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A Study on n-Derivation in Prime Near – Rings

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Abstract
The main purpose of this paper is to show that zero symmetric prime near-rings, satisfying certain identities on n-derivations, are commutative rings.

Keywords: Prime Near-Ring,Semigroup Ideal, n-Derivations.

دراسة على الاشتقاقات-n في الحلقات المقتربة الاولية

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المديرية العامة للتربية في القادسية ، قسم التخصص الاشرافي ، القادسيه ، العراق

الخلاصة
الهدف الاساسي من البحث هو اثبات انه الحلقات المقتربة الاولية تحت تاثير شروط معينة على الاشتقاقات تصبح حلقات ابدالية.

1. INTRODUCTION

A near – ring is a set A together with two binary operations (+ and .) such that (i) (A,+) is a group (not necessarily abelian),(ii) (A, .) is a semi group, and(iii) $\forall$a,b,c ϵA; we have a.(b + c) = a.b + b.c . In this paper, A will be a zero symmetric near-ring (i.e.,A satisfying 0.x = 0$\forall$ x $\in$A) andC = {a ϵ A, ab = ba for all a ϵ A}. If I $\subseteq$ A, I is said to bea semigroup left ideal (semigroup right ideal) if AI $\subseteq$A (IA $\subseteq$ I) and it will be called a semigroup ideal if I is a semigroup left ideal as well as a semigroup right ideal. denote a.b by ab, $\forall$a,b ϵ A, [a, b] = ab–ba, and a$\circ$b =ab + ba. A is called a prime near-ring if aAb = {0},which implies that either a = 0 or b = 0. For moreinformation aboutthe near-rings, we refer to a previous publication [1].

In another article [2], Ashraf defined n-derivations in the near-rings. In our work, we show that the prime near-rings involving n-derivations,aspreviously defined [2],with some conditionsare commutative rings.

2. PRELIMINARY RESULT

Lemma 2.1. [3]. Let N be a prime near-ring, U a nonzero semigroup right ideal (resp. semigroup left ideal), and x is an element of N such that Ux= {0} (resp. xU = {0}), then x = 0.

Lemma 2.2. [3]. Let N be a prime near-ring and Z contains a nonzero semigroup left ideal or nonzero semigroup right ideal, then N is a commutative ring.

Lemma 2.3.[3]. Let N be a prime near-ring and U be a nonzero semigroup ideal of N. If $x, y \in N$ and$xUy = \{0\}$, then$x = 0$ or$y = 0$.

Lemma2.4.[2]. Let N be a prime near-ring, then d is n-derivation of N if and only if
$d(x_1x_1', x_2, \dots, x_n) = x_1 d(x_1', x_2, \dots, x_n) + d(x_1, x_2, \dots, x_n)x_1'$
$\forall\, x_1, x_1', x_2, \dots, x_n \in$ N.

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Lemma 2.5[2]. Let N be a near-ring and d be n-derivation of N. Then for every $x_1, x_1', x_2, , \dots, x_n, y \in N$
(i)$\left(x_1 d\left(x_1', x_2, \dots, x_n\right) + d(x_1, x_2, \dots, x_n) x_1'\right) y =$
$x_1\ d\left(x_1', x_2, \dots, x_n\right) y + \ \ d(x_1, x_2, \dots, x_n) x_1' y$,
(ii) $(d(x_1, x_2, \dots, x_n) x_1' + x_1 d(x_1', x_2, \dots, x_n)) y =$
$d(x_1, x_2, \dots, x_n) x_1' y + x_1 d\left(x_1', x_2, \dots, x_n\right) y$.
Lemma 2.6 [4]. Let d be n-derivation of a near ring N. Then d(Z,N,…,N) ⊆ Z.
Lemma 2.7 [4]. Let N be a prime near ring, d a nonzero n-derivation of N, and $U_1,U_2,...,U_n$ are nonzero semigroup right (left) ideals of N. If $d(U_1,U_2,...,U_n) = \{0\}$, then d = 0.
Lemma 2.8 [4]. Let N be a prime near ring, d a nonzero n-derivation of N, and $U_1,U_2,...,U_n$ be a nonzero semigroup left ideals of N. If $d(U_1,U_2,...,U_n) \subseteq Z$, then N is a commutative ring.

3. MAIN RESULTS

Theorem 3.1. Let A be a prime near ring and $I_1,I_2,...,I_n$ be semigroup ideals of A. If there exists a nonzero n-derivation d of A satisfying one of the following :
(i) $d([a, b],i_2,...,i_n) = a^k[a, b]a^t \forall a,b \in I_1, i_2 \in I_2,...,i_n \in I_n$, or
(ii) $d([a, b],i_2,...,i_n) = - a^k[a, b]a^t \forall a,b \in I_1, i_2 \in I_2,...,i_n \in I_n$,
for some k, t∈ ℕ, then A is a commutative ring.
Proof. (i)Suppose that:
$$d([a, b],i_2,...,i_n) = a^k[a, b]a^t \forall a,b \in I_1, i_2 \in I_2,...,i_n \in I_n. \quad (1)$$
By replacing b by ab in (1), we obtain:
$d([a,ab],i_2,...,i_n) = a^k[a, ab]a^t \forall a,b \in I_1, i_2 \in I_2,...,i_n \in I_n$.
So we have:
$d(a[a,b],i_2,...,i_n) = a^{k+1}[a, b]a^t \ \forall a,b \in I_1, i_2 \in I_2,...,i_n \in I_n$.
By defining the property of d, the previous equation becomes:
$d(a,i_2,...,i_n)[a,b] + ad([a,b],i_2,...,i_n) = a^{k+1}[a, b]a^t \forall a,b \in I_1, i_2 \in I_2,...,i_n \in I_n$.
By using (1) again in the last equation we have:
$$d(a,i_2,...,i_n)ab = d(a,i_2,...,i_n)ba \forall a,b \in I_1, i_2 \in I_2,...,i_n \in I_n. \quad (2)$$
By substituting b by br, where r∈ A in (2) and using (2)again, it implies that:
$d(a,i_2,...,i_n)y[a,r] = 0$ for all$\forall a,b \in I_1, i_2 \in I_2,...,i_n \in I_n$, r∈ A.
Therefore
$$d(a,i_2,...,i_n)I_1[a,r] = 0 \forall a \in I_1,\ i_2 \in I_2,...,i_n \in I_n, r \in A \quad (3)$$
By using Lemma 2.3 in the previous equation, we conclude that, for each $a \in I_1$, either a∈ C or $d(a,i_2,...,i_n) = 0$ for all $i_2 \in I_2,...,i_n \in I_n$. In both cases, byusing Lemm 2.6,we obtain $d(a,i_2,...,i_n) \in C$ for all $a \in U_1, i_2 \in I_2,...,i_n \in I_n$, i.e., $d(I_1,I_2,...,I_n) \subseteq C$. Now, by using Lemma 2.8, we find that Ais acommutative ring. (ii)By using the same techniqu

Corollary 3.2 Let Abe a prime near ring. If there exists k, t∈ ℕ such that Aadmits a nonzero n-derivation d, satisfying either
(i) $d([a, b],a_2,...,a_n) = a^k[a, b]a^t$
$\forall a,b,a_2,...,a_n \in A$, or
(ii) $d([a, b],a_2,...,a_n) = - a^k[a, b]a^t$
$\forall a,b,a_2,...,a_n \in A$,
Then A is a commutative ring.

Theorem 3.3. Let A be a prime near ring and $I_1I_2,...,I_n$ be semigroup ideals of A. If there exists a nonzero n-derivation d of A satisfying one of the following:
(i) $d(a \circ b,i_2,...,i_n) = a^k(a \circ b)a^t \forall a,b \in I_1, i_2 \in I_2,...,i_n \in I_n$, or
(ii) $d(a \circ b,i_2,...,i_n) = - a^k(a \circ b)a^t \forall a,b \in I_1, i_2 \in I_2,...,i_n \in I_n$,
for some k, t∈ ℕ, then A is a commutative ring.
Proof. (i)Assume that:
$d(a \circ b,i_2,...,i_n) = a^k(a \circ b)a^t \forall a,b \in I_1, i_2 \in I_2,...,i_n \in I_n$(4)
Replacing b by abin (4) we get
$d(a \circ ab,i_2,...,i_n) = a^k(a \circ ab)a^t \forall a,b \in I_1, i_2 \in I_2,...,i_n \in I_n$.
So we get:
$d(a(a \circ b),i_2,...,i_n) = a^{k+1}(a \circ b)a^t \forall a,b \in I_1, i_2 \in I_2,...,i_n \in I_n$.
By defining the property of d, the previous equation implies that:
$d(a,i_2,...,i_n)(a \circ b) + ad(a \circ b,i_2,...,i_n) = a^{k+1}(a \circ b)a^t \forall a,b \in I_1, i_2 \in I_2,...,i_n \in I_n$.

By using (4) again in the previous equation, it implies that:

$$d(a,i_2,...,i_n)ba = - d(a,i_2,...,i_n)ab \forall a,b \epsilon I_1, i_2 \epsilon I_2,...,i_n \epsilon I_n. \quad (5)$$

Bu putting bc for b, where cϵA, in (5) and using it again, it leadsto:

$d(a, i_2, . . ., i_n)bca= - d(a, i_2, . . ., i_n)abc$
$= d(a, i_2, . . ., i_n)ab(- c)$
$= d(a, i_2, . . ., i_n)b(-a)(- c)$
$\forall a,b\epsilon I_1, i_2\epsilon I_2,...,i_n \epsilon I_n., c\epsilon A$. Thus, we obtain:
$d(a, i_2, . . ., i_n)b(ca + (-a)c) = 0 \ \forall a,b\epsilon I_1, i_2\epsilon I_2,...,i_n \epsilon I_n., c\epsilon A$.
Therefore:
$d(a, i_2, . . ., i_n)I_1(- c(- a) + (-a) c) = \{0\} \ \forall a,b\epsilon I_1, i_2\epsilon I_2,...,i_n \epsilon I_n., c\epsilon A$.
For each fixed aϵ I_1, Lemma 2.3leads to:

$$-a\epsilon C \ \text{ or } d(a,i_2,. . ., i_n) = 0 = d(-a,i_2,. . ., i_n) \ \ \forall i_2\epsilon I_2,...,i_n\epsilon I_n. \quad (6)$$

If there is an element $a_1\epsilon I_1$ such that $-a_1\epsilon C$, then by Lemma 2.4 and the definition of dwe obtain $\forall r\epsilon A, i_2\epsilon I_2,...,i_n\epsilon I_n$,

$d((-a_1)r, i_2, . . ., i_n) = (-a_1)d(r, i_2, . . ., i_n) + d(-a_1, i_2, . . ., i_n)r$
$= d(r(-a_1), i_2, . . ., i_n)$
$= d(r, i_2, . . .,i_n)(- a_1) +rd(-a_1, i_2, . . ., i_n)$.

This implies that:

$$d(- a_1, i_2, . . ., i_n)r = rd(- a_1, i_2, . . ., i_n) \text{ for all } r\epsilon A, i_2\epsilon I_2,...,i_n\epsilon I_n. \quad (7)$$

From (6) and (7), we secure that:

$$d(- a,i_2, . . .,i_n)r = rd(- a,i_2, . . .,i_n) \text{for all } r\epsilon A,a\epsilon I_1,i_2\epsilon I_2,...,i_n \epsilon I_n. \quad (8)$$

So

$$d(- a, i_2, . . ., i_n)\epsilon C \forall a\epsilon I_1, i_2\epsilon I_2,..., i_n\epsilon I_n. \quad (9)$$

Now,by replacinga by (-a)b, where bϵ I_1, in (9), we obtain
$d(-((-a)b), i_2, . . ., i_n) = d((-a)(-b), i_2, . . ., i_n)\epsilon C\forall \ a, b\epsilon I_1, i_2\epsilon I_2,...,i_n\epsilon I_n$.
Which means that:
$d((-a)(-b), i_2, . . ., i_n)m = md((-a)(-b), i_2, . . ., i_n)\forall \ a, b\epsilon I_1, i_2\epsilon I_2,...,i_n \epsilon I_n, m\epsilon A$.
By using Lemma 2.5(ii) weobtain

$$d(-a, i_2, . . ., i_n)(- b)m + (-a)d(- b, i_2, . . ., i_n)m = md(- a, i_2, . . ., i_n)(- b) +m(- a)d(- b, i_2, . . ., i_n) \quad \forall a, b\epsilon I_1,i_2\epsilon I_2,...,i_n\epsilon I_n, m\epsilon A. \quad (10)$$

Bu taking (- a) instead of m in (10) and using (9)we obtain
$d(- a, i_2, . . ., i_n)A[(-a), (- b)] = \{0\}\forall a, b\epsilon I_1, i_2\epsilon I_2,...,i_n \epsilon I_n$.
By primeness of A we get$\forall$ aϵ I_1
$d(-a, i_2, . . ., i_n) = 0\forall i_2\epsilon I_2,...,i_n \epsilon I_n$
or
$(-a)(- b) = (-b)(-a)\forall b\epsilon I_1$.
If $d(- a, i_2, . . ., i_n) = 0 \ \forall a\epsilon I_1,i_2\epsilon I_2,...,i_n \epsilon I_n$, we secure that $d(I_1,I_2,...,I_n) = 0$ and by using Lemma 2.7, we have that d is a zero derivation, and this result contradictsour hypothesis.
Therefore, there exist $z_1\epsilon I_1,z_2\epsilon I_2,...,z_n\epsilon I_n$with all being nonzero, such that:

$$d(-z_1,z_2,\ldots,z_n) \neq 0 \text{ and}(-z_1)(-y) = (-y)(-z_1)\forall y\epsilon I_1. \quad (11)$$

By replacing y by -yx, where xϵ N in (11),we obtain

$$(-z_1)yx = yx(-z_1)\forall y\epsilon I_1,x\epsilon A. \quad (12)$$

By putting (-s)y, where s$\epsilon$$I_1$, instead of y and $d(-z_1,z_2,\ldots,z_n)$ instead of x in (12),we obtain
$(-z_1)(-s)yd(-z_1,z_2,\ldots,z_n) =(-s)yd(-z_1,z_2,\ldots,z_n)(-z_1) \ \forall s,y\epsilon I_1$.
By using (11) and (9) in the last equation, weobtain
$(-s)[(-z_1),y]Ad(-z_1,z_2,\ldots,z_n) = \{0\}$ for all s, yϵ I_1.
Since $d(- z_1,z_2,\ldots,z_n) \neq 0$, As A is a prime ring, we obtain$(-s)[(-z_1), y] = 0\forall s,y\epsilon I_1$. (13).
By putting -sa, where aϵ A, instead of s in (13), we obtain

$$sA[(-z_1), y] = \{0\}\text{for all } s, y\epsilon I_1. \quad (14)$$

Since $I_1\neq 0$,As A is a prime ring, we obtain

$$(-z_1)y = y(-z_1) \text{ for all } y\epsilon I_1. \quad (15)$$

By replacing y by yq, where qϵA, in (15) and using it again, we obtain
$y[(-z_1), q] = 0$ for all yϵ I_1, qϵA. Which means that:

$U_1[(-z_1), q] = \{0\}$ for all $q\epsilon A$. By Lemma 2.1, we secure that `$-z_1\epsilon C$. Returning to
If we put z_1 instead of ain(10), we obtain
$d(-z_1, i_2, \ldots, i_n)[m, - y] = 0 \forall y\epsilon I_1, i_2\epsilon I_2,\ldots,i_n\epsilon I_n, m\epsilon N$.
In particular,
$d(-z_1, z_2, \ldots, z_n)A[m, - y] = 0$ for allyϵI_1, $m\epsilon N$. Since$d(-z_1, z_2, \ldots,z_n) \neq 0$, the primeness of A implies that $-y\epsilon C$ for allyϵI_1. Which means that $-I_1\subseteq C$. But $-I_1$ is a semigroup left ideal, thenwe conclude that A is a commutative ring by Lemma 2.2.
(ii)We can prove it similarly

Corollary 3.4. Let d be a nonzero n-derivationdefined on a prime near-ring A, satisfying either
(i) $d(x\circ y,x_2,\ldots,x_n) = x^k(x\circ y)x^t \forall x,y,x_2,\ldots,x_n\epsilon A$, or
(ii) $d(x\circ y,x_2,\ldots,x_n) = - x^k(x\circ y)x^t \forall x,y,x_2,\ldots,x_n\epsilon A$,
forsome $k, t\epsilon \mathbb{N}$, then A is a commutative ring.

Theorem 3.5. Let d be a nonzero n-derivation defined asa prime near ring A and $I_1,I_2,\ldots,I_n$ be semigroup ideals of A. If d is satisfying either
(i) $d([x, y],i_2,\ldots,i_n) = x^k(x\circ y)x^t \forall x,y\epsilon I_1 ,i_2\epsilon I_2,\ldots,i_n \epsilon I_n$, or
(ii) $d([x, y],i_2,\ldots,i_n) = - x^k(x\circ y)x^t \forall x,y\epsilon I_1 ,i_2\epsilon I_2,\ldots,i_n \epsilon I_n$,
forsome $k, t\epsilon \mathbb{N}$, t,hen A is a commutative ring.

Proof. (i)Supposethat:

$$d([x, y],u_2,\ldots,u_n) = x^k(x\circ y)x^t \ \forall x,y\epsilon I_1 ,i_2\epsilon I_2,\ldots,i_n \epsilon I_n \quad (16)$$

If we replace y by xy in (16), we imply that:
$d([x,xy],i_2,\ldots,i_n) = x^k(x\circ xy)x^t \forall x,y\epsilon I_1 ,i_2\epsilon I_2,\ldots,i_n \epsilon I_n$.
So
$d(x[x,y],i_2,\ldots,i_n) = x^{k+1}(x\circ y)x^t \forall x,y\epsilon I_1 ,i_2\epsilon I_2,\ldots,i_n \epsilon I_n$.
By defining the property of d, we obtain:
$d(x,i_2,\ldots,i_n)[x,y] + xd([x,y],i_2,\ldots,i_n) = x^{k+1}(x\circ y)x^t$
By using (16) again in the previous equation, it implies that:

$$d(x,i_2,\ldots,i_n)xy = d(x,i_2,\ldots,i_n)yx \forall x,y\epsilon I_1 ,i_2\epsilon I_2,\ldots,i_n \epsilon I_n. \quad (17)$$

which is identical with equation (2) in Theorem 3.1.Following the same way, we secure that A is a commutative ring.
(ii)We can prove it similarly.

Corollary 3.6. Let d be a nonzero n-derivationof a prime near ringA, satisfying either
(i) $d([x, y],x_2,\ldots,x_n) = x^k(x\circ y)x^t$
for all $x,y,x_2,\ldots,x_n\epsilon A$, or
(ii) $d([x, y],x_2,\ldots,x_n) = - x^k(x\circ y)x^t$
for all $x,y,x_2,\ldots,x_n\epsilon A$,
forsome $k, t\epsilon \mathbb{N}$, then A is a commutative ring.

Theorem 3.7. Letd be a nonzero n-derivation of a prime near ring A and $I_1,I_2,\ldots,I_n$ be semigroup ideals of A. If d is satisfying either
(i) $d(x\circ y,i_2,\ldots,i_n) = x^k[x, y]x^t \forall x,y\epsilon I_1 ,i_2\epsilon I_2,\ldots,i_n \epsilon I_n$, or
(ii) $d(x\circ y,u_2,\ldots,u_n) = - x^k[x, y]x^t \forall x,y\epsilon I_1 ,i_2\epsilon I_2,\ldots,i_n \epsilon I_n$,
For some $k, t\epsilon \mathbb{N}$, then A is a commutative ring.

Proof.(i) Assumethat:

$$d(x\circ y,i_2,\ldots,i_n) = x^k[x, y]x^t \forall x,y\epsilon I_1 ,i_2\epsilon I_2,\ldots,i_n \epsilon I_n \quad (18)$$

If we replace y by xy in (18), we have
$d(x\circ xy,i_2,\ldots,i_n) = x^k[x, xy]x^t \forall x,y\epsilon I_1 ,i_2\epsilon I_2,\ldots,i_n \epsilon I_n$.
So
$d(x(x\circ y),i_2,\ldots,i_n) = x^{k+1}[x, y]x^t \forall x,y\epsilon I_1 ,i_2\epsilon I_2,\ldots,i_n \epsilon I_n$.
By defining the property of d, we obtain:
$d(x,i_2,\ldots,i_n)(x\circ y) + xd(x\circ y,i_2,\ldots,i_n) = x^{k+1}[x, y]x^t$
By using (18) again in the previous equation, it implies that:

$$d(x,i_2,\ldots,i_n)xy = - d(x,i_2,\ldots,i_n)yx \forall x,y\epsilon I_1 ,i_2\epsilon I_2,\ldots,i_n \epsilon I_n. \quad (19)$$

whichis identical with equation (5) in Theorem 3.3., and following the same step leads to the result
(ii) We can proveit similarly.

Corollary 3.8. Let d be a nonzero n-derivationof a prime near ring A. If d is satisfying either

(i) $d(x\circ y, x_2, \ldots, x_n) = x^k[x, y]x^t$
for all $x, y, x_2, \ldots, x_n \in A$, or
(ii) $d(x\circ y, x_2, \ldots, x_n) = - x^k[x, y]x^t$
for all $x, y, x_2, \ldots, x_n \in A$,
for some $k, t \in \mathbb{N}$, then A is a commutative ring.

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