SOME GENERALIZATIONS ON δ-LIFTING MODULES

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Abstract

In this note we study the concept δ-lifting and we add some new results. Also we introduce weak δ-lifting modules and FI-δ-lifting modules as two generalizations of δ-lifting modules. We obtain some properties, characterizations and decompositions of weak δ-lifting modules and FI-δ-lifting modules.

Keywords: c-singular submodule, δ-small submodule, δ-lifting module, FI-δ-lifting module, weak δ-lifting module

بعض تعميمات مقاسات الرفع من الصنف (δ)

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الخلاصة

في هذا البحث ندرس مفهوم مقاسات الرفع من الصنف(δ) وأضفنا بعض النتائج الجديدة. كذلك قدمنا تعريف مقاسات الرفع من الصنف(W- δ) ومقاسات الرفع من الصنف (FI- δ) كتعميمات لمقاسات الرفع من الصنف(δ) وحصلنا على نتائج عن بعض الخواص، والمكافئات، وتجزئة المقاسات من الصنف(W- δ) والمقاسات من الصنف(FI- δ).

الكلمات المفتاحية:مقاس جزئي شاذ من الصنف(c)، مقاس جزئي صغير من الصنف(δ)،مقاسات الرفع من الصنف(δ)،مقاسات الرفع من الصنف(W- δ)، مقاسات الرفع من الصنف (FI- δ).

1. Introduction and preliminaries :

Throughout this paper, R is a ring with identity and every R-module is unitary left R-modules. Let M be an R-module, a submodule A of M is called essential (notation $A \subseteq_e M$), if for every nonzero submodule of M has a nonzero intersection with A (see [1]). Let M be an R-module and A be a submodule of M, then annihilator of A (denoted by $Ann(A)$) is defined as follows $Ann(A) = \{r \in R \mid rA = 0\}$, (see [1]). Let M be an R-module, then $Z(M) = \{x \in M : Ann(A) \subseteq_e R\}$ is called the singular submodule of M. If $Z(M) = M$, then M is called the singular module. If $Z(M) = 0$ then M is called nonsingular module, (see [1]). Let M be an R-module. A submodule A of M is called c-singular ($A \underset{c.s}{\subseteq} M$) if $\frac{M}{A}$ is a singular module. Following Zhou [2], a submodule A of a module M is called a δ-small submodule of M ($A<<_{\delta}M$), if $M \neq A+B$, for any proper c-singular submodule B of M. Let $\delta(M)=\sum\{A \subseteq M \mid A$ is δ-small submodule of $M\}$ is the δ-radical of M and $soc(M)$ will indicate the socle of M. Let M be an R-module and let B and A submodules of M such that $B \subseteq A \subseteq M$, then B is called a δ-coessentail submodule of A in M ($B \subseteq_{\delta.ce} A$ in M) if $\frac{A}{B} <<_{\delta} \frac{M}{B}$ following Lomp [3], a submodule A of M is called δ-coclosed submodule of M if $X \subseteq_{c.s} A$ and $X \subseteq_{\delta.ce} A$ in M for some $X \subseteq A$, then $A = X$. An R-module M is called an projective R-module if given any epimorphism $f: A \to B$ and any homomorphism $g: M \to B$, there exists a homomorphism $h: M \to A$ such that $h \circ f = g$. Let M be an R-module, then an R-module P is called projective δ-cover of M, if P is projective and there exists anepimorphism $\varphi: P \to M$ with $\ker(\varphi) <<_{\delta} P$, (see [2]).

Following Kosan [4], a module M is called δ-lifting if for every submodule A of M, there exists a direct summand B of M such that $B \subseteq_{\delta.ce} A$ in M. Let A and B be a submodules of an R-module M. Recall that B is called δ-supplement of A in M, if $M = A+B$, and $A \cap B <<_{\delta} B$. If every submodule of M has a δ-supplement in M, then M is called δ-supplemented module. Recall that a submodule A of M is called fully invariant if $f(A) \subseteq A$ for all $f \in End(M)$. If every submodule of M is fully invariant then M is called a duo-module.

In this note, as two generalizations of δ-lifting modules we introduce weak δ-lifting modules and FI-δ-lifting modules as follows. Any module M is called weak δ-lifting, if for each semisimple submodule A of M, there exists a direct summand B of M such that $B \subseteq_{\delta.ce} A$ in M. Any module M is called FI-δ-lifting, if for each fully invariant submodule A of M, there exists a direct summand B of M such that $B \subseteq_{\delta.ce} A$ in M.

We starting by the following lemmas which one can easily prove it.

Lemma 1.1:

Let A be a submodule of any module M. Then:

1. every submodule of a singular module is c-singular.
2. If $A \subseteq_{c.s} M$ and $f: M \to N$ is an epimorphism, then $f(A) \subseteq_{c.s} N$.
3. If $B \subseteq_{c.s} N$ and $f: M \to N$ is a homomorphism, then $f^{-1}(B) \subseteq_{c.s} M$.

The following lemmas show the properties of c-singular submodules.

Lemma 1.2:

Let A and B be submodules of an R-module M.

1. If $A \subseteq_{c.s} M$ and $B \subseteq_{c.s} M$, then $A \cap B \subseteq_{c.s} M$.
2. If $A \subseteq B$ and $A \subseteq_{c.s} M$, then $B \subseteq_{c.s} M$.
3. If $A \subseteq_{c.s} B$, then $A \cap X \subseteq_{c.s} B \cap X$, for any submodule X of M.

The following lemma shows some properties of δ-small submodules, which is appear in [2].

Lemma 1.3:

Let M be a module.

1. Let $A <<_{\delta} M$ and $M = A+B$. Then $M = A \oplus B$, for projective semisimple submodule Y of A.
2. If $A <<_{\delta} M$ and $f: M \to N$ is a homomorphism, then $f(A) <<_{\delta} N$. In particular, if $A \subseteq M \subseteq N$, then $A <<_{\delta} N$.
3. Let $A_1 \subseteq M_1 \subseteq M$, $A_2 \subseteq M_2 \subseteq M$, and $M = M_1 \oplus M_2$. Then $A_1 \oplus A_2 <<_{\delta} M_1 \oplus M_2$ if and only if $A_1 <<_{\delta} M_1$ and $A_2 <<_{\delta} M_2$.
4. If $M = \bigoplus_{i \in I} M_i$ then $\delta(M) = \bigoplus_{i \in I} \delta(M_i)$.

The following lemma shows some properties of δ-coessential submodules, which one can easily prove it.

Lemma 1.4:

Let M be an R-module and let A, B, C and submodules of M.

1. $X \subseteq_{\delta.ce} B$ in M if and only if $\frac{X}{A} \subseteq_{\delta.ce} \frac{B}{A}$ in $\frac{M}{A}$.
2. If $A \subseteq B \subseteq C \subseteq M$. Then $A \subseteq_{\delta.ce} C$ in M if and only if $A \subseteq_{\delta.ce} B$ in M and $B \subseteq_{\delta.ce} C$ in M.
3. If $A \subseteq_{\delta.ce} B$ in M and $X \subseteq_{\delta.ce} C$ in M, then $A + X \subseteq_{\delta.ce} B + C$ in M.
4. If $A \subseteq_{\delta.ce} B$ in M and $f : M \to N$ be an epimorphism, then $f(A) \subseteq_{\delta.ce} f(B)$ in N.

The following proposition gives some properties of δ-supplements.

Lemma 1.5:

Let A and B be submodules of an R-module M such that B is δ-supplement of A. then:

1. If $M = X + B$, for some submodule X of A, then B is δ-supplement of X.
2. If $C <<_\delta M$, then B is a δ-supplement of $A + C$.
3. For any submodule Y of A, then $\frac{(B+Y)}{Y}$ is a δ-supplement of $\frac{A}{Y}$ in $\frac{M}{Y}$.

2. δ-Lifting Modules

In this section we study the properties of δ-lifting modules. Also we add some new results.

Lemma 2.1: [4]

The following are equivalent for a module M :

1. M is δ-lifting.
2. For every submodule A in M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2 <<_\delta M_2$.
3. Every submodule A of M can be written as $A = B \oplus S$, where B is a direct summand of M and $S <<_\delta M$.

Proposition 2.2: [4]

Any direct summand of δ-lifting module is δ-lifting.

Definition 2.3:

Let M be an R-module. We say that M satisfies the condition (δ *), if for every direct summands M_1 and M_2 of M with $M_1 \cap M_2 <<_\delta M$, then $M_1 \cap M_2 = 0$.

Lemma 2.4:

Let M be an R-module satisfies (δ *) condition, then each direct summand of M satisfies (δ *) condition.

Proof: clear.

Proposition 2.5:

Let M be a δ-lifting module satisfies the condition (δ *). If M_1 and M_2 are direct summands of M, then $M_1 \cap M_2$ is a direct summand of M.

Proof:

Assume that $M_1 \cap M_2 \neq 0$. Since M is a δ-lifting module, then there is a submodule A of $M_1 \cap M_2$. such that $M = A \oplus B$ and $(M_1 \cap M_2) \cap B <<_\delta B$. Hence $(M_1 \cap M_2) \cap B <<_\delta M$, by lemma (1.2). Claim that $(M_1 \cap B)$ and $(M_2 \cap B)$ are direct summand of B. By modular law . Since M_1 is a direct summand of M, then $M_1 = M_1 \cap M = M_1 \cap (A \oplus B) = A \oplus (M_1 \cap B)$ $(M_1 \cap B)$ is a direct summand of M and hence $(M_1 \cap B)$ is a direct summand of B. Similarly, we have $(M_2 \cap B)$ is direct summand of B. But M satisfies (δ *) condition, therefore B satisfies (δ *) condition, by lemma (2.4). Since $(M_1 \cap B) \cap (M_2 \cap B) = (M_1 \cap M_2) \cap B <<_\delta B$ then $(M_1 \cap B) \cap (M_2 \cap B) = 0$. Thus we get $(M_1 \cap M_2) \cap B = 0$. By modular law $(M_1 \cap M_2) = (M_1 \cap M_2) \cap M = (M_1 \cap M_2)$ $\cap (A \oplus B) = A \oplus (M_1 \cap M_2) \cap B = A$. Thus $(M_1 \cap M_2)$ is a direct summand of M.

Theorem 2.6:

Let M be a δ-lifting module. Then $M = M_1 \oplus M_2 \oplus M_3$, where

1. M_1 is semisimple.
2. M_2 is δ-lifting with $\delta(M_2)$ δ-small and essential in M_2.
3. M_3 is δ-lifting module with $\delta(M_3) = M_3$.

Proof:

Clearly M is δ-supplemented. By [4, prop.2.13], we have a decomposition $M = M_1 \oplus A$ where M_1 is semisimple and $\delta(A) \subseteq_e A$. So A is δ-lifting, by proposition (2.2). Hence $A = M_1 \oplus M_2$, where $M_3 \subseteq \delta(A)$ and $\delta(A) \cap M_2 <<_\delta M_2$. But $\delta(A) \cap M_2 = M_2 \cap (\delta(M_2) \oplus \delta(M_3)) = \delta(M_2)$, therefore $\delta(M_2) <<_\delta M$. Now, since $\delta(A) = \delta(M_2) \oplus \delta(M_3) \subseteq_e M_2 \oplus M_3 = A$, then $\delta(M_2) \subseteq_e M_2$, by [5, prop. 5.20]. Since $M = A \oplus M_1 = M_1 \oplus M_2 \oplus M_3$, then M_3 is a direct summand of M. But $M_3 \subseteq \delta(A)$, therefore
$M_3 = M_3 \cap \delta(A) = M_3 \cap (\delta(M_2) \oplus \delta(M_3)) =$
$M_3 \cap \delta(M_3) = \delta(M_3)$.

Proposition 2.7:

Let $M = M_1 \oplus M_2$ be a duo module such that M_1 and M_2 are δ-lifting modules. Then M is δ-lifting module.

Proof:

Assume that $M = M_1 \oplus M_2$ be a duo module and let A be a submodule of M , then by assumption A is fully invariant, hence $A = (A \cap M_1) \oplus (A \cap M_2)$. Since M_1 and M_2 are δ-lifting, there exists a decompositions $A \cap M_1 = A_{11} \oplus A_{12}$ and $A \cap M_2 = A_{21} \oplus A_{22}$, where A_{11} is a direct summand of M_1 and A_{21} is a direct summand of M_2 and $A_{12} <<_\delta M_1$ and $A_{22} <<_\delta M_2$. Then $A_{11} \oplus A_{21}$ is a direct summand of M and by lemma (1.3), $A_{12} \oplus A_{22} <<_\delta M$. Thus M is δ-lifting.

Following [9], let M_1 and M_2 be R-modules, then M_1 is M_2-projective if for every submodule A of M_2 and any homomorphism $f : M_1 \to \frac{M_2}{A}$ there is a homomorphism $g : M_1 \to M_2$ such that $\pi \circ g = f$, where $\pi : M_2 \to \frac{M_2}{A}$ is the natural epimorphism.

Theorem 2.8:

Let $M = M_1 \oplus M_2$, where M_1 be a δ-lifting module and let M_2 is M_1-projective. Then:

1. M is δ-lifting module.
2. for every submodule A of M such that $M \neq A + M_1$, there exists a direct summand X of M such that $X \subseteq_{\delta.ce} A$ in M

Proof:

$(1) \Rightarrow (2)$ Clear.

$(2) \Rightarrow (1)$ Let A be a submodule of M such that $M = A + M_1$. Since M_2 is M_1-projective, then there exists a submodule $A_1 \subseteq A$ such that $M = A_1 \oplus M_1$, by [6, 41.14]. But M_1 is δ-lifting and $\frac{M}{A_1} = \frac{A_1 + M_1}{A_1} \cong \frac{M_1}{A_1 \cap M_1} \cong M_1$, by (the second isomorphism theorem), therefore $\frac{M}{A_1}$ is δ-lifting, so there exists a direct summand $\frac{X}{A_1}$ of $\frac{M}{A_1}$ such that $\frac{X}{A_1} \subseteq_{\delta.ce} \frac{A}{A_1}$ in $\frac{M}{A_1}$. Hence $X \subseteq_{\delta.ce} A$ in M, by lemma (1.4). Now, $X = X \cap M = X \cap (A_1 \oplus M_1) = A_1 \oplus (X \cap M_1)$, by modular law. But $\frac{X}{A_1}$ is a direct summand of $\frac{M}{A_1}$ so $\frac{A_1 \oplus (X \cap M_1)}{A_1}$ is a direct summand of $\frac{A_1 \oplus M_1}{A_1}$ Hence $X \cap M_1$ is a direct summand of M_1, by (the second isomorphism theorem). Let $M_1 = (X \cap M_1) \oplus Y$, for some submodule Y of M. Thus $M = A_1 \oplus M_1 = A_1 \oplus (X \cap M_1) \oplus Y = X \oplus Y$ and hence M is δ-lifting module.

Proposition 2.9:
Let R be a ring. If R is δ-lifting, then every cyclic R-module M has a projective δ-cover.

Proof:
Assume that $M = Ra$, for some $a \in M$. By (the first isomorphism theorem), $\frac{R}{\ker(\varphi)} \cong Ra$. One can easy to show that $\ker(\varphi) = Ann(a)$. Now, put $A = Ann(a)$. Since R is δ-lifting, then there exists an ideal A_1 of R such that $A_1 \subseteq A$, $R == A_1 \oplus A_2$ and $A \cap A_2 <<_\delta A_2$. Let $\pi : R \to \frac{R}{A}$ be the natural epimorphism. Clearly that $\pi|_{A_2} : A_2 \to \frac{R}{A}$ is an epimorphism and $\ker(\pi|_{A_2}) = A \cap A_2 <<_\delta A_2$. So $\pi|_{A_2} : A_2 \to \frac{R}{A}$ is a projective δ-cover of $\frac{R}{A}$. Thus M has a projective δ-cover.

Theorem 2.10:
Let M_1 and M_2 be δ-lifting modules such that M_i is M_j-projective $(i, j = 1,2)$. Then $M = M_1 \oplus M_2$ is δ-lifting.

Proof:
Let A be a submodule of M. Consider the submodule $M_1 \cap (A + M_2)$ of M_1. Since M_1 is δ-lifting, there exists decomposition $M_1 = A_1 \oplus B_1$ such that $A_1 \subseteq M_1 \cap (A + M_2)$ and $[M_1 \cap (A+M_2) \cap B_1] <<_\delta B_1$. Therefore $M = M_1 \oplus M_2 = A_1 \oplus B_1 \oplus M_2 = A + (M_2 \oplus B_1)$. Since $M_2 \cap (A + B_1) \subseteq M_2$ and M_2 is δ-lifting, there exists a decomposition $M_2 = A_2 \oplus B_2$ such that $A_2 \subseteq A + B_1$ and $B_2 \cap (M_2 \cap (A + B_1)) = B_2 \cap (A + B_1) <<_\delta B_2$, we have $M = A + (B_1 \oplus M_2) = A + (B_1 \oplus B_2)$. So $M = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2)$. Since M_i is M_j-projective, then $A_1 \oplus A_2$ is $B_1 \oplus B_2$-projective, by [7, prop.2-1-6, 2-1-7]. Then there exists $Y \subseteq A$ such that $M = Y \oplus (B_1 \oplus B_2)$, by [6, 41.14]. Since $B_1 \cap (A+M_2) <<_\delta B_1$ and $B_2 \cap (A+B_1) <<_\delta B_2$, then $[B_1 \cap (A + M_2) \oplus B_2 \cap (A + B_1)] <<_\delta B_1 \oplus B_2$. But $A \cap (B_1 \oplus B_2) \subseteq (B_1 \cap (A + B_2)) \oplus (B_2 \cap (A + B_1))$, therefore $A \cap (B_1 \oplus B_2) <<_\delta B_1 \oplus B_2$. Thus M is δ-lifting module.

Corollary 2.11:
Let M be a projective R-module such that $M = \bigoplus_{i \in I} M_i$. Then M is δ-lifting if and only if M_i is δ-lifting $(i = 1,...,n)$.

Proof:
By proposition (2.2), M_i is δ-lifting for each $(i = 1,...,n)$. Conversely, assume that each M_i $(i = 1,...,n)$ is δ-lifting modules. Hence each of M_i is δ-supplemented for each $(i = 1,...,n)$. Then by [8, propo.3.2] M is δ-supplemented. But M is projective, therefore M is δ-lifting module, by [4, propo.3.5].

3. weak δ-lifting modules
We introduce the concept of weak δ-lifting with example and basic properties.

Examples 3.1:
Clearly Z as a Z-module is w-δ-lifting, since Z has no semisimple submodule but not δ-lifting.

Proposition 3.2:
Every ring R is w-δ-lifting.

Proof:
First, we show that $soc(R) <<_\delta R$. Let R=soc(R)+I, where $I \subseteq_{c.s} R$, by [1, prop.1-20, p.32], $I \subseteq_e R$. But $soc(R)$ is the intersection of all essential ideal of R, therefore $soc(R) \subseteq I$ and hence $R = I$. Now, let J be a semisimple ideal of R, then $J \subseteq soc(R)$. But $soc(R) <<_\delta R$, therefore $J <<_\delta R$. Thus R is w-δ-lifting.

Proposition 3.3:
Let M be an R-module. If M is nonsingular, then M is w-δ-lifting.

Proof:
Let A be a semisimple submodule of M. Then $A \subseteq soc(M)$. Claim that $A <<_\delta M$, let $M = A + X$ where $X \subseteq_{c.s} M$, then $X \subseteq_e M$,

by [1, prop.1.21, p.32]. Clearly $A \subseteq soc(M) \subseteq X$. Hence $M = X$. Thus M is w-δ-lifting module.

Proposition 3.4:

Any direct summand of a w-δ-lifting module is w-δ-lifting.

Proof:

Let X be a direct summand of M and let A be a semisimple submodule of X, so $A \subseteq M$. Then there exists a direct summand B of M such that $A \subseteq B$ and $A \subseteq_{\delta.ce} C$ in M. Claim that $\frac{X}{A}$ is δ-coclosed submodule of $\frac{M}{A}$, let $\frac{Y}{A}$ be a submodule of $\frac{X}{A}$ such that $\frac{Y}{A} \subseteq_{\delta.ce} \frac{X}{A}$ in $\frac{M}{A}$ with $\frac{Y}{A} \subseteq_{c.s} \frac{X}{A}$ Then $Y \subseteq_{c.s} X$, by (the third isomorphism theorem) and $Y \subseteq_{\delta.ce} X$ in M, by lemma (1.4). But X is direct summand of M, then X is δ-coclosed and $Y \subseteq_{c.s} X$ hence $X = Y$. Thus $\frac{X}{A}$ is δ-coclosed in $\frac{M}{A}$ Since $\frac{B}{A} \subseteq \frac{X}{A} \subseteq \frac{M}{A}$, therefore $\frac{B}{A} <<_{\delta} \frac{X}{A}$ by lemma (1.3). Thus X is w-δ-lifting.

Proposition 3.5:

The following statements are equivalent for an R-module M:

1. M is w-δ-lifting.
2. For every semisimple submodule A in M there is a decomposition such that $M_1 \subseteq A$ and $A \cap M_2 <<_{\delta} M_2$.
3. Every semisimple submodule A of M can be written as $A = B \oplus S$ with B is a direct summand of M and $S <<_{\delta} M$.

Proof:

$(1) \Rightarrow (2)$ let A be a semisimple submodule of M, then there exists a direct summand $M_1 \subseteq A$ and $M_1 \subseteq_{\delta.ce} A$ in M. Hence $M = M_1 \oplus M_2$ for some submodule M_2 of M. By modular law $A = A \cap M = A \cap (M_1 \oplus M_2) = M_1 \oplus (A \cap M_2)$.

Now, let $\varphi : \frac{M}{M_1} \to M_2$ be a map defined by $\varphi((m_1 + m_2) + M_1) = m_2$, for all $m_1 \in M_1$ and $m_2 \in M_2$. Clearly that φ an isomorphism. Since $\frac{A}{M_1} <<_{\delta} \frac{M}{M_1}$, then $\varphi(\frac{A}{M_1}) <<_{\delta} M_2$, by lemma (1.3). But $\varphi(\frac{A}{M_1}) = A \cap M_2$.

$(2) \Rightarrow (3)$ Let A be a semisimple submodule of M, then by (2) there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2 <<_{\delta} M_2$. By modular law $A = A \cap M = A \cap (M_1 \oplus M_2) = M_1 \oplus (A \cap M_2)$.

$(3) \Rightarrow (1)$ Let A be a semisimple submodule of M. By (3) A can be written as $A = B \oplus S$, where B is a direct summand of M and $S <<_{\delta} M$. To show that $B \subseteq_{\delta.ce} A$ in M, let $\pi : M \to \frac{M}{B}$ be the natural epimorphism. Since $S <<_{\delta} M$, then $\pi(S) = \frac{S+B}{B} = \frac{A}{B} <<_{\delta} M$, by lemma (1.3). Thus M is w-δ-lifting.

Following [10], an R-module M is called an injective module if given any monomorphism $f : A \to B$ and any homomorphism $g : A \to M$, there exists a homomorphism $h : B \to M$ such that $h \circ f = g$.

Proposition 3.6:

Let $M = M_1 \oplus M_2$ be an R-module. If M_1 is w-δ-lifting and M_2 is injective w-δ-lifting, then M is w-δ-lifting.

Proof:

Let A be a semisimple submodule of M. Then $A = (A \cap M_2) \oplus A_1$, for some submodule A_1 of A. Hence by (the second isomorphism theorem) $\frac{A_1 + M_2}{A_1} \cong \frac{M_2}{M_2 \cap A_1} \cong M_2$. Now, consider the short exact sequence

$$0 \to \frac{A_1 + M_2}{A_1} \xrightarrow{i} \frac{M}{A_1} \xrightarrow{\pi} \frac{M/A_1}{A_1 + M_2/A_1} \to 0$$

Where i is the inclusion map and π is the natural epimorphism. By [6, 16.3], this short exact sequence split. Let

$\frac{M}{A_1} = \frac{A_1 + M_2}{A_1} \oplus \frac{M_3}{A_1}$ for some $M_3 \subseteq M$. Then $M = A_1 + M_2 + M_3 = M_2 + M_3$. Since $M = M_1 \oplus M_2 = M_2 \oplus M_3$, then $M_3 \cong M_1$ and hence M_3 is w-δ-lifting module. So there exists a direct summand Y of M_3 such that $Y \subseteq M_3$ and $Y \subseteq_{\delta.ce} A_1$ in M_3. Now, since $A \cap M_2$ is semisimple submodule of M_2, there exists a direct summand X of M_2 such that $X \subseteq_{\delta.ce} A \cap M_2$ in M_2. It is clear that $X \oplus Y$ is a direct summand of M. Now, let $f_1 : \frac{M}{X} \to \frac{M}{X \oplus Y}$ and $f_2 : \frac{M}{Y} \to \frac{M}{X \oplus Y}$ be a maps defined as follows $f_1(m + X) = m + (X \oplus Y)$ and $f_2(m + Y) = m + (X \oplus Y)$. Since $\frac{A \cap M_2}{X} <<_\delta \frac{M}{X}$ and $\frac{M_1}{Y} <<_\delta \frac{M}{Y}$, then $f_1(\frac{A \cap M_2}{X}) = \frac{(A \cap M_2) \oplus Y}{X \oplus Y} <<_\delta \frac{M}{X \oplus Y}$ and $f_2(\frac{A_1}{Y}) = \frac{A_1 \oplus X}{X \oplus Y} <<_\delta \frac{M}{X \oplus Y}$ by lemma (1.3). Hence $\frac{A}{X \oplus Y} = \frac{A \cap M_2}{X \oplus Y} \oplus \frac{A_1}{X \oplus Y} <<_\delta \frac{M}{X \oplus Y}$, by lemma (1.3). Thus M is w-δ-lifting module.

Proposition 3.7:

Let $M = M_1 \oplus M_2$ be an R-module. If M_1 is a w-δ-lifting module and M_2 is a semisimple module, then M is w-δ-lifting.

Proof:

Let A be a semisimple submodule of M. By modular law $A + M_1 = (A + M_1) \cap (M_1 \oplus M_2) = M_1 \oplus [(A + M_1) \cap M_2]$. Since M_2 is semisimple then $(A + M_1) \cap M_2$ is a diect summand of M_2. So $(A + M_1) \cap M_2$ is a direct summand of M. Therefore $A + M_1$ is a direct summand of M. Since A is semisimple, then there exists submodule X of A such that $A = (A + M_1) \oplus X$. Hence $A + M_1 = [(A \cap M_1) \oplus X] + M_1 = X + M_1$. Now, since M_1 is w-δ-lifting, then there exists a direct summand B of M_1 such that $B \subseteq_{\delta.ce} (A \cap M_1)$ in M_1 and hence $B \subseteq_{\delta.ce} (A \cap M_1)$ in M, by lemma (1.4). Clearly $B \oplus X$ is a direct summand $M_1 \oplus X$, since $A + M_1 = X \oplus M_1$ and $A + M_1$ is a direct summand of M, then $B \oplus X$ is a direct summand of A.

Claim that $\frac{A}{B \oplus X} = \frac{[(A \cap M_1) \oplus X]}{B \oplus X} <<_\delta \frac{M}{B \oplus X}$.

Let $\frac{M}{B \oplus X} = \frac{[(A \cap M_1) \oplus X]}{B \oplus X} + \frac{Y}{B \oplus X}$ where $\frac{Y}{B \oplus X} \subseteq_{c.s} \frac{M}{B \oplus X}$. Then $M = [(A + M_1) \oplus X] + Y = (A \cap M_1) + Y$ and hence $\frac{M}{B} = \frac{(A \cap M_1)}{B} + \frac{Y}{B}$. Since $\frac{Y}{B \oplus X} \subseteq_{c.s} \frac{M}{B \oplus X}$, then by (the third isomorphism theorem) $Y \subseteq_{c.s} M$ and hence $\frac{Y}{B} \subseteq_{\delta.ce} \frac{M}{B}$. But $\frac{(A \cap M_1)}{B} <<_\delta \frac{M}{B}$, therefore $M = Y$. Thus $B \oplus X \subseteq_{\delta.ce} A$ $B \oplus X$ in M and hence M is w-δ-lifting.

Lemma 3.8:

Let M be a w-δ-lifting module. Then $M = M_1 \oplus M_2$, where M_1 is semisimple module and M_2 is w-δ-lifting module with $soc(M_2) <<_\delta M_2$.

Proof:

Assume that M is w-δ-lifting. Since $soc(M)$ is semisimple submodule of M, then there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq soc(M)$ and $M_2 \cap soc(M) = soc(M_2) <<_\delta M_2$. Thus M_1 is semisimple by [8, lemma3.1], and M_2 is w-δ-lifting.

Proposition 3.9:

Let M be an indecomposable and not simple module. Then M is w-δ-lifting if and only if $soc(M) <<_\delta M$.

Proof:

Assume that $soc(M) \neq 0$. Since $soc(M)$ is semisimple submodule of M, then $soc(M) = A \oplus S$, where A is a direct summand of M and $S <<_\delta M$. But M is indecomposable, therefore $A = 0$. Thus $soc(M) <<_\delta M$. Conversely, assume that

$soc(M) \ll_{\delta} M$ and let A be a semisimple submodule of M. Clearly $A \subseteq soc(M) \ll_{\delta} M$, hence $A \ll_{\delta} M$, by lemma (1.3). Thus M is w-δ-lifting module.

Proposition 3.10:

Let P be a projective module. Then the following statements are equivalent:

1. P is w-δ-lifting.
2. For every semisimple submodule A of P, $\frac{P}{A}$ has a projective δ-cover.

Proof:

$(1) \Rightarrow (2)$ Let A is a semisimple submodule of P. Then there exists a submodule X of A such that $P = X \oplus Y$, for some $Y \subseteq P$ and $A \cap Y \ll_{\delta} Y$. Now, consider the following short exact sequence:

$$0 \rightarrow A \cap Y \xrightarrow{i} Y \xrightarrow{\pi} \frac{Y}{A \cap Y} \rightarrow 0$$

Where i is the inclusion map and π be the natural epimorphism. By (the second isomorphism theorem), $\frac{P}{A} = \frac{A+Y}{A} \cong \frac{Y}{A \cap Y}$ Since P is projective and Y is a direct summand of M, then Y is projective. But $\ker(\pi) = A \cap Y \ll_{\delta} Y$, therefore Y is a projective δ-cover of $\frac{Y}{A \cap Y}$ Since $\frac{P}{A} \cong \frac{Y}{A \cap Y}$ Thus $\frac{P}{A}$ has a projective δ-cover.

$(2) \Rightarrow (1)$ let A be a semisimple submodule of P and let $\pi : P \rightarrow \frac{P}{A}$ be the natural epimorphism. By (2), $\frac{P}{A}$ has a projective δ-cover. Thus by [2, lemma 2-3], there exists a decomposition $P = P_1 \oplus P_2$ such that $\pi|_{P_2} : P_2 \rightarrow \frac{P}{A}$ is a projective δ-cover and $P_1 \subseteq \ker(\pi)$. This implies that $P_1 \subseteq A$ and $\ker(\pi|_{P_2}) = A \cap P_2 \ll_{\delta} P_2$. Thus P is w-δ-lifting module.

Proposition 3.11:

Let P be a projective module with $\delta(P) \ll_{\delta} P$. Then P is w-δ-lifting if and only if for every semisimple submodule X of P, there exists a direct summand A of P such that $\overline{X} = \overline{A}$ (where $\overline{X} = X + \delta(P)/\delta(P)$).

Proof:

Assume that X is a semisimple submodule of P. Then $X = A \oplus S$, where A is a direct summand of P and $S \ll_{\delta} P$. So $S \subseteq \delta(P)$ and hence $X + \delta(P) = A + A + \delta(P) = A + \delta(P)$. Thus $\overline{X} = \overline{A}$. Conversely, let X be a semisimple submodule of P, then there exists a direct summand A of P such that $\overline{X} = \overline{A}$. Let $P = A \oplus B$ for some $B \subseteq P$. Since $\frac{P}{\delta(P)} = \frac{A+\delta(P)}{\delta(P)} \oplus \frac{B+\delta(P)}{\delta(P)} = \frac{X+\delta(P)}{\delta(P)} \oplus \frac{B+\delta(P)}{\delta(P)}$ then $P = X + B + \delta(P)$. Since $\delta(P) \ll_{\delta} P$, then by lemma (1.3), $P = (X + B) \oplus Y$ for projective semisimple submodule Y of $\delta(P)$. By modular law $X + B = (X+B) \cap P = (X+B) \cap (A \oplus B) = ((X+B) \cap A) \oplus B$. Since P is projective, then $X + B$ is projective and hence $X + B$ is B-projctive, by [9, p.68]. So $(X+B) \cap A$ is B-projective by [7, prop.2-1-6]. So there exists $X_1 \subseteq X$ such that ?, by [6, 41.14]. So $P = (X + B) \oplus Y = X_1 \oplus B \oplus Y$. Now, $X \cap (B \oplus Y) \subseteq X \cap (B + \delta(P)) \subseteq \delta(P) \ll_{\delta} P$, hence $X \cap (B \oplus Y) \ll_{\delta} P$. Thus P is w-δ-lifting module.

Following [9], an R-module M is called quasi-projective if M is M-projective.

Theorem 3.12:

Let M be quasi-projective module. Then the following statements are equivalent:

1. M is w-δ-lifting.
2. Every semisimple submodule A of M has a δ-supplement which is a direct summand.

Proof:

$(1) \Rightarrow (2)$ Let A be a semisimple submodule of M, then there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2 \ll_{\delta} M_2$. Clearly $M = A + M_2$, then M_2 is a δ-supplement of A in M which is a direct summand. $(2) \Rightarrow (1)$ Assume that every semisimple submodule has a δ-supplement which is a direct summand and let A is a semisimple submodule of M, then there exists a direct summand B of M such that $M = B \oplus B_1 = B + A$ and $A \cap B <<_\delta B$ for some submodule B_1 of M. Let $\psi : M \to \frac{B}{A \cap B}$ where $\psi = \varphi \circ \pi$ and $\pi : M \to \frac{M}{A}$ be the natural epimorphism and $\varphi : \frac{M}{A} \to \frac{B}{A \cap B}$ be an isomorphism and let $\alpha : B \to \frac{B}{A \cap B}$ be an epimorphism. Now, since M is M-projective, then by [7, prop.2-1-5] M is B-projective and hence there exists a homomorphism $h : M \to B$ such that $\psi = \alpha \circ h$. So $\alpha \circ h(M) = \varphi \circ \pi(M)$, $\alpha(h(M)) = \varphi(\pi(M))$, $\frac{h(M)}{A \cap B} = \frac{B}{A \cap B}$, therefore $h(M) = B$. Thus h is epimorphism. Since B is M-projective, by [7, pro.2-1-6], then h is split by [7, pro.2-1-8]. Hence there exists a homomorphism $g : B \to M$ such that $h \circ g = I_B$. By [10, coro.3-4-10], $M = \ker(h) \oplus \mathrm{Im}(g)$. $\ker(h) \subseteq A$. Clearly $A \cap \mathrm{Im}(g) = g(A \cap B)$. Since $A \cap B <<_\delta B$, then $g(A \cap B) = A \cap \mathrm{Im}(g) <<_\delta \mathrm{Im}(g)$, by lemma (1.3). Thus M is w-δ-lifting module.

4. FI-δ-lifting modules

We introduce the concept of FI-δ-lifting with example and some basic properties.

Example 4.1:

Consider the Z-module $M = Z_8 \oplus Z_2$. One can easy show that M is FI-δ-lifting, but not δ-lifting.

Proposition 4.2:

The following statements are equivalent for an R-module M.

1. M is FI-δ-lifting module.
2. Every fully invariant submodule A in M can be written as $A = X \oplus S$ with X is a direct summand of M and $S <<_\delta M$.
3. Every fully invariant submodule A of M can be written as $A = X + S$ with X is a direct summand of M and $S <<_\delta M$.

Proof:

(1) $\Rightarrow$ (2) Let A is a fully invariant submodule of M, then there exists a direct summand X of M such that $Y \subseteq_{\delta.ce} A$ in M. So $M = X \oplus X_1$, for some $X_1 \subseteq M$. By modular law $A = A \cap M = A \cap (X \oplus X_1) = X \oplus (A \cap X_1)$. To show that $A \cap X_1 <<_\delta X_1$, let $X_1 = (A \cap X_1) + Y$ where $Y \subseteq_{c.s} X_1$, then $M = A + Y$. Now, $\frac{M}{X} = \frac{A}{X} + \frac{Y}{X}$, by (the isomorphism theorems) $\frac{M/X}{Y + X/X} \cong \frac{M}{Y + X} = \frac{X \oplus X_1}{Y + X} \cong \frac{X_1}{Y}$. Since $Y \subseteq_{c.s} X_1$, then $\frac{Y + X}{X} \subseteq_{c.s} \frac{M}{X}$ But $\frac{A}{X} <<_\delta \frac{M}{X}$, therefore $M = Y + X$. Since $M = X + X_1$ and $Y \subseteq X_1$, then $Y = X_1$. let $S = A \cap X_1$. Thus $A = X \oplus S$, where X is a direct summand of M and $S <<_\delta M$. (2) $\Rightarrow$ (3) Clear. (3) $\Rightarrow$ (1) Let A be a fully invariant submodule of M. Then $A = X + S$, where X is a direct summand of M and $S <<_\delta M$. So $M = X \oplus Y$ for some $Y \subseteq M$. Since Y is a δ-supplement of X in M and $S <<_\delta M$, then Y is a δ-supplement of $X + S = A$ in M, by lemma (1.5). Hence $M = A + Y$ and $A \cap Y <<_\delta Y$. To show that $X \subseteq_{\delta.ce} A$ in M, let $\varphi : Y \to \frac{M}{X}$ be a map defined by $\varphi(y) = y + X$. Clearly φ is an isomorphism. Since $A \cap Y <<_\delta Y$, then $\varphi(A \cap Y) = \frac{A}{X} <<_\delta \frac{M}{X}$ by lemma (1.3). Thus M is FI-δ-lifting module.

Theorem 4.3:

The following statements are equivalent for an R-module M.

1. M is FI-δ-lifting module.
2. Every fully invariant submodule A of M has a δ–supplement B in M such that $A \cap B$ is a direct summand in A.

Proposition 4.4:
Let M be FI-δ-lifting R-module and A be a fully invariant direct summand of M, then A is FI-δ-lifting.

Proof:
Suppose that $M = A \oplus B$ is FI-δ-lifting module where A is a fully invariant submodule of M. Now, let X be a fully invariant submodule of A, so X is a fully invariant submodule of M, by [11, lemma2.1]. Then $X = Y \oplus S$, where Y is a direct summand of M and $S <<_\delta M$ and hence $S <<_\delta A$ and clearly Y is adirect summand of A. Thus A is FI-δ-lifting.

Proposition 4.5:
Let M be an indecomposable R-module. If M is FI-δ-lifting, then for every fully invariant submodule A of M, $\delta(A) <<_\delta M$.

Proof:
Let A be a fully invariant submodule of M. Since $\delta(A)$ is a fully invariant submodule of A, then $\delta(A)$ is a fully invariant submodule of M, by [11, lemma 2.1]. Hence $\delta(A) = B \oplus S$, where B is a direct summand of M and $S <<_\delta M$. But M is an indecomposable, therefore $B = 0$. Thus $\delta(A) = S$ and hence $\delta(M) <<_\delta M$.

Theorem 4.6:
Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of FI-δ-lifting modules. Then M is FI-δ-lifting.

Proof:
Let A be fully invariant submodule of M, then $A = \bigoplus_{i \in I}(A \cap M_i)$ and $A \cap M_i$ is a fully invariant submodule of M_i, by [11, lemma 2.1]. Since each of M_i is FI-δ-lifting, then $A \cap M_i = X_i \oplus S_i$, where X_i is a direct summand of M_i and $S_i <<_\delta M_i$. Let $X = \bigoplus_{i \in I} X_i$ and $S = \bigoplus_{i \in I} S_i$. It is clear that X is a direct summand of M and $S <<_\delta M$.

Proposition 4.7:
Let M be FI-δ-lifting module satisfies the condition (δ*). If M_1 and M_2 are fully invariant direct summands of M, then $(M_1 \cap M_2)$ is a direct summand of M.

Proof:
Assume that $M_1 \cap M_2 \neq 0$. Since M_1 and M_2 are fully invariant, then $M_1 \cap M_2$ fully invariant, by [11, lemma 2.1]. Now, since M is FI-δ-lifting module, then there exists a submodule X of M such that $M = (M_1 \cap M_2) + X$, $(M_1 \cap M_2) \cap X <<_\delta X$, hence $(M_1 \cap M_2) \cap X <<_\delta M$, by lemma (1.3) and $(M_1 \cap M_2) = [(M_1 \cap M_2) \cap X] \oplus Y$, for some $Y \subseteq (M_1 \cap M_2)$, by theorem (3.13). Clearly $M = X \oplus Y$. Claim that $(M_1 \cap X)$ and $(M_2 \cap X)$ are direct summand of X. By modular law
$M_1 = M_1 \cap M = M_1 \cap (X \oplus Y) = (M_1 \cap X) \oplus Y$.
Since M_1 is a direct summand of M, then $(M_1 \cap X)$ is a direct summand of M and hence $(M_1 \cap X)$ is a direct summand of X. Similarly, we have $(M_2 \cap X)$ is direct summand of X. But M satisfies (δ*) condition, therefore X satisfies (δ*) condition, by lemma (2.4). Since $(M_1 \cap X) \cap (M_2 \cap X) = (M_1 \cap M_2) \cap X <<_\delta X$, then $(M_1 \cap X) \cap (M_2 \cap X) = 0$. Thus we get $(M_1 \cap M_2) \cap X = 0$. By modular law $(M_1 \cap M_2) = (M_1 \cap M_2) \cap M = (M_1 \cap M_2) \cap (X \oplus Y)$ $((M_1 \cap M_2) \cap X) \oplus Y = Y$. Thus $(M_1 \cap M_2)$ is a direct summand of M.

Proposition 4.8:
Let P be a projective module. Then the following statements are equivalent:
1. P is FI-δ-lifting module.
2. For every fully invariant submodule A of P, $\frac{P}{A}$ has a projective δ-cover.

Proof:
$(1) \Rightarrow (2)$ Let A be a fully invariant submodule of P. Then $A = X \oplus S$, where X is a direct summand of P and $S <<_\delta P$. So

$P = X \oplus Y$, for some $Y \subseteq P$. By modular law $A = A \cap P = A(X \oplus Y) = X \oplus (A \cap Y)$. Now, let $\pi : P \to \frac{P}{X}$ be the natural epimorphism. Since $S <<_\delta P$, then $\pi(S) \frac{S+X}{X} = \frac{A}{X} <<_\delta \frac{P}{X}$.

Let $f : \frac{P}{X} \to \frac{P}{S+X} = \frac{P}{A}$ be an epimorphism. One can easily show that $\ker(\varphi|_X) = \frac{A}{X} <<_\delta \frac{P}{X}$. Thus $\frac{P}{X}$ has a projective δ-cover. $(2) \Rightarrow (1)$ Let A be a submodule of P and let $\pi : P \to \frac{P}{A}$ be the natural epimorphism and let $\varphi : M \to \frac{P}{A}$ be a projective δ-cover of $\frac{P}{A}$ for every fully invariant submodule A of M. Then by [11, lemma 2-1], there exists a decomposition $P = X \oplus Y$ such that $\varphi|_X : X \to \frac{P}{A}$ is a projective δ-cover and $Y \subseteq \ker(\varphi)$, this implies that $Y \subseteq A$ and $\ker(\varphi|_X) = A \cap X <<_\delta X \subseteq P$, then $A \cap X <<_\delta P$. Thus P is FI-δ-lifting.

References:

1. Goodearl, K.R.**1976**. Ring theory, Non-Singular Rings and Modules, Mercel Dekker, New York.pp.10-40.
2. Zhou, Y. **2000**. Generalization of Perfect, Semiperfect and Semiregular Rings, Algebra colloquium, **7**(3):305-318.
3. Lomp, C. and Büyükasik, E. **2009**, When δ-Semiperfect Rings are Semiperfect, Turk. J. Math. 33:1-8.
4. Kosan, M. T. **2007**, δ-Lifting and δ-Supplemented Modules, Algebra colloquium, **14 (**1):53-60.
5. Anderson, F. W. and Fuller, K. R. **1974**. Rings and Categories of Modules, Sipringer-Verlag, New York.pp.65-105.
6. Wisbauer, R. **1991**. Foundations of Modules and Rings theory, Gordon and Breach, Philadelphia.pp.350-359.
7. Abdelkader, B.H. **2001**. On Lifting Modules, M.Sc. Thesis. Department of math, college of scince, University of Baghdad.pp.29-43.
8. Wang, Y. **2007**. δ-Small Submodules and δ-Supplemented Modules, International J. Math. and Mathematical Sciences. pp.1-8.
9. Mohamad, S. H. and Muller, B. J. **1990**. Continuous and Discrete Modules, London, Math. Soc. LNS 174 Cambridge University, press, Cambridge.pp.55-79.
10. Kasch, F. **1982**. Modules and Rings, Academic press., London.pp.61-116.
11. Birkenmeier, G. F. Müller, B. J. and Rizvi, S. T. **2002**. Modules in which every fully invariant submodules is essential in a direct summand, Comm. Algebra, **30**:1395-1415.