, \dots, 254$, so the hyperplanes have the following forms:

H_1: 2, 3, 4, 5, 6, 7, 8, 15, 16, 18, 20, 21, 23, 26, 28, 34, 35, 36, 38, 41, 42, 44, 45, 46, 47, 49, 51, 52, 53, 55, 56, 58, 59, 61, 63, 65, 67, 68, 71, 72, 74, 78, 80, 81, 82, 86, 89, 90, 96, 99, 101, 102, 103, 104, 105, 106, 108, 109, 110, 111, 112, 114, 119, 123, 128, 129, 130, 131, 133, 134, 138, 140, 143, 144, 145, 150, 151, 154, 157, 166, 168, 170, 174, 175, 179, 180, 182, 183, 186, 188, 190, 191, 192, 193, 196, 200, 201, 202, 203, 204, 207, 208, 209, 210, 214, 215, 216, 219, 220, 223, 224, 225, 227, 228, 229, 232, 234, 235, 240, 242, 243, 245, 246, 247, 249, 251, 254.

H_2: $H_0[j] + 2$, if $H_0[j] + 2 \geq 256$ will be replaced by $H_0[j] + 2 - 255$, $j = 1, \dots, 255$.

$\vdots$

H_k: $H_0[j] + k$, if $H_0[j] + k = 256$ will replace by $H_0[j] + k - 255$, $j = 1, \dots, 255$.

$\vdots$

H_{255}: 1, 2, 3, 4, 5, 6, 13, 14, 16, 18, 19, 21, 24, 26, 32, 33, 34, 36, 39, 40, 42, 43, 44, 45, 47, 49, 50, 51, 53, 54, 56, 57, 59, 61, 63, 65, 66, 69, 70, 72, 76, 78, 79, 80, 84, 87, 88, 94, 97, 99, 100, 101, 102, 103, 104, 106, 107, 108, 109, 110, 112, 117, 121, 126, 127, 128, 129, 131, 132, 136, 138, 141, 142, 143, 148, 149, 152, 155, 164, 166, 168, 172, 173, 177, 178, 180, 181, 184, 186, 188, 189, 190, 191, 194, 198, 199, 200, 201, 202, 205, 206, 207, 208, 212, 213, 214, 217, 218, 221, 222, 223, 225, 226, 227, 230, 232, 233, 238, 240, 241, 243, 244, 245, 247, 249, 252, 255.

Theorem 5.1: In $P_2(7)$ the following number are satisfied.
(i) Any hyperplane has 127 points and Π_5;
(ii) any hyperplane has 2667 lines and Π_4;
(iii) any hyperplane has 11811 planes and Π_3;
(iv) any Π_5 has 63 points and Π_4;
(v) any Π_5 has 672 lines;
(vi) any Π_5 has 1395 planes
(vii) any Π_5 has 651 Π_3;
(viii) any Π_5 there are 3 hyperplanes through it;
(ix) any Π_4 has 31 points and Π_3;
(x) any Π_4 has 155 lines and a plane;
(xi) from any Π_4, there are 7 hyperplanes through it;
(xii) any Π_3 has 15 points and plane;
(xiii) any Π_3 has 35 lines;
(xiv) from any Π_3, there are 15 hyperplanes through it
(xv) any plane has 7 points and lines;
(xvi) from any plane, there are 31 hyperplanes through it;
(xvii) any line has 3 points;
(xviii) from any point, there are 127 lines and hyperplanes through it;
(xix) any point, there are 2667 planes and Π_5 through from it;
(xx) from any point, there is 11811 Π_3,$\Pi_4$through from it;
(xxi) any line, there are 63 planes and hyperplanes through it;
(xxii) any line there, are 651 Π_3 and Π_5 through from it;
(xxiii) from any line, there are 1395 Π_4 through from it;

(xxiv) any plane, there are 31 Π_3 and hyperplanes through from it;
(xxv) any plane, there are 155 Π_4 and Π_5 through from it;
(xxvi) from any Π_3, there are 15 Π_4 and hyperplanes through from it;
(xxvii) from any Π_3, there are 35 Π_5 through from it;
(xxviii) from any Π_4, there are 7 Π_5 and hyperplanes through from it;
(xxix) from any Π_5, there are 3 hyperplanes through from it.

Corollary 5.2: In $P_2(7)$,
(i) There are 42 cycles of lines of length 255 and one cycle of lines of length 85.
(ii) There are 381 cycles of planes of length 255.
Proof:
(i) By Theorem 4.4(ii), $\mathrm{M}_1 = 42$; so there are 42 cycles of lines of length 255; the first line l_i from each cycle is as follows;

$l_1 = \{1,2,100\}$, $l_2 = \{1,3,199\}$, $l_3 = \{1,4,206\}$, $l_4 = \{1,5,142\}$,
$l_5 = \{1,6,88\}$, $l_6 = \{1,7,156\}$, $l_7 = \{1,8,159\}$, $l_8 = \{1,9,28\}$,
$l_9 = \{1,10,145\}$, $l_{10} = \{1,11,175\}$, $l_{11} = \{1,12,68\}$, $l_{12} = \{1,13,56\}$,
$l_{13} = \{1,14,45\}$, $l_{14} = \{1,15,62\}$ $l_{15} = \{1,16,238\}$, $l_{16} = \{1,17,55\}$,
$l_{17} = \{1,18,69\}$, $l_{18} = \{1,21,94\}$ $l_{19} = \{1,22,182\}$, $l_{20} = \{1,23,135\}$,
$l_{21} = \{1,24,166\}$, $l_{22} = \{1,25,111\}$ $l_{23} = \{1,26,155\}$, $l_{24} = \{1,27,89\}$,
$l_{25} = \{1,29,123\}$, $l_{26} = \{1,30,210\}$ $l_{27} = \{1,31,220\}$, $l_{28} = \{1,33,109\}$,
$l_{29} = \{1,35,137\}$, $l_{30} = \{1,36,184\}$ $l_{31} = \{1,38,176\}$, $l_{32} = \{1,40,104\}$,
$l_{33} = \{1,41,187\}$, $l_{34} = \{1,42,126\}$ $l_{35} = \{1,43,108\}$, $l_{37} = \{1,50,128\}$,
$l_{38} = \{1,53,177\}$, $l_{39} = \{1,59,164\}$ $l_{40} = \{1,61,184\}$, $l_{41} = \{1,64,179\}$,
$l_{42} = \{1,71,167\}$.

From Theorem 4.4(i), $N = 85$; there is one cycle of lines of length 85, the first line is l_1^*={1,86,171}.
(ii) Since $\gcd(2+1,8) = 1$, so there is no cycle of length less than θ_2^7 (Theorem 4.3); that is, the matrix $\mathcal{M}$ will permute the each plane θ_2^7 in one cycle. Therefore, $\varphi_2^7(2)/\theta_2^7 = 97155/255 = 381$. ∎

6. Conclusion

The finite 7- dimensional projective space can be spread by lines of cycle $N = d^6 + d^4 + d^2 + 1 < \theta_d^7$. There are $\mathrm{M}_1 = \frac{d^{16}-d^{15}-d^{10}+d^9-d^8+d^7+d^2-d}{d^{11}-d^{10}-d^9+d^8-d^3+d^2+d-1}$ numbers of lines' partition of $P_d(7)$ of length θ_d^7. Through our research, we recommend conducting further analytical research on projective spaces with dimensions of more than 7 so that the programming and computational work for these spaces becomes easier and more accurate.

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References
[1]. J. W. P. Hirschfeld, *Projective Geometries Over Finite Fields*, *2nd ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press,* 1998.
[2]. G. Kiss and T. Szonyi, *Finite Geometries*, 1st ed., Chapman and Hall/CRC Press, 2020.
[3]. A. M. Khalaf and N. Y. K. Yahya, "New Examples in Coding Theory for Construction Optimal Linear Codes Related with Weight Distribution", *AIP Conf. Proc.*, vol. 2845, p. 050043, 2023.

[4]. E. B. Al-Zangana and E. A. Shehab, "Certain Types of Linear Codes Over the Finite Field of order Twenty-Five", *Iraqi Journal of Science (IJS)*, vol. 62, no. 11, pp. 4019-4031, 2021.
[5]. E. B. Al-Zangana, "Projective MDS Codes Over $GF(27)$", *Baghdad Sci. J.*, vol. 18, no. 2(Suppl.), pp. 1125-1132, 2021.
[6]. F. N. Abdullah and N. Y. K. Yahya, "Bounds on Minimum Distance for Linear Codes Over $GF(q)$", *Italian Journal of Pure and Applied Mathematics*, no. 45, pp. 894-903, 2021.
[7]. A. A. Davydov, S. Marcugini, and F. Pambianco, "On the Weight Distribution of the Costs of MDS Codes," *Advances in Mathematics of Communications*, vol. 17, no. 5, pp. 1115-1138, 2023.
[8]. A. A. Davydov, S. Marcugini, and F. Pambianco, "On Cosets Weight Distributions of the Doubly-Extended Reed-Solomon Codes of Codimension 4", *IEEE Trans. Inform. Theory*, vol. 67, no. 8, pp. 5088-5096, 2021.
[9]. T. L. Alderson, *"n-Dimensional Optical Orthogonal Codes, Bounds and Optimal Constructions"*, Springer-Verlag GmbH Germany, part of Springer Nature, vol. 30, no. 5, pp. 373-386, 2019.
[10]. V. Starodub, R. V. Skuratovskii, and S. S. Podpriatov, "Triangle Conics, Cubics and Possible Applications in Cryptography", *Mathematics and Statistics*, vol. 9, no. 5, pp. 749-759, 2021.
[11]. R. Skuratovskii and V. Osadchyy, "Criterions of Supersingularity and Groups of Montgomery and Edwards Curves in Cryptography", *WSEAS Transactions on Mathematics*, vol. 19, pp. 709-722, 2021.
[12]. A.A. Bruen, J.W.P. Hirschfeld, and D.L. Wehlau, "Cubic Curves, Finite Geometry and Cryptography", Acta Appl Math, vol. 115, pp. 265-278, 2011.
[13]. F. H. Ali and R. N. Jawad, "Using Evolving Algorithms to Cryptanalysis Nonlinear Cryptosystems", *Baghdad Science Journal*, vol. 17, no. (2 Special Issue), pp. 682-688, 2020.
[14]. A. A. Ghazi and F. H. Ali, "Robust and Efficient Dynamic Stream Cipher Cryptosystem, " *Iraqi Journal of Science*, vol. 59, no.2C, pp. 1105-1114, 2018.
[15]. C. Alonso-González, M. Á. Navarro-Pérez, and X. Soler-Escrivà, "Flag Codes from Planar Spreads in Network Coding", Finite Fields and Their Applications, vol.73, pp. 1 -21, 2020.
[16]. M. Greferath, M. O. Pavcevic, N. Silberstein, and M. A. Vazquez-Castro. *Network Coding and Subspace Designs. 1st edn., Springer Publishing Company, Incorporated,* 2018.
[17]. N. A. M. Al-Seraji and R. I. K. Al-Humaidi, "On the Embedding of An Arc into a Cubic Curve in a Finite Projective Plane of Order Seven", *Al-Mustansiriyah Journal of Science*, vol. 30, no. 1, pp. 158-168, 2019.
[18]. N. A. M. Al-Seraji and R. A. B. AL-Ogali, "Classification of Arcs in Finite Projective Plane of Order Sixteen", *Al-Mustansiriyah Journal of Science*, vol. 29, no. 1, pp. 138-149, 2018.
[19]. R. N. Daskalov, "New Large $(n;r)$-Arcs in $PG(2,q)$", *Iranian Journal of Mathematical Sciences and Informatics*, vol. 17, no. 1, pp. 125-133, 2022.
[20]. J.W.P. Hirschfeld, *Finite Projective Spaces of Three Dimensions: Oxford University Press,* 1985.
[21]. A. S. Al-Mukhtar, *Complete Arcs and Surfaces in Three-Dimensional Projective Space Over Galois Field*, Ph.D. Thesis, University of Technology, Baghdad, Iraq, vol. 7, no. 4, pp. 1-15, 2008.
[22]. A. J. Al-Rikabi, *Construction of Some Types of Sets in Three-Dimensional Projective Space over Finite Field of Order Eight*, Ph.D. Thesis, Mustansiriyah University, Baghdad, Iraq, 2022.
[23]. V. Abatangelo, "Classification of Transitive Caps in $PG\,(3,5)$," *Pure Mathematics and Applications*, vol. 11, pp. 1-11, 2000.
[24]. A. E. M. Sulaimaan and N. Y. K. Yahya, "The Reverse Construction of Complete $(k;n)$-Arcs in Three-Dimensional Projective Space $PG(3,4)$", *Journal of Physics: Conference Series*, vol. 1591, no. 1, p. 012078, 2020.
[25]. D. Bartoli, A. A. Davydov, S. Marcugini, and F. Pambianco, "On Planes Through Points of the Twisted Cubic in $PG(3,q)$ and Multiple Covering Codes", *Finite Fields and Their Applications*, vol. 67, p. 101710, 2020.
[26]. G. Donati and N. Durante, "A Generalization of The Quadratic Cone of $PG(3,Q^N)$ and its Relation with the Affine Set of the Lüneburg Spread", *Journal of Algebraic Combinatorics*, vol. 49, no. 2, pp. 169-177, 2019.
[27]. G. Donati and N. Durante, "Absolute Points of Correlations of $PG(3,q^n)$", *Journal of Algebraic Combinatorics*, vol. 54, pp. 109–133, 2021.

[28].V. Napolitano, "On Baer Cones in $PG\ (3, q)$", *Journal of Geometry*, vol. 112, no. 37, pp. 1-6, 2021.
[29].S. Innamorati and F. Zuanni, "A Combinatorial Characterization of the Baer and the Unital Cone in $PG(3, q^2)$," *Journal of Geometry*, vol. 111, no. 3, pp. 45, 2020.
[30].A. Aguglia and F. Pavese, "On Non-Singular Hermitian Varieties of $PG(4, q^2)$", *Discrete Mathematics*, vol. 343, no. 1, pp. 1-5, 2020.
[31].N. Durante and G. G. Grimaldi, "Absolute Points of Correlations of $PG(4, q^n)$", *Journal of Algebraic Combinatorics*, vol. 56, pp. 873-887, 2022.
[32].D. Bartoli and N. Durante, "On the Classification of Low-Degree Ovoids of $Q(4, q)$", *Combinatorica*, vol. 42, suppl. 1, pp. 953-969, 2022.
[33].F. Aguglia and F. Pavese, "On Non-Singular Hermitian Varieties of $PG(4, q^2)$", *Discrete Mathematics*, vol. 343, no. 1, pp. 111634, 2020.
[34].N. Durante and G. G. Grimaldi, "Absolute Points of Correlations of $PG(5, q^n)$", *Discrete Mathematics*, vol. 346, no. 9, pp. 1-14, 2023.
[35].A. Aguglia, L. Giuzzi, and M. Homma, "On Hermitian Varieties in $PG(6, q^2)$", *Ars Mathematica Contemporanea* , vol. 21, no. 1, pp. 1-11, 2021.
[36].S. G. Barwick, "Ruled Quintic Surfaces in $PG(6, q)$", *Innovations in Incidence Geometry*, vol. 17, pp. 25-41, 2019.
[37].R. Shaw, N. Gordon, and H. Havlicek, "Tetrads of Lines Spanning $PG(7,2)$", *Bulletin of the Belgian Mathematical Society - Simon Stevin*, vol. 20, pp. 735-752, 2013.
[38].T. P. McDonough, R. Shaw, and S. Topalova, "Classification of Book Spreads in $PG(5,2)$", *Note di Matematica*, vol. 33, no. 2, pp. 43-64, 2013.
[39].G. Marino and V. Pepe, "On Symplectic Semifield Spreads of $PG(5, q^2), q$ Odd", *Forum Mathematicum*, vol. 30, pp. 497-512, 2018.
[40].Z. T. Mateva and S. T. Topalova, "Line Spreads of $PG(5,2)$", *Journal of Combinatorial Designs*, vol. 17, pp. 90-102, 2009.
[41].R. Shaw and S. T. Topalova, "Book Spreads in $PG(7,2)$", *Discrete Mathematics*, vol. 330, pp. 76-86, 2014.
[42].N. L. Johnson, *Combinatorics of Spreads and Parallelisms*. 1st edn., CRC Press, 2010.