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# Semi-$T$-maximal sumodules

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**Abstract**

Let $R$ be a commutative ring with identity and $M$ be an $R$-module. In this work, we present the concept of semi-$T$-maximal sumodule as a generalization of $T$-maximal submodule.

We present that a submodule $K$ of an $R$-module $M$ is a semi-$T$-maximal (sortly $S$-$T$-max) submodule if $\frac{T+K}{K}$ is a semisimple $R$-module (where $T$ is a submodule of $M$). We investigate some properties of these kinds of modules.

**Key Words:** $T$-maximal submodule, Semi maximal, T-Radical, $T$-cosemisimple

# المقاسات الجزئيه شبه العظمى من النمط T

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**الخلاصه**

لتكن Rحلقة ابداليه ذات عنصر محايد و M مقاسا معرفا على الحلقة Rفي هذا البحث قدمنا مفهوم المقاس الجزئي شبه الاعظم من النمط T كتعمييم للمقاس الاعظم من النمط T . يدعى المقاس الجزئي K من المقاس M المعرف على الحلقة R بانه مقاس جزئي شبه اعظم من النمط T اذا كان $\frac{T+K}{K}$ مقاس شبه بسيط على الحلقة R . قدمنا بعض الخصائص لهذا النوع من المقاسات.

## 1. Introduction

Throughout this paper, $R$ is a ring with identity and every $R$-module is unitary left $R$-module, unless otherwise stated. A proper submodule $N$ of $M$ is called maximal if and only if there is no proper submodule of $M$ different from $N$ containing $N$ properly **[1]**. The concept of semimaximal submodules was initially introduced [2] where a submodule $N$ of an $R$-module $M$ is called semimaximal submodule if and only if $M/N$ is a semisimple $R$-module. A previous report [3] introduced the concept of $T$-maximal submodule, where a submodule $K$ of $M$ is called $T$-maximal submodule of $M$ of $\frac{T+K}{K}$ is simple. This concept leads to introduce the following concept; if $K$ and $T$ are two submodules of an $R$-module $M$, $K$ is said to be semi-$T$-maximal (shortly $S$-$T$-max) submodule of $\frac{T+K}{K}$ is a semisimple $R$-module.

The paper contains three parts. In part two, we investigate the concept of semi-$T$-maximal submodule and provide the basic properties of this concept. We observe that the intersection of two semi-$T$-maximal submodules is also semi-$T$-maximal (Prop. 2.2), and the homomerphic image of semi-$T$-maximal submodule is semi-$T$-maximal under certain conditions (Prop. 2.9).

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Also we define a new concept which is called semi-T- Radical and we prove some relations about it. S.2 Semi-T-maximal submodules

In this section, we present the concept of semi-$T$-maximal submodules as a generalization of $T$-maximal submodules.

**Definition 2.1:** If $K$ and $T$ are submodules of an $R$-module $M$. $K$ is said to be a semi-$T$-maximal (shortly $S$-$T$-max ) submodule if $\frac{T+K}{K}$ is a semisimple $R$-module.

**Remarks and Examples 2.2**

(1) Obviously, every $T$-maximal submodule is a $S$-$T$-max subomdule. However, for the $Z$-module $Q$. If $K = 6Z, T = 5Z$ then $\frac{T+K}{K} = \frac{5Z+6Z}{6Z} = \frac{Z}{6Z} \simeq Z_6$ which is $Z$-semisimple that is $K$ is $S$-$T$-max, and $K$ is not $T$- maximal because $Z_6$ is not simple.

(2) For any $R$-module $M$, If $T = 0$, then every submodule $K$ of $M$ is $S$-$T$-max, since $\frac{K+0}{K} = \frac{K}{K} \simeq (0)$ is semisimple.

(3) For any $R$-module $M$, $M$ is $S$ - $T$-max .

(4) It is clear that every semi-maximal submodule of an $R$-module $M$ is $S$ - $T$-max, where a submodule $U$ of $M$ is semimax if $\frac{M}{U}$ is semisimple [4]

On the other hand, consider $M = \langle\bar{2}\rangle \subseteq Z_{24}$ as a $Z$-module and $K = \langle\bar{8}\rangle \leq M$. $\frac{M}{K} \simeq Z_4$ is not semisimple, so $K$ is not semimaximal.

If $T = \langle\bar{4}\rangle \leq M$, then $\frac{T+K}{K} = \frac{\langle\bar{4}\rangle+\langle\bar{8}\rangle}{\langle\bar{8}\rangle} \simeq \frac{\langle\bar{4}\rangle}{\langle\bar{8}\rangle} \simeq Z_2$

Which is semisimple, that is $K$ is $S$-$T$-max submodule of $M$ .

Beside these, if $T = M$, then every $S$-$T$-max submodule of $M$ is semisimple.

(5) If $A$ and $T$ are submodules of $M$ with $A + T = M$, then $A$ is semimaximal if and only if $A$ is $S$-$T$-max.

(6) $A$ and $T$ are submodules of $M$ and $T$ is semisimple, then $A$ is $S$-$T$-max.

**Proof:** Since $\frac{T+A}{A} \simeq \frac{T}{T\cap A}$ ( by 2nd Fundamental theorem ) and $\frac{T}{T\cap A}$ is semisimple (because $T$ is semisimple), hence $\frac{T+A}{A}$ is semisimple and $A$ is $S$-$T$-max.

(7) If $K$ is an $S$-$T$-max submodule of $M$ and $K_1 \leq K$, then $K_1$ is not $S$-$T$-max in general as: when $M$ be $Z$-module $Q$, $T = Z$, $K = 6Z$, $K_1 = 12Z$, $K_1 \subseteq K$, $\frac{T+K}{K} = \frac{Z+6Z}{6Z} \simeq Z_6$ which is semisimple, so that $K$ is $S$-$T$-max, however $\frac{T+K_1}{K_1} \simeq \frac{Z+12Z}{12Z} \simeq Z_{12}$ is not semisimple and this means $K_1$ is not $S$-$T$-max.

(8) Let $T, K \leq M$. If $K$ is $S$-$T$-max, then $K$ is $S$-A-max, for all $A \subseteq T$.

**Proof**: Since $K$ is $S$-$T$-max, then $\frac{T+K}{K}$ is semisimple. As $A \subseteq T$ impllies $\frac{A+K}{K} \leq \frac{T+K}{K}$ and so $\frac{A+K}{K}$ is semisimple, that is $K$ is $S$-$A$-max.

**Proposition 2.3:** If $N$ is an $S$-$T$-max submodule of $N \leq K \leq M$, then $K$ is $S$-$T$-max.

**Proof:** Let $f: \frac{M}{N} \to \frac{M}{K}$ defined by $f(m + N) = m + K$ for all $m \in M$. One can easily check that $f$ is well-define and epimorphic. Since $N$ is $S$-$T$-max, $\frac{T+N}{N}$ is semisimple so that $f\left(\frac{T+N}{K}\right) = \frac{T+K}{K}$ is semisimple see [5, cor. (8.1.5), 2, 192]. Thus $K$ is $S$-$T$-max.

**Corollary 2.4:** If $N$ is an $S$-$T$-max submodule of $M$, then $N + A$ is $S$-$T$-max, for all $A \leq M$.

**Corollary 2.5:** Let $N$ be an $S$-$T$-max submodule of $M$. Then $[N:_M I]$ is $S$-$T$-max, for each ideal $I$ of $R$.

**Proof:** Since $N \leq [N:_M I]$, the result follows directly by Prop.2.3.

**Proposition 2.6:** Let $T$, $K$are two submodules of an $R$-module $M$. Then $T\cap K$ is $S$-$T$-max, if and only if $K$ is $S$-$T$-max.

**Proof:**⇒) It follows directly by Prop.2.3.

⇐) Since $K$ is $S$-$T$-max., then $\frac{T+K}{K}$ is semisimple. By 2nd iso.Th., $\frac{T+K}{K} \simeq \frac{T}{T\cap K}$ and $\frac{T}{T\cap K} = \frac{T+(T\cap K)}{T\cap K}$, so that $\frac{T+(T\cap K)}{T\cap K}$ is semisimple. Thus $(T \cap K)$ is $S$-$T$-max.

**Proposition 2.7**: Let $M$ be an $R$-module and let $A, B$ are two submodules of $M$ then $\frac{M}{A\cap B}$ isomorphic to a submodule of $\frac{M}{A}\oplus\frac{M}{B}$.

**Proof:** Define $f:\rightarrow\frac{M}{A}\oplus\frac{M}{B}$ by $f(m)=(m+A, m+B)$, $\forall m\in M$. Then $f$ is a well-defined $R$-homomorphism and

$$kerf=\{m\in M: f(m)=(0,0)\}=\{m\in M:(m+A, m+B=(0,0)\}$$
$$=\{m\in M: m\in A\cap B\}=A\cap B$$

Thus by 1st Fund. Th. $\frac{M}{kerf}\simeq Imf\leq\frac{M}{A}\oplus\frac{M}{B}$; that is $\frac{M}{A\cap B}$ isomorphic to a submodule of $\frac{M}{A}\oplus\frac{M}{B}$.

**Proposition 2.8:** if $A$ and $B$ are $S$-$T$-max submodules of an $R$-module $M$, then $A\cap B$ is an $S$-$T$-max submodule of $M$.

**Proof**: Since $A$ and $B$ are $S$-$T$-max submodules, then $\frac{T+A}{A}$ and $\frac{T+B}{B}$ are semisimple of $\frac{M}{A}$ and $\frac{M}{B}$, respectively. By 2nd Fund. Th, $\frac{T+A}{A}\cong\frac{T}{T\cap A}$ and $\frac{T+B}{B}\cong\frac{T}{T\cap B}$. Now, $\frac{T+(A\cap B)}{A\cap B}\cong\frac{T}{A\cap B\cap T}=\frac{T}{(A\cap T)\cap(B\cap T)}$. So that by lemma (2.7) $\frac{T}{(A\cap T)\cap(B\cap T)}$ isomorphic to submodule of $\frac{T}{(A\cap T)}\oplus\frac{T}{(B\cap T)}$. On the other hand, $\frac{T}{(A\cap T)}\oplus\frac{T}{(B\cap T)}$ is semisimple [ 5. Cor.8.1.5(3), 192 ] so that any submodule of it is semisimple. Thus $\frac{T}{(A\cap T)\cap(B\cap T)}$ is isomorphic semisimple, and hence $\frac{T+(A\cap B)}{(A\cap T)}$ is semisimple. Therefore $(A\cap B)$ is $S$-$T$-max.

Since every $T$-maximal submodule of $M$ is $S$-$T$-max, then the intersection of any two $T$-maximal submodules is $S$-$T$-max.

**Proposition 2.9:** Let $f: M\rightarrow N$ be an R-homorphism, and $K$, $T$ are two submodules of $M$, such that $\ker f\subseteq K$. If $K$ is $S$-$T$-max, then $f(K)$ is $S$-$f(T)$-max.

**Proof:** Since $K$ is $S$-$T$-max, then $\frac{T+K}{K}$ is a semi simple $R$-module. To prove $f(K)$is $S$-$f(T)$-max, we must show that $\frac{f(T)+f(K)}{f(K)}$ is semisimple. Submodule of $\frac{N}{f(K)}$. Define $\tilde{f}:\frac{M}{K}\rightarrow\frac{N}{f(K)}$ by $\tilde{f}(m+K)=f(m)+f(K)$ for each $m\in M$ clearly $\tilde{f}$ is a well-defined R- homomorphism. Hence, $\tilde{f}\left(\frac{T+K}{K}\right)$ is semisimple by [5, cor.(8.1.5)(2),p.192 ] and $\frac{f(T)+f(K)}{f(K)}$ is semisimple. Thus $f(K)$ $S$-$f(T)$-max.

The following result follows directly by Prop. 2.9.

**Corollary 2.10:** Let $N$ be an $S$-$T$-max submodule on R-module and let $K\subseteq N$. Then $\frac{N}{K}$ is an $S-\frac{T+K}{K}$-max submodule of $\frac{M}{K}$.

**Proposition 2.11**: Let $T$, $K$ and $N$ be submodules of an $R$- module $M$. Then $K$ is $S$-$T$-max, whenever $K\leq N\leq T+K$ and $N$ is a direct summand of $T+K$.

**Proof**: To prove that $K$ is $S$-$T$-max, we must show that $\frac{T+K}{K}$ is semisimple. Let $\frac{N}{K}<\frac{T+K}{K}$. Then $K\leq N\leq T+K$, and so by hypothesis $N$ is a direct. It summand of $T+K$. Hence $T+K=N\oplus U$ for some $U\leq T+K$ it follows that $\frac{N}{K}\oplus\frac{U+K}{K}=\frac{T+K}{K}$ and therefore $\frac{N}{K}$ is a direct summand of $\frac{T+K}{K}$, which implies that $\frac{T+K}{K}$ is semisimple.

**Proposition 2.12**: Let $A, T$ be submodules of an $R$-module $M$ with $A\nsubseteq T$. If $A$ is $S$-$T$-max then $A+T=xR+K$ for some $K\supseteq A$ and $\forall x\in A+T$, $x\notin A$.

**Proof:** let $x\in A+T$ and $x\notin A$. Then $A\subsetneqq A+xR\subseteq A+T$, and so $\frac{A+xR}{A}\leq\frac{A+T}{A}$. As $\frac{A+T}{A}$ is semisimple because $A$ is $S$-$T$-max, it follows that $\frac{A+xR}{A}\leq^{\oplus}\frac{A+T}{A}$. Hence $\frac{A+T}{A}=\frac{A+xR}{A}\oplus\frac{K}{A}$, for some $\frac{K}{A}\subseteq\frac{A+T}{A}$, this implies that $A+T=A+xR+K=xR+K$ therefore $A+T=xR+K$, $K\supseteq A$.

**Proposition 2.13**: Let $M=M_1\oplus M_2$ where $M_1$ and $M_2$ are $R$-modules, let $A\leq M_1$ $B\leq M_2$. Then $A\oplus B$ is an $S$-$T_1\oplus T_2$-max submodule of $M$ if and only if $A$ is $S$-$T_1$-max in $M_1$ and $B$ is $S$-$T_2$-max in $M_2$.

**Proof:** $\Rightarrow$) if $A\oplus B$ is an $S$-$T_1\oplus T_2$-max in $M$, then $\frac{T_1\oplus T_2+(A\oplus B)}{A\oplus B}$ is semisimple. Since $\frac{T_1\oplus T_2+(A\oplus B)}{A\oplus B}=\frac{(T_1+A)\oplus(T_2+B)}{A\oplus B}$ which is isomorphic to $\frac{T_1+A}{A}\oplus\frac{T_2+\boldsymbol{B}}{B}$ it follows that $\frac{T_1+A}{A}$ and $\frac{T_2+B}{B}$ are semisimple by [5 , cor 8.1.5(1),192]. Thus $A$ is $S$-$T_1$-max and $B$ is $S$-$T_2$-max.
$\Leftarrow$) If $A$ is $S$-$T_1$-max and $B$ is $S$-$T_2$-max, then $\frac{T_1+A}{A}$ and $\frac{T_2+B}{B}$ are semisimple modules. Hence $\frac{T_1+A}{A}\oplus\frac{T_2+\boldsymbol{B}}{B}$ is semisimple [ 5, cor.8.1.5(3),192]. But $\frac{T_1+A}{A}\oplus\frac{T_2+\boldsymbol{B}}{B}\simeq\frac{(T_1+A)\oplus(T_2+B)}{A\oplus B}=\frac{(T_1\oplus T_2)+(A\oplus B)}{A\oplus B}$. Therefore $A\oplus B$ is $S$-$T_1\oplus T_2$-max.
"A submodule $A$ of an R-module $M$ is called essential (large) in $M$ (shortly $A\leq_{ess} M$) if whenever $A\cap C=0$, $C\leq M$, then two $C=(0)$" [4].
The next two results are characterizations of an $S$-$T$-max submodule of a module.
**Theorem 2.14:** Let $M$ be an R-module and let $N\lneqq M$ and $T\leq M$. Then $N$ is $S$-$T$-max if and only if $S$-$T$-max if and only if there are $A$, $B\leq T+N$ with $B\supseteq N$ such that $T+N=A\oplus B$ where $A$ is semisimple, $N\leq_{ess} B$ and $N$ is an $S-(T\cap B)$-max submodule of $B$, also $N$ is semimaximal in $B$.
**Proof:** $\Rightarrow$) Assume $N$ is an $S$-$T$-max submodule of $M$, so $\frac{T+N}{N}$ is semisimple. Let $A$ be a complement of $N$ in $T+N$. Hence $A\oplus B\leq_{ess} T+N$. Now $A\cong\frac{A\oplus N}{N}\leq\frac{T+N}{N}$, and so $\frac{A\oplus N}{N}\leq^{\oplus}\frac{T+N}{N}$ by [5. Th.8.1.3 (4), p.191]. It follows that $\frac{T+N}{N}=\frac{A\oplus N}{N}\oplus\frac{B}{N}$ for some $\frac{B}{N}\leq\frac{T+N}{N}$. Hence $T+N=(A\oplus N)+B=A+B$ (since $N\subseteq B$). We claim that $A\cap B=(0)$, if $x\in A\cap B$, then $x+N\in\frac{A+N}{N}\cap\frac{B}{N}=0_{\frac{T+N}{N}}$ , hence $x+N=N$ and so $x\in N$. Thus $A\cap B\subseteq N$. But $A\cap(A\cap B)\subseteq A\cap N=0$, which implies $A\cap B\subseteq(0)$, that is $A\cap B=(0)$. Therefore $A\oplus B=T+N$. Now, since $A\simeq\frac{A\oplus N}{N}\leq\frac{T+N}{N}$.
Hence $\frac{A+N}{N}$ is semisimple and so $A$ is semisimple. To prove $N\leq_{ess} B$.
Let $C\leq B$ and $C\cap N=0$. As $A$ is a complement of $N$, so $C\leq A$. It follows that $C\subseteq A\cap B$ (since $C\leq$B). Thus $C=0$ and $N\leq_{ess} B$.
To show that $N$ is $S$-$T\cap B$-max: we have $\frac{(T\cap B)+N}{N}\leq\frac{T+N}{N}$, but $\frac{T+N}{N}$ is semisimple , so $\frac{(T\cap B)+N}{N}$ is semisimple and $N$ is an $S$-$T\cap B$-max submodule of $B$. Moreover $\frac{B}{N}\leq\frac{T+N}{N}$, so $\frac{B}{N}$ is semisimple (i.e. $N$ is semimax in $B$).
$\Leftarrow$) Since $T+N=A\oplus B$, where $A$ is semisimple, $N\leq_{ess} B$.
Hence, $\frac{T+N}{N}=\frac{A+N}{N}+\frac{B}{N}$
But $\frac{A+N}{N}\simeq\frac{A}{A\cap B}$ and since $A$ is semisimple, so $\frac{A}{A\cap B}$ is semisimple thus $\frac{A+N}{N}$ is semisimple. Also $N$ is semimaximal in $B$, so $\frac{B}{N}$ is semisimple. Thus $\frac{A+N}{N}+\frac{B}{N}$ is semisimple, that is $\frac{T+N}{N}$ is semisimple and $N$ is an $S$-$T$-max submodule of $M$.
**Theorem 2.15**: Let $N<M$ and $T\leq M$. Then $N$ is $S$-$T$-max if and only if for each $A\leq T+N$, there exists $B\leq M$, $B\supseteq N$, $B\subseteq T+N$ such that $A+B=T+N$ and $A\cap B\subseteq N$.
**Proof:** $\Rightarrow$) Since $N$ is $S$-$T$-max, $\frac{T+N}{N}$ is semisimple and as $\frac{A+N}{N}\leq\frac{T+N}{N}$ for each $A\leq T+N$. Hence $\frac{A+N}{N}\leq^{\oplus}\frac{T+N}{N}$ and so $\frac{T+N}{N}=\frac{A+N}{N}\oplus\frac{B}{N}$ for some $B\leq T+N$ and $B\supseteq N$. It follows that $T+N=A+N+B=A+B$. Now, let $x\in A\cap B$. Then $+N\in\frac{A+N}{N}\cap\frac{\boldsymbol{B}}{N}=O_{\frac{T+N}{N}}$ , so that $x\in N$. Thus $A\cap B\subseteq N$.
$\Leftarrow$) Let $\frac{A}{N}<\frac{T+N}{N}$ then $A<T+N$. By hypothesis, there is $B\leq T+N$, $B\supseteq N$ such that $A+B=T+N$ and $A\cap B\subseteq N$. Hence
$\frac{A}{N}+\frac{B}{N}=\frac{T+N}{N}$. But $A\cap B\supseteq N$ (since $A\supseteq N$ and $B\supseteq N$), hence $A\cap B=N$ and $\frac{A}{N}\cap\frac{B}{N}=\frac{A\cap B}{N}=O_{\frac{T+N}{N}}$.
Thus $\frac{A}{N}\leq^{\oplus}\frac{T+N}{N}$ and $\frac{T+N}{N}$ is semisimpore therefore $N$ is $S$-$T$-max.
As we mentioned in Rem. & exp. 2.2, every $T$-maximal submodule is an $S$-$T$-maximal submodule, but not conversely. However the know definition are nedeed "A proper submodule $N$ of $M$ is said to be prime if whenever $rx\in N$, $r\in R$, $x\in M$, then $x\in N$ or $r\in(N{:}M)$" [6].
"An R-module $M$ is called prime if (0) is a prime submodule of $M$"

**Proposition 2.16:** If $N$ is an $S$- $T$-max submodule of $M$ and $P$ be a prime submodule of $M$ containing $N$, then $P$ is a $T$-maximal submodule.

**Proof:** By hypothesis, $N$ is $S$- $T$-max, so $\frac{T+N}{N}$ is semisimple. Also $P \supseteq N$ implies $P$ is $S$- $T$-max by Prop (2.3) , and hence $\frac{T+P}{P}$ is semisimple. Since $P$ is a prime submodule of $M$, then $\frac{T+P}{P}$ is a prime submodule of $M/P$ can be obtained as follows:

If $r(x+P) = P = O_{\frac{P+T}{P}}$, $r \in R$, $x \in T$, then $rx \in P$ which implies either $x \in P$ or $r \in (P: M)$. Hence either $x + P = P$ or $rM \subseteq P$. If $X + P = P$, we are done, if $rM \subseteq P$, then $r(P+T) \subseteq P$ which implies $r \in [P: P+T]$, i.e $r \in ann(\frac{T+P}{P})$. Therefore $\frac{T+P}{P}$ is a prime module. But $\frac{T+P}{P}$ is semisimple, $\frac{T+P}{P}$ is simple. Thus $P$ is a T-maximal submodule.

**Corollary 2.17:** If $N$ is $S$- $T$-max and prime submodule, then $N$ is $T$-maximal.

**Proof:** It follows directly by Prop. (2.15).

"A module $M$ over an integral domain is called Torsion free if $T(M) = 0$, where $T(M) = \{m \in M: there\ exists\ r \in R, r \neq 0, mr = 0\}$ " [ 4]

"A submodule $U$ of $M$ is pure if $MI \cap U = UI$ for each ideal $I$ of $R$" [7].

Corollary 2.18: Let $M$ be a torsion free module over integral domain. If $N \leq P \leq M$ such that $N$ is $S$-$T$-max, and $P$ is pure, then $P$ is $S$- $T$-maximal.

**Proof:** Since $P \supseteq N$ and $N$ is $S$- $T$-max, then by (Prop. 2.3), $P$ is $S$- $T$-max. Now we can show that $P$ is a prime submodule as follows:

Let $xr \in P$, $r \in R$, $x \in M$. Then $xr \in Mr \cap P = Pr$, so that $xr = wr$ for some $wr \in P$. It follows that $(x-w)r = 0$ and hence $x - w = 0$ since $M$ is torsion free. Thus $x = w \in P$ and therefore $P$ is prime. Then $P$ is $T$-maximal by (Cor. (2.17)).

**Corollary 2.19:** Let $M$ be a module over integral domain and let $N \leq T(M) < M$ and $N$ is $S$-$T$-max. Then $T(M)$ is a $T$-maximal.

**Proof:** $T(M) \supseteq N$, so $T(M)$ is an $S$-$T$-max by Prop. 2.3. On the other hand, since $T(M) < M$, $T(M)$ is a prime submodule, thus $T(M)$ is a $T$-maximal submodule of $M$, by Cor (2.17).

"It is known that every primary submodule $N$ with $(N:_R M)$ is a prime ideal, is a prime submodule " [8, prop.(2.10) ], where " a submodule $N$ of $M$ is prime if whenever $r \in R$, $x \in M$ , $rx \in N$ implies $x \in N$ or $r^n \in (N: M)$ for some $n \in Z_+$ " [9].

**Corollary 2.20:** Let $N$ be a primary submodule of $M$ with $(N:_R M)$ is a prime ideal. If $N \supseteq W$ and $W$ is $S$-$T$-max, then $N$ is $T$-maximal.

**S3 Semi- $T$-Redical**

Authors of a previous work [3] denoted the intersection of $T$-maximal submodule in an $R$-module $M$ (where $T \leq M$ ) by $Rad_T M$.

We introduce the following:

**Definition 3.1:** let $M$ be an $R$-module and $T \leq M$. The intersection of all $S$-$T$-max submodules of $M$ by $S{-}Rad_T M$.

Note that, for any $T \leq M$, $M$ is $S$-$T$-max.

**Examples 3.2:**

1- Consider the $Z$- module $Z_p\infty$. If $T = Z_p\infty$, then for each proper submodule $A$ of $Z_p\infty$, $\frac{A+T}{A} = \frac{A+Z_p\infty}{A} \simeq \frac{Z_p\infty}{A} \simeq Z_p\infty$ is not semisimple, so $A$ is not $S$-$T$- max. Hence $S - Rad_T Z_p\infty = Z_p\infty$. Also $Rad_T Z_p\infty = Z_p\infty$ by [ 10, Ex. 1.3.18].

2- Consider $M = Z_{12}$ as a $Z$-module. Let $T = \langle\bar{4}\rangle$ since $\frac{(\bar{0})+T}{T} = \frac{(\bar{0})+\langle\bar{4}\rangle}{(\bar{0})} \simeq \langle\bar{4}\rangle$ which is semisimple, so that $(\bar{0})$ is $S$-$T$- max. Thus $S - Rad_T Z_{12} = (\bar{0})$ on the other $(\bar{0})$ is not $T$- maximal sice $\frac{(\bar{0})+T}{T}$ is not simple. Now it is easy to notice that $N_1 = \langle\bar{2}\rangle$ is not $T$- maximal, $N_2 = \langle\bar{3}\rangle$ is $T$- maximal, $N_3 = \langle\bar{4}\rangle$ is not $T$-maximal, $N_4 = \langle\bar{6}\rangle$ is $T$-maximal, $N_6 = M$ is not $T$-maximal. Thus $Rad_T M = \langle\bar{3}\rangle \cap \langle\bar{6}\rangle = \langle\bar{6}\rangle$ and hence $S - Rad_T M \subsetneq Rad_T M$.

To prove the next result, we need the following:

**Proposition 3.3:** Let $M$ and $N$ be $R$-modules, let $f: M \to N$ be an epimorphism, let $T \leq M$ and $K \leq N$. If $K$ is an $S - f(T)$-max submodule of $N$ then $f^{-1}(K)$ is an $S - T$-max submodule of $M$.

**Proof:** Since $K$ is $S-f(T)$-max, then $\frac{f(T)+K}{K}$ is semisimple. To prove that $f^{-1}(K)$ is $S-T$-max, we must show that $\frac{f^{-1}(K)+T}{f^{-1}(K)}$ is semisimple. Let $\frac{A}{f^{-1}(K)} \leq \frac{f^{-1}(K)+T}{f^{-1}(K)}$, then $A \subseteq f^{-1}(K)+T$ and so $f(A) \subseteq ff^{-1}(K)+f(T)=K+f(T)$.

But $f^{-1}(K) \subseteq A$ implies $K=ff^{-1}(K) \subseteq f(A)$. Hence $\frac{f(A)}{K} \leq \frac{K+T}{K}$ (which is semisimple). It follows that $\frac{f(A)}{K} \leq^{\oplus} \frac{K+f(T)}{K}$ and so $\frac{K+f(T)}{K}=\frac{f(A)}{K} \oplus \frac{W}{K}$ for some $\frac{W}{K} \leq \frac{K+f(T)}{K}$. Hence $K+f(T)=f(A)+W$. This implies that $f^{-1}\big(K+f(T)\big)=f^{-1}(f(A)+w)$ and so $f^{-1}\big(K)+f^{-1}f(T)\big)=f^{-1}f(A)+f^{-1}(w)$ (since $f$ is epi.) Thus $f^{-1}(K)+T+kerf=A+\ker f+f^{-1}(W)$, but $kerf=f^{-1}\{0\} \subseteq f^{-1}(K)$, $kerf=f^{-1}\{0\} \subseteq f^{-1}(W)$, then $f^{-1}(K)+T=A+f^{-1}(W)$. Therefore $\frac{f^{-1}(K)+T}{f^{-1}(K)}=\frac{A}{f^{-1}(K)}+\frac{f^{-1}(W)}{f^{-1}(K)}$. Moreover, we can see that: $\frac{A}{f^{-1}(K)} \cap \frac{f^{-1}(W)}{f^{-1}(K)}=(0)$
as follows:-

Let $a+f^{-1}(K) \in \frac{A}{f^{-1}(K)} \cap \frac{f^{-1}(W)}{f^{-1}(K)}$. Then $a \in A \cap f^{-1}(W)$ and $f(a) \in f(A) \cap W$, hence $f(a)+K \in \frac{f(A)}{K} \cap \frac{W}{K}=(0)$, so that $f(a) \in K$; that is $a \in f^{-1}(K)$. Thus $a+f^{-1}(K)=f^{-1}(K)=(0)_{M/f^{-1}(K)}$. Therefore $\frac{A}{f^{-1}(K)} \cap \frac{f^{-1}(W)}{f^{-1}(K)}=(0)$ and $\frac{A}{f^{-1}(K)} \leq^{\oplus} \frac{f^{-1}(K)+T}{T}$, that is $f^{-1}(K)$ is $S-T$-max.

**Theorem 3.4**: Let $M$ and $N$ be $R$-modules, $f: M \rightarrow N$ be an epimorphism such that $\ker f \leq S-Rad_T\, M$. Then f( $S{-}Rad_T\, M)=S{-}Rad_{f(T)}\, N$

**Proof**: Since $S{-}Rad_T\, M=\bigcap_{i \in \Lambda} A_i$, $A_i$ is $S$-$T$-max, $\forall i \in \Lambda$, $f(S{-}Rad_T\, M)=f(\bigcap_{i \in \Lambda} A_i) \subseteq \bigcap_{i \in \Lambda} f(A_i)$. But $kerf \subseteq A_i$ , $\forall i \in \Lambda$ by hypothesis,
$\cap_{i \in \Lambda} f(A_i)=f(\cap_{i \in \Lambda} A_i)$, that is $f(S{-}Rad_T\, M)=\bigcap_{i \in \Lambda} f(A_i)$. By (Prop.(3.3)) and (Prop. 2.9), $A$ is $S$-$T$-max, implies $f(A)$ is $S$-$f(T)$-max and $B$ is $S$-$f(T)$-max, implies $f^{-1}(B)$ is $S$-$T$-max. Therefore $f(S{-}Rad_T\, M)=\bigcap_{i \in \Lambda} f(A_i)=S{-}Rad_{f(T)}\, N$.

**Proposition 3.5:** Let $M=M_1 \oplus M_2$ where $M_1$ and $M_2$ are submodules of $M$, with $annM_1+annM_2=R$ and $T=T_1 \oplus T_2 \leq M$ then $S-Rad_T M=S-Rad_{T_1} M_1 \oplus S-Rad_{T_2} M_2$.

**Proof:** $S-Rad_T M=\bigcap_{i \in \mathrm{I}} A_i$, $A_i$ $is$ $S-T$-max submodule of $M$. since $annM_1+annM_2=R$, then $A_i=B_i \oplus W_i$ for some $B_i \leq M_1$ and $C_i \leq M_2$. Hence $S-Rad_T M=\bigcap_{i \in \mathrm{I}} B_i \oplus \bigcap_{i \in \mathrm{I}} W_i$. Moreover, for each $i \in \mathrm{I}$, $A_i$ is an $S-T$-max submodule of $M=M_1 \oplus M_2$, implies $B_i$ is an $S-T_1$-max submodule of $M_1$ and $C_i$ is an $S-T_2$-max submodule of $M_2$. $\forall i \in \mathrm{I}$, by Prop. 2.12, $S-Rad_T M=S-Rad_{T_1} M_1 \oplus S-Rad_{T_2} M_2$.

Let $M$ be an $R$-module and $T$ be a nonzero submodule of $M$. “$M$ is said to be $T$-cosemisimple if every submodule of $M$ is the intersection of $T$-maximal submodules” [3].

We state that $M$ is semi $T$-cosemisimple if every submodule is the intersection of $S-T$-max submodules.

**Remarks 3.6:** Let $M$ be an $R$-module and $T$ is a semisimple submodule of $M$. Then $S-Rad_T M=0$.

**Proof:** By [3, Prop. 39], $Rad_T M=0$. But $S-Rad_T M \subseteq Rad_T M$ by [Ex.3.2(2)]. Thus $S-Rad_T M=0$.

We conclude the paper with the following proposition.

**Proposition 3.7:** Let $M$ be an $R$-module and $T$ be a nonzero submodule of $M$, Then:

1- If $M$ is semi- $T$-cosemisimple, then every submodule of $M$ containing $T$ is semi- $T$-cosemisimple module and $\frac{M}{N}$ is semi-$\frac{T+K}{-K}$-cosemisimple.

2- If $M$ is semi-$T$-cosemisimple if and only if $S$-$Rad_{\frac{T+K}{K}} \frac{M}{K}=0$.

**Proof (1):** Suppose $T \subseteq N \subseteq M$ and $M$ is semi-$T$-cosemisimple. If $L \leq N$, then $L=L \cap N$. Since $L \leq M$, $L=\cap_A S$, where $A$ is a set of $S$-$T$-maximal submodule of $M$. Hence $L=(\cap_A S) \cap N=\cap_A (S \cap N)$.

But $\frac{(S \cap N)+T}{S \cap N} \simeq \frac{T}{S \cap N \cap T} \simeq \frac{T}{S \cap T} \simeq \frac{S+T}{T}$ which is semisimple. Thus $S \cap N$ is an $S$-$T$-max submodule of $M$ and $N$ is semi-$T$-cosemisimple. Now, let $\frac{L}{N} \leq \frac{M}{N}$. Then $L \leq M$ and $L=\cap_A S$, where $A$ is a set of $S$-$T$-

max submodules of $M$. Thus $\frac{L}{N}=\frac{\cap_A S}{N}=\cap_A\left(\frac{S}{N}\right)$. But each $S$ is $S$-$T$-max, hence $\frac{S}{N}$ is an $S$-$\frac{T+N}{N}$-max, by cor. 2.10, therefore $\frac{M}{N}$ is a semi $\frac{T+N}{N}$ cosemisimple module.

(2) Suppose that $M$ is semi-$T$-cosemisimple and $K \leq M$. $S\text{-}Rad_{\frac{T+K}{K}}M = \cap_A \frac{S}{N}$, where $A=\left\{\frac{S}{K}\leq\frac{M}{K}:\frac{S}{K}\ is\ an\ S-\frac{T+K}{K}-\max submodule\ of\ \frac{M}{K}\right\}$

Since $\frac{S}{K}$ is an $S-\frac{T+K}{K}$ max if $S$ is an $S$-$T$-max submodule of $M$, by (cor. (2.10)).

Hence $\frac{S}{K}\in A^*$ if and only if $S\in \acute{A}$, where

$$A^*=\{S\supseteq K: S\leq M\ and\ S\ is\ S-T-\max submodule\ of\ M\}$$

Now $K\leq M$ implies $K=\cap_B S$ where $B$ is a set of $S$-$T$-max submodule of $M$, so $B\subseteq A^*$. Hence $\cap_A\frac{S}{N}=\frac{\cap_{A^*}S}{\cap_B S}=0$, that $S\text{-}Rad_{\frac{T+K}{K}}M=0$.

Conversely, suppose that $S\text{-}Rad_{\frac{T+K}{K}}\frac{M}{K}=0$ for all $K\leq M$ then $S\text{-}Rad_{\frac{T+K}{K}}(\frac{M}{K})=\cap_A\frac{S}{K}=\frac{\cap_{A^*}S}{K}=0$, where

$$A=\left\{\frac{S}{K}\leq\frac{M}{K}:\frac{S}{K}\ is\ an\ S-\frac{T+K}{K}-\max submodule\ of\ \frac{M}{K}\right\}$$

$$\acute{A}=\{S\supseteq K: S\leq M\ and\ S\ is\ an\ S-T-\max submodule\ of\ M\}$$

Therefore $\cap_{\acute{A}} S=K$ and so $M$ is semi-$T$-cosemisimple.

Next, the $Z$-module $Z_{12}$ is semi -$T$-cosemisimple where $=\langle\bar{4}\rangle$ , since every submodule is an $S$-$T$-max.

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