



REPRESENTATION OF GROUPS BY HOMOMORPHISM

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Abstract

Issacs [1] shows that if $\lambda : G \rightarrow F^*$ is a homomorphism (F^* is the multiplicative group of the field F) and G is a group, then one can define $\rho(g) = (\lambda(g))$ which is an F -representation of G of degree 1 affords λ as character. [2,3,4] inspires us to do the following: given a homomorphism $f : G \rightarrow G^*$ then we define a representation of G by the homomorphism f from a given F -representation $\rho^* : G^* \rightarrow GL(n, F)$ of G^* as $\rho = \rho^* \circ f$. We study this kind of representations and their associated characters.

Keywords: Representation theory, Character theory, Irreducible characters.

تمثيل الزمر بواسطة التشاكلات

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الخلاصة

Isaacs [1] بين باناه اذا كان $\lambda : G \rightarrow F^*$ تشاكل (F^* هي زمرة الضرب للحقل F و G زمرة ما)، فانه من الممكن تعريف $\rho(g) = (\lambda(g))$ وهو تمثيل ل G ل F من الدرجة 1 وله λ كشخص . المصادر [2,3,4] الهمتنا لفعل ما يلي: ليكن $f : G \rightarrow G^*$ تشاكل زمري ، نعرف تمثيل ل G بواسطة التشاكل f من التمثيل F $\rho^* : G^* \rightarrow GL(n, F)$ ل G^* بالشكل $\rho = \rho^* \circ f$ ، ندرس هذه الانواع من التمثيلات مع شواخصها.

الكلمات المفتاحية: نظرية التمثيل، نظرية الشواخص، الشواخص الغير قابلة للحل

1.Introduction

Representation theory reduces problems in abstract algebra to problems in linear algebra it provides a very useful technique in studying a group by represent it in terms of something familiar and concrete. Issacs [1] shows that if

$\lambda : G \rightarrow F^*$ is a homomorphism (F^* is the multiplicative group of the field F) and G is a group, then one can define $\rho(g) = (\lambda(g))$ which is an F -representation of G of degree 1 affords λ as character. [2,3,4] inspires us to do the following: given a groups homomorphism

$f : G \rightarrow G^*$ then we define a representation of G by the homomorphism f from a given F -representation $\rho^* : G^* \rightarrow GL(n, F)$ of G^* as $\rho = \rho^* \circ f$. We study this kind of representations and then define their corresponding characters. We show that most of the associated characters properties are consistent with the general known properties of characters.

2.Preliminaries

Definition(2.1): A (linear) representation of a group G on a vector space V is a homomorphism $\rho : G \rightarrow GL(V)$, if we choose a basis for V , we can view a representation as a homomorphism $\rho : G \rightarrow GL(n, F)$ where $GL(n, F)$ is the group of invertible matrices with entries in the field F , in this situation we call it a matrix representation of G

Example (2.2): Let G be the cyclic group C_n of order n with generator a . It has two-dimensional real representation defined by

$$\rho : C_n \rightarrow CL(2, \mathbb{R}), \quad a^k \rightarrow \begin{pmatrix} \cos(2k\pi/n) & -\sin(2k\pi/n) \\ \sin(2k\pi/n) & \cos(2k\pi/n) \end{pmatrix}$$

Definition (2.3): Let G be a group ρ be an F -representation then an F -character χ of G afforded by ρ is the function given by $\chi(g) = trace(\rho(g))$

Definition(2.4): Two F -representations ρ and μ of a group G are said to be equivalent if they arise from isomorphic FG -modules (FG is the group algebra of G over F) M and N that is let $\delta : M \rightarrow N$ be an isomorphism of FG -modules then $\delta(ag) = \delta(a)g$ for all $a \in M$ and $g \in G$ then for the representations ρ and μ we have $\delta(\rho(g)a) = \delta(a)\mu(g)$ thus $\delta^{-1}(\delta(\rho(g))) = \mu(g) \quad \forall g \in G$.

Remark(2.5): (I) We denote by $Irr(G)$ for the set of all irreducible characters of G . For their definition and properties see [5].
 (II) Characters are class functions that is constant on conjugacy classes.

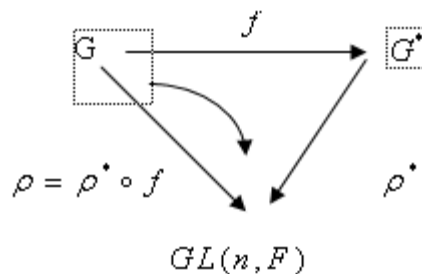
Theorem(2.6): Every class function \mathcal{G} of G can be uniquely expressed in the form $\mathcal{G} = \sum_{\chi \in Irr(G)} b_\chi \chi$ where $b_\chi \in F$.

Furthermore \mathcal{G} is a character if and only if b_χ are non-negative integers and $\mathcal{G} \neq 0$.

Proof: see [6]

3. Representation of groups by homomorphism

Definition(3.1): Let G, G^* be two groups $f : G \rightarrow G^*$ be a homomorphism and ρ^* is a representation of G^* , $\rho^* : G^* \rightarrow GL(n, F)$ we define a representation $\rho : G \rightarrow GL(n, F)$ by $\rho(g) = \rho^* \circ f(g) = \rho^*(f(g)) \quad \forall g \in G$ as in the following diagram;



We Call ρ a representation of G by f

Definition(3.2): Let G, G^* be two groups $f : G \rightarrow G^*$ be a homomorphism and ρ^* be a representation of the group G^* $\rho(g) = \rho^* \circ f(g) = \rho^*(f(g)) \quad \forall g \in G$ be the associated representation of G by f we define a character of G afforded by ρ as $\chi(g) = trace(\rho(g)) = trace(\rho^*(f(g))) = \chi'(g)$ for some $\chi' \in Irr(G')$

Note that for the above definition the degree of the characters takes the form $\chi(1) = trace(\rho^*(f(1))) = trace(\rho^*(1'))$ where 1 and $1'$ are the identities of G, G^* respectively and since f is a homomorphism $f(1) = 1'$.

Proposition(3.3): Let G, G^* be two groups $f : G \rightarrow G^*$ be a homomorphism and ρ^* is a

representation of G^* , $\rho^* : G^* \rightarrow GL(n, F)$ then for a given finite n homomorphisms of G into G^* there exist at least one representation of G .

Proof: Let $f_1, f_2 \dots f_n$ be an n homomorphisms of G into G^* . We can write

$$\rho(g) = \begin{pmatrix} \rho' \circ f_1(g) & & 0 \\ & \ddots & \\ 0 & & \rho' \circ f_n(g) \end{pmatrix} \quad \forall g \in G$$

which is a representation of G .

In the situation of the above proposition characters takes the form

$$\chi(g) = \text{trace}(\rho(g)) = \text{trace}(\rho' \circ f_1(g)) + \dots + \text{trace}(\rho' \circ f_n(g))$$

We can define the product of two associated characters for the same homomorphism as;

Definition(3.3): Let G, G^* , f as in definition(3.2) and $\chi_1 = \chi'_1 \circ f$, $\chi_2 = \chi'_2 \circ f$ then the product of χ_1, χ_2 defined by:

$$[\chi_1, \chi_2] = \sum_{g \in G} \sum_{f(g) \in G^*} \chi'_1(f(g)) \cdot \overline{\chi'_2(f(g))}$$

the following properties are satisfied for the product of characters so defined

- 1- $[\chi_1, \chi_2] = \overline{[\chi_2, \chi_1]}$
- 2- $[\chi, \chi] \geq 0$, and $[\chi, \chi] = 0$ if $\chi = 0$
- 3- $[c_1\chi_1 + c_2\chi_2, \psi] = c_1[\chi_1, \psi] + c_2[\chi_2, \psi]$
- 4- $[\chi, c_1\psi_1 + c_2\psi_2] = c_1[\chi, \psi_1] + c_2[\chi, \psi_2]$

Theorem(3.4): Let G, G^* be two groups $f : G \rightarrow G^*$ be a homomorphism and ρ^* be a representation of the group G^* , $\rho(g) = \rho^*(f(g))$, $g \in G$ be a representation of G affording the character $\chi = \chi' \circ f$ and $n = O(g)$ then:

a) $\rho(g)$ is similar to a diagonal matrix $\text{diag}(t_1, \dots, t_l)$ (l is the degree of the representation ρ)

b) $t_i^n = 1$

c) $\chi(g) = \sum t_i$ and $|\chi(g)| \leq \chi(1)$

d) $\chi(g^{-1}) = \overline{\chi(g)}$

proof:

(a) We can use the restriction of ρ to the cyclic group $\langle g \rangle$ which form a representation of $\langle g \rangle$ therefore no loss if we assume $G = \langle g \rangle$, ρ is similar to a representation in block diagonal form with irreducible representations of G appearing as the diagonal blocks, since $G = \langle g \rangle$ is abelian the irreducible representations have degree 1 hence ρ is similar to a diagonal representation and ρ is diagonal.

(b) Since $I = \rho(g^n) = \rho(g)^n = \text{diag}(t_1^n, \dots, t_l^n)$

I $|t_i| = 1$ and $|\sum t_i| \leq \sum |t_i| = l = \chi(1)$ so $\chi(g) = \sum t_i$ thus $|\chi(g)| \leq \chi(1)$

(d) $\rho(g^{-1}) = \rho(g)^{-1} = \text{diag}(t_1^{-1}, \dots, t_l^{-1})$ therefore $\chi(g^{-1}) = \sum t_i^{-1}$ but $|t_i| = 1$ we have $t_i^{-1} = \overline{t_i}$ and $\chi(g^{-1}) = \overline{\chi(g)}$

Proposition(3.5): Let $\rho = \rho^* \circ f$ be a representation of group G by a homomorphism $f : G \rightarrow G^*$ which affords the character $\chi = \chi' \circ f$ then $g \in \ker(\rho)$ if and only if $\chi(g) = \chi'(1')$

Proof: Let $g \in \ker(\rho)$ then $\rho(g) = I = \rho(1)$ and $\chi(g) = \chi'(f(g)) = \chi(1) = \chi'(f(1)) = \chi'(1')$ Conversely, by theorem(3.4) $\chi(g) = t_1 + \dots + t_l$ where t_i is a root of unity and $l = \chi(1)$ since $|t_i| = 1$ the equation $\chi(g) = l$ gives $t_i = 1$ for all i

now, $\rho(g)$ is similar to $\text{diag}(t_1, \dots, t_l) = 1$ thus $\rho(g) = I$ which means $g \in \ker(\rho)$.

References

1. Isaacs. I.M. **1976**. *Character theory of finite groups*. Academic press, INC. pp.12-75.
2. Adan-Bante, E. **2004**. Products of characters and finite p-groups, *J. Algebra*, **277** : 236-255.
3. Adan-Bante, E; M. Loukaki, and Moreto, A. **2004**. Homogeneous products of characters, *J. Algebra*. **274** : 587-593.
4. Adan-Bante, E. **2007**. Squares of characters and finite groups. *J. Algebra*, **310**: 619-623.
5. C.W.Curtis, **1981**. *Methods of representations theory with applications to finite groups and order*. John and sons, New York, .pp.201-290.
6. Fulton, W. and Harris, J. **1991**. *Representations theory*. GTM 129, Springer-Verlag, pp.307-388.