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A Study of Stability of First-Order Delay Differential Equations Using Fixed Point Theorem Banach

Zaid A.A. Mahdi Monje*, Buthainah A.A. Ahmed
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

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Abstract

In this paper we investigate the stability and asymptotic stability of the zero solution for the first order delay differential equation

$$\dot{y}(t) = -\sum_{j=1}^{N} a_j(t)\, y\left(t - \tau_j(t)\right) + f(t, y(t - \tau(t)))$$

where the delay is variable and by using Banach fixed point theorem. We give new conditions to ensure the stability and asymptotic stability of the zero solution of this equation.

دراسة استقراريه المعادلات التفاضلية التباطؤيه من الرتبة الاولى باستخدام نظرية بناخ للنقطة الصامدة

زيد عبد القادر مهدي منجي* ، بثينه عبد الحسن احمد
قسم الرياضيات ، كلية العلوم ، جامعة بغداد، بغداد، العراق

الخلاصة

في هذا البحث ندرس استقرارية واستقرارية تقارب الحل الصفري للمعادلة التفاضلية التباطؤية من الرتبة الاولى

$$\dot{y}(t) = -\sum_{j=1}^{N} a_j(t)\, y\left(t - \tau_j(t)\right) + f(t, y(t - \tau(t)))$$

حيث ان معامل التباطؤ متغير باستخدام نظرية بناخ للنقطة الصامدة. مع اعطاء شروطا جديدة لضمان الاستقرارية والاستقرارية التقاربية للحل الصفري لهذه المعادلة.

1. Introduction

In applied science, many practical problems concerning heat flow, species interaction microbiology, neural networks, and many more are linked with delay differential equation. Burton [1 - 5] was among the first who studied the stability of delay differential equation using the fixed point theory, instead of Liapunov method. Many researchers studied the stability for many types of delay differential equations. For example, in 2010, Menc Fan [6] studied the stability of the delay differential equation

$$\dot{x}(t) = -a(t, x_t)x(t) + f(t, x_t)$$

where $x_t(\theta) = x(t + \theta)$ and $\theta \in [-r, 0]$, $r \geq 0$, that is with constant delay using fixed point theory.

While Bo Zhang [7] studied fixed points and stability in differential equations with variable delays

$$\dot{x}(t) = -b(t)x(t - \tau(t))$$

and its generalization

*Email: zaidqader@yahoo.com

$$\dot{x}(t) = -\sum_{j=1}^{N} b_j(t)x(t-\tau_j(t))$$

where $b_j \in C(\mathbb{R}^+, \mathbb{R})$ and $\tau, \tau_j \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $t-\tau(t) \to \infty$ and $t-\tau_j(t) \to \infty$ as $t \to \infty$.

While Ramazan Yazgan [8] studied the global asymptotic stability of solutions to neutral equations of first order

$$\dot{x}(t) = -a(t)x(t) + b(t)g(x(t)) + c(t)f(\dot{x}(t-\tau(t))) + q(t, x(t), x(t-\tau(t)))$$

where $a, b, c \in C([0,\infty), \mathbb{R}), g, f \in C(\mathbb{R}, \mathbb{R}), q \in C([0,\infty), \mathbb{R} \times \mathbb{R}, \mathbb{R})$

In this paper we study the stability of the following delay differential equation

$$\dot{y}(t) = -\sum_{i=1}^{M} a_i(t)\, y(t-\tau_i(t)) + f(t, y(t-\tau(t))) \tag{1.1}$$

where $\mathbb{R} = (-\infty, \infty), \mathbb{R}^+ = [0, +\infty)$ and $\mathbb{R}^- = (-\infty, 0]$, respectively.

and $a_i \in C(\mathbb{R}^+, \mathbb{R})$ and $\tau, \tau_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $t-\tau(t) \to \infty$ and $t-\tau_i(t) \to \infty$ as $t \to \infty, f(t, y(t-\tau(t))) \in C(\mathbb{R}^+ \times \mathbb{A}, \mathbb{R})$.

Here we use

$\mathbb{A} = C([-r, 0], \mathbb{R})$ to denote the space of continuous function from $[-r, 0]$ to $\mathbb{R}$.

Assume that $f(t, 0) = 0$ and there exists a positive constant D and a continuous function $b_1(t) \in C(\mathbb{R} . \mathbb{R}^+)$ such that by Lipchitz function

$$|f(t, \Phi) - f(t, \Psi)| \le b_1(t)|\Phi - \Psi|$$

For all Φ and $\Psi \in \mathbb{A}_D := \{y \in \mathbb{A} : \|y\| \le D\}$.

2. Preliminaries

We now give some basic information.

Theorem 2.1. Banach fixed point theorem [9]

If $T: X \to X$ is a contraction and X is a Banach space, then there is a unique point $x^* \in X$ which is fixed by T (That is $T(x) = x$). Moreover, if $x_\circ$ is any point in X,

then the sequence defined by $x_1 = T(x_\circ), x_2 = T(x_1), \ldots$ converges to x^*.

Corollary 2.2. [9]

If S is a closed subset of the Banach space X, and $T: S \to S$ is a contraction, then T has a unique fixed point in S.

3. Stability of the zero solution for the delay differential equation by Banach fixed point theorem

Let $C(S_1, S_2)$ denotes the set of all continuous functions $\Phi: S_1 \to S_2$

For each $t_\circ$, define

$$m_i(t_\circ) = inf\{s - \tau_i(s): s \ge t_\circ\}, \qquad m(t_\circ) = \min\{m_i(t_\circ): 1 \le i \le M\}$$

And $\mathbb{A}(t_\circ) = C([m(t_\circ), t_\circ], \mathbb{R})$ with the supremum norm $\|\cdot\|$. That is

$$\|f\| = sup\{|f(x)|: f\ is\ continuous\ and\ t \in [m(t_\circ), t_\circ]\}$$

Define the inverse of $t - \tau_i(t)$ by $I_i(t)$ if it exists and set

$$H(t) = \sum_{i=1}^{M} a_i(I_i(t)),$$

$$\mu(t) = \sum_{i=1}^{M} \int_0^t e^{-\int_s^t H(u)du} |a_i(s)||\dot{\tau}_i(s)|ds$$

If $\tau_i(t)$ is differentiable. For each $(t_\circ, \Phi) \in \mathbb{R}^+ \times \mathbb{A}(t_\circ)$, by using existence fixed point theorem, a solution of (1.1) through $(t_\circ, \Phi)$ is a continuous function $y: [m(t_\circ), t_\circ + \sigma) \to \mathbb{R}^n$ for some positive constant $\sigma > 0$ such that $y(t)$ satisfies (1.1) on $[t_\circ, t_\circ + \sigma)$ and $y(s) = \Phi(s)$ for $s \in [m(t_\circ), t_\circ]$. We denote such a solution by $y(t) = y(t, t_\circ, \Phi)$. For each $(t_\circ, \Phi) \in \mathbb{R}^+ \times \mathbb{A}(t_\circ)$, there exists a unique solution $y(t) = y(t, t_\circ, \Phi)$ of (1.1) defined on $[t_\circ, +\infty)$ (see [1]).

For fixed $t_\circ$, we define

$$\|\Phi\| = sup\{|\Phi(s)| : m(t_\circ) \le s \le t_\circ\}$$

Theorem 3.1

Suppose that τ_i is differentiable, the inverse function $I_i(t)$ of $t - \tau_i$ exists and there exists a constant $\sigma \in (0, 1)$ such that for $t \ge 0$.

1. $\lim_{t\to\infty} inf \int_0^t H(s)ds > -\infty$.
2. $\sum_{i=1}^{M} \begin{bmatrix} \int_{t-\tau_i(t)}^{t} |a_i(I_i(s))| ds + \int_0^t e^{-\int_s^t H(u)du} |H(s)| \\ \times \int_{s-\tau_i(s)}^{s} |a_i(I_i(v))| dv ds \end{bmatrix} + \mu(t) \le \sigma$

Then the zero solution of (1.1) is asymptotically stable if and only if

3. $\int_0^t H(s)ds \to \infty \ \ as \ \ t \to \infty$.

Proof:

Suppose that (3) holds, for each $t_\circ \ge 0$.

We set $k = sup_{t\ge 0}\left\{e^{-\int_0^t H(s)ds}\right\}$ ... (3.1)

Let $\Phi \in \mathbb{A}(t_\circ)$ be a fixed and define

$$S = \left\{ \begin{matrix} y \in C([m(t_\circ),\infty),\mathbb{R}): y(t) \to 0 \ as \ t \to \infty, y(s) = \Phi(s) \\ for \ s \in [m(t_\circ), t_\circ] \end{matrix} \right\}$$

Then S is a complete metric space with metric

$$\rho(y,z) = sup_{t\ge t_\circ}\{|y(t) - z(t)|\}$$

Define $T: S \to S$ by $T_y(t) = \Phi(t)$ for $t \in [m(t_\circ), t_\circ]$ and

$$T_y(t) = [\Phi(t_\circ) - \sum_{i=1}^{M} \int_{t_\circ-\tau_i(t_\circ)}^{t_\circ} a_i(I_i(s))\Phi(s)ds] e^{-\int_{t_\circ}^t H(u)du}$$
$$+ \sum_{i=1}^{M} \int_{t-\tau_i(t)}^{t} a_i(I_i(s)) y(s) ds$$
$$- \int_{t_\circ}^{t} e^{-\int_s^t H(u)du} \sum_{i=1}^{M} a_i(s)\dot{\tau}_i(s) y(s-\tau_i(s)) ds$$
$$- \int_{t_\circ}^{t} e^{-\int_s^t H(u)du} H(s) \left[\sum_{i=1}^{M} \int_{s-\tau_i(s)}^{s} a_i(I_i(v)) f(v, y(v-\tau(v))) dv \right] ds \ \ ... (3.2)$$

For $t \ge t_\circ . T_y \in C([m(t_\circ),\infty), R)$, because the total continuous functions are continuous.

We now show that $T_y(t) \to 0$ as $t \to \infty$. Since $y(t) \to 0$ and $t - \tau_i(t) \to \infty$, for each $\varepsilon > 0$, there exists $t_1 > t_\circ$ such that $s \ge t_1$ implies $|y(s - \tau_i(s)| < \varepsilon$ for $i = 1,2,\ldots,M$.

Thus, for $t \ge t_1$, the first term A_1 in (3.2) satisfies

$$A_1 = \left[\Phi(t_\circ) - \sum_{i=1}^{M} \int_{t_\circ-\tau_i(t_\circ)}^{t_\circ} a_i(I_i(s))\Phi(s)ds\right] e^{-\int_{t_\circ}^t H(u)du}$$

$A_1 \to 0$, because by (3) we have, $\int_0^t H(s)ds \to \infty$ as $t \to \infty$, and $e^{-t} \to 0$ as $t \to \infty$.

And since $y(t) \to 0 \ as \ t \to \infty$, we get $A_2 \to 0$. Also $A_3 \to 0$ because $y(t - \tau_i(t)) \to 0 \ as \ t - \tau_i(t) \to \infty$, i=1, 2, …, M.

Now, we show that the fourth term $A_4 \to 0$

$$|A_4| = \left| \int_{t_\circ}^{t} e^{-\int_s^t H(u)du} H(s) \left[\sum_{i=1}^{M} \int_{s-\tau_i(s)}^{s} a_i(I_i(v)) f(v, y(v-\tau(v))) dv \right] ds \right|$$
$$\le \int_{t_\circ}^{t_1} e^{-\int_s^t H(u)du} |H(s)| \left[\sum_{i=1}^{M} \int_{s-\tau_i(s)}^{s} |a_i(I_i(v))| |f(v, y(v-\tau(v)))| dv \right] ds$$
$$+ \int_{t_\circ}^{t} e^{-\int_s^t H(u)du} |H(s)| \left[\sum_{i=1}^{M} \int_{S-\tau_i(S)}^{S} |a_i(I_i(v))| |f(v, y(v-\tau(v)))| dv \right] ds$$

$$\leq \int_{t_\circ}^{t_1} e^{-\int_s^t H(u)du}|H(s)|\left[\sum_{i=1}^{M}\int_{s-\tau_i(s)}^{s}\left|a_i\big(I_i(v)\big)\right|b_1(v)|y(v-\tau(v))|dv\right]ds$$
$$+\int_{t_1}^{t} e^{-\int_s^t H(u)du}|H(s)|\left[\sum_{i=1}^{M}\int_{S-\tau_i(S)}^{s}|a_i(I_i(v))|b_1(v)|y(v-\tau(v))|dv\right]ds$$
$$\leq sup_{\alpha\geq m(t_\circ)}|y(\alpha-\tau(\alpha)|\int_{t_\circ}^{t_1} e^{-\int_s^t H(u)du}|H(s)|\left[\sum_{i=1}^{M}\int_{s-\tau_i(s)}^{s} b_1|a_i(I_i(v))|dv\right]ds$$
$$+sup_{\alpha\geq m(t_\circ)}|y(\alpha-\tau(\alpha)|\int_{t_1}^{t} e^{-\int_s^t H(u)du}|H(s)|\left[\sum_{i=1}^{M}\int_{s-\tau_i(s)}^{s} b_1(v)|a_i(I_i(v))|dv\right]ds$$
$$\leq sup_{\alpha\geq m(t_\circ)}|y(\alpha-\tau(\alpha)|\int_{t_\circ}^{t_1} e^{-\int_s^t H(u)du}|H(s)|\left[\sum_{i=1}^{M}\int_{s-\tau_i(s)}^{s} b_1(v)|a_i(I_i(v))|dv\right]ds$$
$$+\,\varepsilon\int_{t_1}^{t} e^{-\int_s^t H(u)du}\,|H(s)|\left[\sum_{i=1}^{M}\int_{s-\tau_i(s)}^{s} b_1(v)|a_i(I_i(v))|dv\right]ds$$

By (3), there exists $t_2 > t_1$ such that $t \geq t_2$ implies

$$sup_{\alpha\geq m(t_\circ)}|y(\alpha-\tau(\alpha)|\int_{t_\circ}^{t_1} e^{-\int_s^t H(u)du}\,|H(s)|\left[\sum_{i=1}^{M}\int_{s-\tau_i(s)}^{s} b_1(v)|I_i(v))|dv\right]ds < \varepsilon$$

Apply (2) to obtain $|A_4| \leq \varepsilon + \varepsilon\sigma < 2\varepsilon$, $A_4 \to 0$ as $t \to \infty$. Similarly, we can show that the rest of the terms in (3.2) approach zero as $t \to \infty$.

This yields $T_y(t) \to 0$ as $t \to \infty$, and hence $T_y \in S$. Also , by (2) , T is a contraction mapping with a contraction constant σ. By the contraction mapping principle (Smart [2 , p . 2]) T has a unique fixed point y in S.

Which is a solution of (1.1) with $y(s) = \Phi(s)$ on $[m(t_\circ), t_\circ]$ and

$y(t) = y(t, t_\circ, \Phi) \to 0$ as $t \to \infty$.

To obtain the asymptotic stability, we need to show that the zero solution of (1.1) is stable.

Let $\varepsilon > 0$ be given and choose $\delta > 0$ $(\delta < \varepsilon)$ satisfying

$$2\delta k e^{\int_0^{t_\circ} H(u)du} + \sigma\varepsilon < \varepsilon$$

If $y(t) = y(t, t_\circ, \Phi)$ is a solution of (1.1) with $\|\Phi\| < \delta$, then $y(t) = T_y(t)$ defined in (3.2).

We claim that $|y(t)| < \varepsilon$ for all $t \geq t_\circ$.

Notice that $|y(s)| < \varepsilon$ on $[m(t_\circ), t_\circ]$. If there exists $t^* > t_\circ$ such that $|y(t^*)| = \varepsilon$ and $|y(s)| < \varepsilon$ for $m(t_\circ) \leq \delta < t^*$, then it follows from (3.2) that

$$|y(t^*)| \leq \|\Phi\|\left[1+\sum_{i=1}^{M}\int_{t_\circ-\tau_i(t_\circ)}^{t_\circ}\left|a_i\big(I_i(s)\big)\right|ds\right]e^{-\int_{t_\circ}^{t^*} H(u)du}$$
$$+\,\varepsilon\sum_{i=1}^{M}\int_{t^*-\tau_i(t^*)}^{t^*}|a_i(I_i(s))|ds$$
$$+\,\varepsilon\int_{t_\circ}^{t^*} e^{-\int_s^{t^*} H(u)du}\sum_{i=1}^{M}|a_i(s)||\dot{\tau}_i(s)|ds$$
$$+\,\varepsilon\int_{t_\circ}^{t^*} e^{-\int_s^{t^*} H(u)du}|H(s)|\left[\sum_{i=1}^{M}\int_{s-\tau_i(s)}^{s} b_1(v)|a_i(I_i(v))|dv\right]ds$$
$$\leq 2\delta k e^{\int_0^{t_\circ} H(u)du} + \sigma\varepsilon < \varepsilon \qquad \ldots(3.3)$$

Which contradicts the definition of t^*. Thus $|y(t)| < \varepsilon$ for all $t \geq t_\circ$, and the zero solution of (1.1) is stable. This shows that the zero solution of (1.1) is asymptotically stable if (3) holds.
Conversely, suppose that (3) fails. Then by (1) there exists a sequence $\{t_n\}$,
$t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \int_0^{t_n} H(u)du = d \qquad \dots (3.4)$$

for some $d \in \mathbb{R}$.
We may also choose a positive constant J satisfying

$$-J \leq \int_0^{t_n} H(s)ds \leq J$$

for all $n \geq 1$. To simplify expressions, we define

$$\beta(s) = \sum_{i=1}^{M} \left[|a_i(s)||\dot{\tau}_i(s)| + |H(s)| \int_{s-\tau_i(s)}^{s} \left|a_i\big(I_i(v)\big)\right| dv \right]$$

for all $\delta \geq 0$. By (2), we have

$$\int_0^{t_n} e^{-\int_s^{t_n} H(u)du} \beta(s)ds \leq \sigma$$

This yields

$$\int_0^{t_n} e^{\int_0^s H(u)du} \beta(s) \leq \sigma e^{\int_0^{t_n} H(u)du} \leq e^J$$

The sequence $\left\{\int_0^{t_n} e^{\int_0^s H(u)du} \beta(s)ds\right\}$ is bounded, so there exists a convergent subsequence.
By equation (3.4), we may assume

$$\lim_{n\to\infty} \int_0^{t_n} e^{\int_0^s H(u)du} \beta(s)ds = \theta$$

for some $\theta \in \mathbb{R}^+$ and choose a positive integer w so large that

$$\int_{t_w}^{t_n} e^{\int_o^s H(u)du} \beta(s)ds \leq \frac{\delta_\circ}{4k}$$

for all $n \geq w$, where $\delta_\circ > 0$ satisfies

$$4\delta_\circ k e^J + \sigma < 1$$

By (1) , k in (3.1) is well defined. We now consider the solution
$y(t) = y(t , t_w , \Phi)$ of (1.1) with $\Phi(t_w) = \delta_\circ$ and $|H(s)| \leq \delta_\circ$ for $s \leq t_w$.
An argument similar to that in (3.3) shows $|y(t)| \leq 1$ for $t \geq t_w$.
We may choose Φ so that

$$\Phi(t_w) - \sum_{i=1}^{M} \int_{t_w-\tau_i(t_w)}^{t_w} a_i\big(I_i(s)\big)\Phi(s)ds \geq \frac{1}{2}\, \delta_\circ$$

It follows from (1.3) with $y(t) = (T_y)_{(t)}$ that for $n \geq t_w$

$$\left| y(t_n) - \sum_{i=1}^{M} \int_{t_n-\tau_i(t_n)}^{t_n} a_i\big(I_i(s)\big)y(s)ds \right|$$

$$\geq \frac{1}{2}\delta_\circ e^{-\int_{t_w}^{t_n} H(u)du} - e^{-\int_0^{t_n} H(u)du} \int_{t_w}^{t_n} e^{\int_0^s H(u)du} \beta(s)ds$$

$$= \frac{1}{2}\delta_\circ e^{-\int_{t_w}^{t_n} H(u)du} - e^{-\int_0^{t_n} H(u)du} \int_{t_w}^{t_n} e^{\int_0^s H(u)du}\beta(s)ds$$

$$= e^{-\int_{t_w}^{t_n} H(u)du}\left[\frac{1}{2}\delta_\circ - e^{-\int_0^{t_w} H(u)du} \int_{t_w}^{t_n} e^{\int_0^s H(u)du}\beta(s)ds\right]$$

$$\geq e^{-\int_{t_w}^{t_n} H(u)du}\left[\frac{1}{2}\delta_\circ - k \int_{t_w}^{t_n} e^{\int_0^s H(u)du}\beta(s)ds\right]$$

$$\geq \frac{1}{4}\delta_\circ e^{\int_{t_w}^{t_n} H(u)du} \geq \frac{1}{4}\delta_\circ e^{-2J} > 0 \qquad \dots (3.5)$$

On the other hand, if the zero solution of (1.1) is asymptotically stable ,then $y(t) = y(t, t_w, \Phi) \to 0$ as $t \to \infty$. Since $t_n - \tau_i(t_n) \to \infty$ as $n \to \infty$ and (2) holds, we have

$$y(t_n) - \sum_{i=1}^{M} \int_{t_n-\tau_i(t_n)}^{t_n} a_i\big(I_i(s)\big)y\big(s-\tau(s)\big)ds \to 0 \qquad as\ t \to \infty$$

Which contradicts (3, 5). Hence condition (3) is necessary for the asymptotic stability of the zero solution if (1.1).

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