PURE – SUPPLEMENTED MODULES

Sahira Mahmood Yasen، Wasan Khalid Hasan
Department of Mathematics, College of Science, University of Baghdad, Baghdad-Iraq.
sahira.mahmood@gmail.com.

Abstract.
Let R be an associative ring with identity and M be unital non zero right R-module . M is called H– supplemented module if given any submodule A of M there exist a direct summend submodule D of M such that M = A+X iff M= D+X where X is a submodule of M. In this paper we will give a generalization for H– supplemented which is called pure– supplemented module. An R- module M is called pure– supplemented module if given any submodule A of M there exists a pure submodule P of M such that M = A+X iff M= P+X .Equivalently , for every submodule A of M there exist a pure submodule P of M such that $\frac{A+P}{P} << \frac{M}{P}$ and $\frac{A+P}{A} << \frac{M}{A}$.

Key words: Small submodule, Supplementod module, Pure module, lifting module..

(المقاسات النقية المكملة)

وسن خالد حسن ، ساهرة محمود ياسين
جامعة بغداد، كلية العلوم، قسم الرياضيات،بغداد – العراق.

الخلاصة
لتكن R حلقة تجميعية ذات عنصر محايد وليكن M مقاسا احاديا غير صفري ايمن معرفا على R المقاس ألجزئيN منM يقال بأنه مكمل من النوع H أذا أعطينا مقاس جزئي A فيوجد مقاس جزئي مجموع مباشر D منM بحيث M = A+X اذاD+X =M في هذا البحث سنقوم بدراسة المقاسات النقية المكملة حيث قدمنا هذا التعريف كتعميم لمفهوم المقاسات المكملة من النوع H. يقال للمقاس M بانه مكمل نقي أذاأعطينا مقاس جزئي A فيوجد مقاس جزئي نقي P منM بحيث
M = A+X اذاM= P+X

Introduction :

Let R be an associative ring with identity and M be a non zero unital right R-module. A submodule N of M is called a small submodule of M , denoted by N << M , if N + L ≠ M for any proper submodule L of M [1]. let U be a submodule of M, a submodule V of M is called supplement of U if V is minimal element in the set of submodules L<M with U+L = M equivalently U+V= M and U∩V<<V. An R – module M is called supplemented if every submodule of M has supplement in M [2].

M is called H– supplemented module if given any submodule A of M there exist a direct summand D of M such that M = A+X iff M= D+X[3] . M is called weakly – supplemented module if for each submodule A of M there exists a submodule D of M such that M = A+D and A∩D<<M[2].

An R-module M is called lifting if for every submodule N of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 << M$ [2] .

In this work we generalize the H-supplemented module, by introduce the pure-supplemented module . M is called pure–supplemented module if given any submodule A of M there exists a pure submodule P of M such that M = A+X iff M= P+X .Equivalently , for every submodule A of M there exist a pure submodule P of M such that $\frac{A+P}{P} << \frac{M}{P}$ and $\frac{A+P}{A} << \frac{M}{A}$.

Since every direct summand submodule is pure then it is clear that each H–supplemented module is pure–supplemented .

In this work ,we review the concept of pure–supplemented module and we discuss some of the basic properties of this types of modules.

In section two we introduce the completely pure– supplemented module as a generalization of completely H– supplemented. We call a module M is completely pure- supplemented module If every direct summand of M is pure-supplemented .

1.Pure– supplemented modules

In this section we introduce the pure–supplemented module as a generalization of H–supplemented.

Definition1.1:[2] Let M be an R- module . M is called H– supplemented module if given any submodule A of M there exists a direct summand submodule D of M such that M = A+X iff M= D+X .

Definition 1.2:[3] Let M be an R-module . P is called pure submodule of M if KM ∩P=KP for every ideal K in R.

Remarks1.3:[3]

1) Any direct summand submodule is pure submodule in M.

2)If H ≤M and K ≤H such that H is pure in M and K is pure in H then K is pure in M.

3)If A is pure submodule of M and K is pure submodule of N then A⊕ K is pure of M ⊕ N.

As a generalization of H-supplemented module, we introduce the pure-supplemented module .

Definition 1 .4: Let M be a module . M is called pure– supplemented module if given any submodule A of M there exists a pure submodule P of M such that M = A+X iff M= P+X .

Since every direct summand submodule is pure then it is clear that each H–supplemented module is pure– supplemented .

Remark 1.5:

1-Every hollow module is pure- supplemented module .

2-Every lifting module is pure- supplemented module .

3-Every P-supplemented is weakly supplemented.

Proof :

1)Since every hollow module is H-supplemented module then is pure-supplemented module .

2)Let A be submodule of M there exists K≤A ,M=N⊕K where N≤M ,and N∩ A<<M then M=A+N iff M=K+N where K is pure since K is direct summand submodule .

3)Let N be submodule of M , such that N+X=M ,to show that N∩ X<<M .Let (N∩X)+L=M,since M pure – supplemented there exists a pure submodule P of M such that N∩X+L=M, iff M= P+L iff

N∩ X+P ≤ L then N∩ X ≤ L hence L=M and N∩ X<<M.

In this section we introduce the pure – lifting module as a generalization of lifting module.

Definition 1.6: Let M be a module . M is called pure– lifting module if for every submodule A of M there exists a pure submodule P of M , P≤ A such that M = P+X with A∩X<<X.

It is clear that every lifting and simesimple module are pure– lifting module .

Theorem 1.7: The following are equivalent for an R- module M .

1) M is pure– lifting module

2)Every submodule N of M can be written as N=A+S where A is pure in M and S << M.

3)For every submodule N of M there exists a pure submodule A of N such that M= A +K and $\frac{N}{A} << \frac{M}{A}$.

Proof :1→2) Let N be submodule of M, then M = P+X ,P≤N pure in M and N∩X<<X.,hence N∩X<<M.

N=N∩M = N ∩(P+X)=

N∩ P+N∩ X=P+(N∩X) .take A= P,

S= N∩X.

2→3) Let N be a submodule of M by (2) N=A+S where A is pure in M and S << M suppose $\frac{M}{A} = \frac{N}{A} + \frac{L}{A}$ .Then $\frac{M}{A} = \frac{A+S}{A} + \frac{L}{A}$, Thus A+S+L=M, by(2) since S << M then A+L=M.

3→1) Let N be submodule of M ,there exists a pure submodule A of Nsuch that M= A +K and $\frac{N}{A} << \frac{M}{A}$ to prove that N∩K<<K. Suppose that N∩K+B=K where B≤Kthen M=A+K=A+ N∩K+B thus

$$\frac{M}{A} = \frac{A+(N\cap K)+B}{A} =$$

$$\frac{N\cap K+A}{A} + \frac{A+B}{A} = \frac{N}{A} + \frac{A+B}{A}$$,since

$\frac{N}{A} << \frac{M}{A}$ then $\frac{A+B}{A} = \frac{M}{A}$ thus A+B=M then and hence B=K thus N∩K<<K.

Proposition 1.8: Every pure – lifting is pure-supplemented module .

Proof :let M be pure – lifting and A be a submodule of M suppose that

M=A+Y then M=K+L where K≤ A and K pure in M and A∩L<<M, now

A= A∩M = A∩(K+L)=K+ A∩L then

Let M= A+X =K+A ∩L +X since A∩L<<M then M=K+X thus M=A+X since K≤L then M is pure- supplemented module .

Proposition 1.9: Let M be an R- module M is pure– supplemented module iff for every submodule A of M there exist a pure submodule P of M such that $\frac{A+P}{P} << \frac{M}{P}$ and $\frac{A+P}{A} << \frac{M}{A}$.

Proof :(⇒ Let M be a pure- supplemented module .and A≤ M then there exist a pure submodule P such that

M = A+X iff M= P+X . suppose that $\frac{A+P}{P} + \frac{L}{P} = \frac{M}{P}$ then

$\frac{A+P+L}{P} = \frac{M}{P}$ thus $\frac{A+L}{P} = \frac{M}{P}$

Then A+L=M M is pure - supplemented ,A+L=M=P+X, P≤L ,then A+X≤L, then M≤L thus L=M $\frac{L}{P} = \frac{M}{P}$ therefore $\frac{A+P}{P} << \frac{M}{P}$ similarly $\frac{A+P}{A} << \frac{M}{A}$.

⇐) Let A be submodule of M ,then there exists a pure submodule P of M suchthat $\frac{A+P}{P} << \frac{M}{P}$ and

$\frac{A+P}{A} << \frac{M}{A}$.If M=A+X then $\frac{A+X}{P} = \frac{M}{P}$,then $\frac{M}{P} = \frac{A+P}{P} + \frac{X+P}{P}$

but $\frac{A+P}{P} << \frac{M}{P}$ .Then $\frac{M}{P} = \frac{X+P}{P}$ thus M=X+P .In the same way one can show that if M= X+P then M=A+X.

Proposition 1.10 Let M be pure– supplemented module and A be a submodule of M .If for every pure submodule P of M ,

$\frac{A+P}{A}$ is pure in $\frac{M}{A}$ then $\frac{M}{A}$ is pure–supplemented module .

Proof : Let $\frac{N}{A} \leq \frac{M}{A}$, and let $\frac{M}{A} = \frac{N}{A} + \frac{X}{A}$ where A ≤ X then M= N+X iff M=P+X where P is pure in M, (M pure –supplemented) .Then $\frac{M}{A} = \frac{P+X}{A} = \frac{P+A}{A} + \frac{X}{A}$ by assamption $\frac{A+P}{A}$ is pure in $\frac{M}{A}$ hence $\frac{M}{A}$ **is** pure – supplemented.

Recall that a submodule A of R-module M is called fully invariant if for every f $\in End_R(M)$,f(X)⊆ X. A module M is called dou module if every submodule is fully invariant.And M is called distributive iff for every submodules K,L,N, of M we have N+(K∩L)=(N+K) ∩(N+L) or N∩(K+L)=(N∩K)+(N∩L)[2].

Corollary1.11: Let M be a distributive pure–supplemented module then $\frac{M}{A}$ is pure–supplemented module for every submodule A of M .

Proof : Let D be direct summand of M, then M= D⊕K for some K submodule of M .
$\frac{M}{A} = \frac{D+A}{A} + \frac{K+A}{A}$ and A=A+(D∩K)=(A+D) ∩(A+K) (M is distributive) then
$\frac{M}{A} = \frac{D+A}{A} \oplus \frac{K+A}{A}$ hence $\frac{D+A}{A}$ is direct summand of $\frac{M}{A}$,
then is pure in $\frac{M}{A}$ thus by proposition (1.10) we get $\frac{M}{A}$ **is** pure– supplemented.

Corollary 1.12:Let A be a submodule of M and eA⊆ A for all $e^2=e \in End_R(M)$ then $\frac{M}{A}$ **is** pure– supplemented .In particular for every fully invariant submodule Y of M , $\frac{M}{Y}$ is pure – supplemented.

Proof :
Let D is a direct summand of M consider the projection map e:M→D then $e^2=e \in End_R(M)$,eA ⊆ A and
hence eA=A ∩D .Since D is a direct summand of M thenM=D⊕K , K ≤ M hence A= (A∩D) ⊕ (A∩K) now $\frac{D+A}{A} = \frac{D \oplus (A \cap K)}{A}$ and
$\frac{K+A}{A} = \frac{K \oplus (A \cap D)}{A}$ hence
M=D⊕K=D+A+K+A=(D⊕ (A∩K))+K+A
Then $\frac{M}{A} = \frac{D \oplus (A \cap K)}{A} + \frac{K+A}{A}$
Since (D ⊕ (A∩K)) ∩K+A= (A⊕ K) ∩(A∩D)=A .Then $\frac{M}{A} = \frac{D \oplus (A \cap K)}{A} \oplus \frac{K+A}{A}$.
Hence $\frac{K+A}{A}$ is direct summand of $\frac{M}{A}$ then is pure in $\frac{M}{A}$ and by (prop. 1.10) $\frac{M}{A}$ is pure-supplemented .

2–Completely pure - supplemented Modules

We call a module M is completely pure-supplemented module .If every direct summand of M is pure- supplemented.

Proposition 2.1: Every lifting is completely pure- supplemented module .

Proof :
Let N is a direct summand of M, and L ≤ N since M is lifting then L=K⊕A where K is a direct summand of M and A <<M.[2], Let X ≤ M with N=L+X=K+A+X since A direct summand of L and A <<M then by(1.3,2) A <<L .Then N =K+X hence N=L+X iff N=K+X.

An R-module M has PSP if the sum of any two pure submodule of M is pure [4]. .

Proposition 2.2
Let M be pure– supplemented module and M has PSP then M is completely pure-supplemented module .

Proof : Let N is a direct summand of M We show that N is pure– supplemented . M=N⊕K , K ≤ M assume P is pure in M then by assumption N+P pure in M ,
M= N+P⊕K then

$$\frac{M}{K} = \frac{K+P}{K} + \frac{K+N}{K}$$ then $\frac{M}{K}$ is pure-supplemented (prop. 1.10) , but $\frac{M}{K} \cong N$ then N pure- supplemented module .

Proposition 2.3: If an R-module M has PSP and $M=M_1 \oplus M_2$ is duo module ,then M is pure- supplemented iff M_1 and M_2 are pure-supplemented modules.

Proof : ⇒) Since M_1 and M_2 are fully invariant submoduleo , hence M_1 and M_2 are pure- supplemented modules (coro.1.12).

⇐) Assume M_1 and M_2 are pure-supplemented modules and let L≤ M ,then there exist a pure submodule P_1 of M such that $M_1 = (L\cap M_1)+X$ iff $M_1 = P_1+X$ for any submodule X of M_1. And there exist a pure submodule P_2 of M such that
$M_2 = (L\cap M_2)+Y$ iff $M_2 = P_2+Y$ for any submodule Y of M_2.

Claim $M=(P_1\oplus P_2)+Z$ iff M=L+Z for any submodule Z of M. Assume $M=(P_1+P_2)+Z$ then $M_1= P_1+(M_1\cap (P_2+Z))=(L\cap M_1)+(M_1\cap(P_2+Z)= M_1\cap [(L\cap M_1)+(P_2+Z)]$ then $M_1 \leq (L\cap M_1)+Z$, since let $m_1 \in M$,
$m_1= x+ a_2+ z$ where $a_2 \in P_2$,$x\in L\cap M_1$,$z\in Z$
since $Z =(Z\cap M_1)\oplus (Z\cap M_2)$ then $z=z_1+z_2$ where $z_1 \in Z\cap M_1$ and $z_2 \in Z\cap M_2$ clearly $m=x+z_1$ then
$M_1 \leq (L\cap M_1)+Z$ .Similarly
$M_2 \leq (L\cap M_2)+Z$ then $M=(L\cap M_1)+(L\cap M_2)+Z$ by modularity
$M_1=(L\cap M_1)+[M_1\cap(L\cap M_2)+Z]$ then $M_1= P_1+[M_1\cap(L\cap M_2)+Z] = M_1\cap[P_1+(L\cap M_2)+Z]$ (modular low) then $M_1\leq [P_1+(L\cap M_2)+Z]$ hence $M_1\leq P_1+Z$ .In the same way $M_2 \leq P_2+Z$ then $M=(P_1+P_2)+Z$, P_1+P_2 is pure (sum of two pure is pure) PSP.

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