CHARACTERIZATIOS and PROPERTIES of b-$T_{1/2}$-SPACES

Adea Khaleafa Hussain and Ali Saadi Abd Alatif
Department of Maths. College of Basic Education,Al- Mustansiriyah University,Baghdad-Iraq.
alobiadi@yahoo.com

Abstract.
In this paper we introduce a new class of spaces, namely b-$T_{1/2}$-space, which is strictly between b-T_0 and b-T_1 spaces, and weaker than T_{gs}-space. Several properties and characterizations for this space are investigated.

Key words: *gb-open sets bg- open sets, gb-continuous functions, b-$T_{1/2}$-space.*

تمييزات وخواص فضاءات $b-T_{1/2}$

عذية خليفة حسين ، علي سعدي
قسم الرياضيات، كلية التربية الاساسية، الجامعة المستنصرية،بغداد، العراق.

الخلاصة
الهدف من هذا البحث تقديم نوع جديد من الفضاءات أضعف من الفضاءات $T_{1/2}$وهي فضاءات $b-T_{1/2}$ ودراسة العلاقات بينها وبين بديهيات الفصل من النمط b− ومن جهة اخرى دراسة العلاقة بينها وبين الفضاءات T_{gs} كما أعطينا العديد من التمييزات لهذا النوع من الفضاءات.

Introduction.
In 1996, Andrijevic **[1]** introduce a new class of generalized open sets into field of the topology, the so-called b-open sets. A subset A of a topological space (X,τ) is said to be b-open if $A \subseteq cl(int(A)) \cup int(cl(A))$. The complement of b-open set is said to be b-closed. Thus A is b-closed if $int(cl(A)) \cap cl(int(A)) \subseteq A$. The family of all b-open (resp. b-closed) subsets of X is denoted by BO(X) (resp. BC(X)).

By using this notion, several research papers in different respects came to existence. If A is a subset of a topological space (X, τ), then the b-closure of A (abbreviated bcl(A)) is the smallest b-closed set containing A.

Definition 1.2 [2]
Let (X, τ) be a topological space. Then X is said to be:
1- b-T_0 if for each pair of distinct points x, y of X, there exists a b-open set containing one of the two points but not the other.
2- b-T_1 if for each pair of distinct points x, y of X, there exists a pair of b-open sets, one contains x but not y and the other contains y but not x.
3- b-T_2 if for each pair of distinct points x, y of X, there exists a pair of disjoint b-open sets, one contains x and the other contains y.

Theorem 1.3 [2]

A topological space (X, τ) is b-T_1 if and only if the singletons are b-closed sets.

Definition 1.4 [3]

A subset A of a topological space (X, τ) is said to be b-generalized closed set (abbreviated bg-closed) if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is b-open. The complement of b-generalized closed set is said to be b-generalized open (abbreviated bg-open). The family of all bg-closed (resp. bg-open) subsets of X is denoted by BGC(X) (resp. BGO(X)).

Definition 1.5 [4]

A subset A of a topological space (X, τ) is said to be generalized b-closed set (abbreviated gb-closed) if $bcl(A) \subseteq U$ wherever $A \subseteq U$ and U is open. The complement of generalized b-closed set is said to be generalized b-open (abbreviated gb-open). The family of all gb-closed (resp. gb-open) subsets of X is denoted by GBC(X) (resp. GBO(X)).

Remark 1.6

For any topological space (X, τ), we have $\tau^c \subseteq BC(X) \subseteq BGC(X) \subseteq GBC(X)$ (resp. $\tau \subseteq BO(X) \subseteq BGO(X) \subseteq GBO(X)$).

The following example shows that gb-closed set is not necessarily bg- closed.

Example 1.7

Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}\}$, then $BGC(X) = \{\varnothing, X, \{b\}, \{c\}, \{b, c\}\}$ and $GBC(X) = \{\varnothing, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, c\}\}$. It is clear that $\{a, b\}$ is gb-closed subset of X, but it is not bg-closed.

2. b-$T_{1/2}$-Space

In this section we introduce and study the b-$T_{1/2}$ space.

Definition 2.1

A topological space (X, τ) is said to be b-$T_{1/2}$ if every bg-closed subset of X is a b-closed.

Definition 2.2 [5]

A topological space (X, τ) is said to be T_{gs} if every gs-closed subset of (X, τ) is a sg-closed.

Lemma 2.3 [4]

Every gb-closed set is a b-closed if and only if (X, τ) is a T_{gs}.

Next, we show that T_{gs} –space is stronger than b-$T_{1/2}$-space

Theorem 2.4

Every T_{gs}-space is a b-$T_{1/2}$.

Proof*:* Let A be a bg-closed subset of (X, τ), then A is gb-closed. Since (X, τ) is T_{gs}, so by Lemma 2.3, A is b-closed. Hence (X, τ) is b-$T_{1/2}$. ■

The converse of the above theorem need not be true as seen from the following example.

Example 2.5

Let $X = \{a, b, c\}$ with $\tau = \{\varnothing, X, \{a\}\}$, then $BC(X) = BGC(X) = \{\varnothing, X, \{b\}, \{c\}, \{b, c\}\}$ and $GBC(X) = \{\varnothing, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, c\}\}$. So (X, τ) is b-$T_{1/2}$-space, but it is not T_{gs}.

Lemma 2.6 [2]:

A topological space (X, τ) is b-T_1 if and only if the singletons are b-closed sets.

Lemma 2.7 [3]

Let A be a bg-closed subset of (X, τ). Then $bcl(A) - A$ dose not contain any non-empty b-closed.

The next results show that b-$T_{1/2}$-space is placed strictly between b-T_1-space and b-T_0-space.

Theorem 2.8

Every b-T_1-space is a b-$T_{1/2}$.

Proof: Suppose that A is not b-closed subset of (X, τ) and let $x \in bcl(A) - A$. Then $\{x\} \subseteq bcl(A) - A$. Since (X, τ) is b-T_1. So, by Lemmas 2.6, $\{x\}$ is b-closed. Thus A is not bg-closed, by Lemma 2.7. ■

The converse of the above theorem is not true in general as shown by the following example.

Example 2.9

Let $X = \{a, b, c\}$ with $\tau = \{\varnothing, X, \{c\}\}$, so $BC(X) = BGC(X) = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is b-$T_{1/2}$-space, but it is not b-T_1.

Next, we give example about space which is T_{gs} but it is not b-T_1.

Example 2.10

Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$, so $BC(X) = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ and $GBC(X) = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Then (X, τ) is T_{gs}-space, but it is not b-T_1.

Theorem 2.11

Every b-$T_{1/2}$-space is a b-T_0.

Remark 2.12

It seems that a b-T_0-space need not to be b-$T_{1/2}$, but we could not prove or disprove it.

Corollary 2.13

Every T_{gs}-space is a b-T_0.

Proof: This is a direct consequence of Theorem 2.4 and Theorem 2.11.

Definition 2.14 [6]

A subset A of a topological space (X, τ) is said to be g-closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open set.

Definition 2.15 [6]

A topological space (X, τ) is said to be $T_{1/2}$ if every g-closed subset of (X, τ) is a closed or equivalently if every singleton is open or closed.

Lemma 2.16 [7]

Let A be a gb-closed subset of (X, τ). Then $bcl(A) - A$ does not contain any non-empty closed sets.

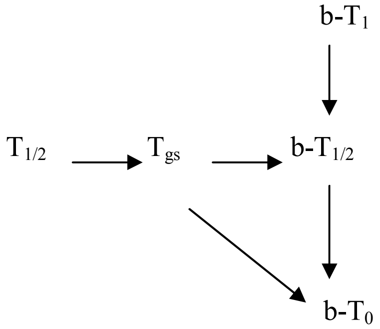
Theorem 2.17 [7]

Every $T_{1/2}$-space is a b-$T_{1/2}$

The converse of the above theorem need not be true as shown in**[7].**

Remark 2.19

From preceding theorems, remarks and examples, we have the following diagram in which no other implications hold.



Definition 2.20

A map f: $(X, \tau) \longrightarrow (Y, \sigma)$ is said to be (1) b-continuous **[8]** if for each open set U of Y, the inverse image $f^{-1}(U)$ is a b-open set in X**.**

(2) generalized b-continuous (abbreviated gb-continuous) **[7]** if for each closed set F of Y, the inverse image $f^{-1}(F)$ is a gb-closed set in X.

A sufficient condition for a bg-continuous map to be b-continuous is given in the following.

Theorem 2.21

If a map $f : (X, \tau) \longrightarrow (Y, \sigma)$ is bg-continuous and (X, τ) is a b-$T_{1/2}$, then f is b-continuous.

Proof: Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a bg-continuous and let $A \subseteq Y$ be a closed, then $f^{-1}(A)$ is bg-closed subset of (X, τ). Since (X, τ) is b-$T_{1/2}$, so $f^{-1}(A)$ is b-closed. Hence f is b-continuous. ■

4. Characterizations of b-$T_{1/2}$-Space.

In this section several characterizations of b-$T_{1/2}$-space are given.

In the following definition, we introduce a new version of b-closure operator of a set and a new version of b-openness which includes the collection BO(X).

Definition 3.1

For any subset A of a topological space (X, τ), $bcl^*(A) = \cap\{F \subseteq X : F \in BGC(X); A \subseteq F\}$ and $BO(X, \tau)^* = \{U \subseteq X: bcl^*(U^c) = U^c\}$.

Proposition .3.2

For any topological space (X, τ), we have $BO(X, \tau) \subseteq BO(X, \tau)^*$.

Proof: Let $U \in BO(X, \tau)$, then U^c is b-closed subset of X. Since every b-closed set is a bg-closed, so U^c is bg-closed and thus $bcl^*(U^c) = U^c$. Then $U \in BO(X, \tau)^*$. Hence $BO(X, \tau) \subseteq BO(X, \tau)^*$. ■

Theorem 3.3

A topological space (X, τ) is b-$T_{1/2}$ if and only if $BO(X, \tau) = BO(X, \tau)^*$ holds.

Proof*:* Necessity. Since the b-closed sets and the bg-closed sets coincide by the assumption, $bcl(A) = bcl^*(A)$ holds for every subset A of (X, τ). Therefore, we have that $BO(X,\tau)= BO(X, \tau)^*$.

Sufficiency. Let A be a bg-closed set of (X,τ). Then, we have $A = bcl^*(A)$ and hence $A^c \in BO(X, \tau)$. Thus A is b-closed. Therefore, (X, τ) is b-$T_{1/2}$. ■

Theorem 3.4

A topological space (X, τ) is a b-$T_{1/2}$ if and only if the singletons are b-open or b-closed.

Proof: Necessity. Suppose that for some $x \in X$, $\{x\}$ is not b-closed. Since X is the only b-open set containing $\{x\}^c$, the set $\{x\}^c$ is bg-closed and so it is b-closed in the b-$T_{1/2}$-space. Therefore, $\{x\}$ is b-open.

Sufficiency. Since $BO(X, \tau) \subseteq BO(X, \tau)^*$ holds, by Proposition 3.2, it is enough to prove that $BO(X, \tau)^* \subseteq BO(X, \tau)$. Let $A \in BO(X, \tau)^*$. Suppose that $A \notin BO(X, \tau)$. Then $bcl^*(A^c) = A^c$ and $bcl(A^c) \neq A^c$ hold. There exists a point x of X such that $x \in bcl(A^c)$ and $x \notin A^c = bcl^*(A^c)$. Since $x \notin bcl^*(A^c)$, there exists a bg-closed set F such that $x \notin F$ and $A^c \subseteq F$, by Definition 3.1. By the hypothesis, the singleton $\{x\}$ is b-open or b-closed. We have two cases:

Case (1). $\{x\}$ is b-open: Since $\{x\}^c$ is b-closed set $A^c \subseteq \{x\}^c$, we have $bcl(A^c) \subseteq \{x\}^c$, i.e., $x \notin$

$bcl(A^c)$. This contradicts the fact that $x \in bcl(A^c)$. Therefore, $A \in BO(X, \tau)$.

Case (2). $\{x\}$ is b-closed: Since $\{x\}^c$ is b-open set containing the bg-closed set $F(\supset A^c)$, we have $bcl(A^c) \subseteq bcl(F) \subseteq \{x\}^c$. Therefore, $x \notin bcl(A^c)$. This is a contradiction. Therefore, $A \in BO(X, \tau)$.

Hence in both cases, we have $A \in BO(X, \tau)$, i.e., $BO(X, \tau)^* \subseteq BO(X, \tau)$. Thus $BO(X, \tau) = BO(X, \tau)^*$ and so (X, τ) is b-$T_{1/2}$, by Theorem 3.3. ■

As a consequence of Theorem 3.4, we have the following characterization.

Corollary 3.5

A topological space (X, τ) is a b-$T_{1/2}$ if and only if every subset of X is the intersection of all b-open sets and all b-closed sets containing it.

Proof: Necessity. Let (X, τ) be a b-$T_{1/2}$-space with $A \subset X$ arbitrary. Then, by Theorem 3.3.4, $A = \cap\{\{x\}^c; x \notin A\}$ is an intersection of b-open sets and b-closed sets. The result follows.

Sufficiency. For each $x \in X$, $\{x\}^c$ is the intersection of all b-open sets and all b-closed sets containing it. Thus $\{x\}^c$ is either b-open or b-closed and hence (X, τ) is b-$T_{1/2}$. ■

In order to obtain more characterization of b-$T_{1/2}$-space, we introduce the following new concepts.

Definition 3.6

A map $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be approximately-b-continuous (abbreviated ap-b-continuous) if $bcl(A) \subseteq f^{-1}(U)$ whenever $A \subseteq f^{-1}(U)$ where $A \in BGC(X)$ and $U \in BO(Y)$.

Example 3.7

Let $X = \{a, b, c\}$ with $\tau = \{\varnothing, X, \{b\}, \{c\}, \{b, c\}\}$, so $BO(X) = \{\varnothing, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, c\}\}$ and $BGC(X) = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. Let $f : (X, \tau) \longrightarrow (X, \tau)$ be the identity map. Then f is ap-b-continuous, since every bg-closed set is a b-closed in this example.

Definition 3.8

A map $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be approximately-b-closed (abbreviated ap-b-closed) if $f(F) \subseteq bint(A)$ whenever $f(F) \subseteq A$ where $F \in BC(X)$ and $A \in BGO(Y)$.

Example 3.9

Let $X = \{a, b, c\}$ with $\tau = \{\varnothing, X, \{a\}\}$, then $BC(X) = \{\varnothing, X, \{b\}, \{c\}, \{b, c\}\}$. Let $Y = \{a, b, c\}$ and $\sigma = \{\varnothing, Y, \{a\}, \{a, b\}\}$, so $BGO(Y) = \{\varnothing, Y, \{a\}, \{a, b\}\{a, c\}\}$. Let $f : X \longrightarrow Y$ be the identity map. Then f is ap-b-closed, since the only bg-open subset of (Y, σ) containing the image of b-closed F in X is Y.

Theorem 3.10

For a topological space (X, τ), the following statements are equivalent:

(1) (X, τ) is b-$T_{1/2}$.

(2) For every space (Y, σ) and every map $f: (X, \tau) \longrightarrow (Y, \sigma)$, f is ap-b-continuous.

Proof: (1) $\Rightarrow$ (2). Suppose that $A \subseteq f^{-1}(U)$, where $A \in BGC(X)$ and $U \in BO(X)$. Since (X, τ) is b-$T_{1/2}$, then A is b-closed. Therefore, $bcl(A) = A \subseteq f^{-1}(U)$. Then f is ap-b-continuous.

(2) $\Rightarrow$ (1). Let $B \subseteq X$ be a bg-closed and let Y be the set X with the topology $\sigma = \{\phi, Y, B\}$. Let $f : X \longrightarrow Y$ be the identity map. By assumption, f is ap-b-continuous. Since B is bg-closed in X and b-open in Y and $B \subseteq f^{-1}(B)$, it follows that $bcl(B) \subseteq f^{-1}(B) = B$. Thus B is b-closed in X and hence (X, τ) is b-$T_{1/2}$-space. ■

Theorem 3.11

For a topological space (Y, σ), the following statements are equivalent:

(1) (Y, σ) is b-$T_{1/2}$.

(2) For every space (X, τ) and every map $f : (X, \tau) \longrightarrow (Y, \sigma)$, f is ap-b-closed.

Proof: (1) $\Rightarrow$ (2). Suppose that $f(F) \subseteq A$, where $F \in BC(X)$ and $A \in BGO(X)$. Since (Y, σ) is b-$T_{1/2}$, then A^c is b-closed. Thus, A is b-open. Therefore, $f(F) \subseteq A = bint(A)$. Then f is ap-b-closed.

(2) $\Rightarrow$ (1). Let $B \subseteq Y$ be a bg-closed and let X be the set Y with the topology $\tau = \{\varnothing, X, B\}$. Let $f : X \longrightarrow Y$ be the identity map. By assumption, f is ap-b-closed. Since B^c is bg-open in Y and b-closed in X and $f(B^c) \subseteq B^c$. It follows that $f(B^c) = B^c \subseteq bint(B^c)$. Then B^c is b-open in Y. Thus, B is b-closed and hence (Y, σ) is b-$T_{1/2}$-space. ■

Next we recall the following.

Definition 3.12 [2]

A topological space (X, τ) is said to be b-symmetric-space if for x and y in X, $x \in bcl(\{y\})$ implies $y \in bcl(\{x\})$.

Theorem 3.13 [2]

Let (X, τ) be a b-symmetric-space. Then the following are equivalent:

(1) (X, τ) is b-T_0.
(2) (X, τ) is b-T_1.

Corollary 3.14

Let (X, τ) be a b-symmetric-space. Then the following statements are equivalent:

(1) (X, τ) is b-T_1.
(2) (X, τ) is b-$T_{1/2}$.
(3) (X, τ) is b-T_0.

Proof:

(1) ⇒ (2). Theorem 2.8.
(2) ⇒ (3). Theorem 2.11.
(3) ⇒ (1). Theorem 3.13. ■

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