UNSTEADY NON-NEWTONIAN FLUID FLOW PROBLEM IN PLANE SOLVING BY MAC METHOD

Mohammed S. Hussein
Mohammedsabah95@yahoo.com
Department of Mathematics, College of Science, University of Baghdad. Baghdad-*Iraq*

Abstract

In this paper consideration is given to viscose, incompressible, time-depended and non-Newtonian fluid flowing in a straight pipe with square cross-section under the action of pressure gradient. In particular consideration is given to second order fluid flow which can be represented by the equation of state of the form:

$$T_{ij} = 2\eta\, e_{ij} + 4\zeta \sum_{k=1}^{2} e_{ik} e_{kj} \quad \text{i,j = 1,2}$$

Where η viscosity coefficient and ζ is normal stress coefficient and, T_{ij} and e_{ij} , i,j =1,2 are the stress and rate of strain respectively. Cartesian coordinate system has been used to describe the fluid motion and it is found that equations of motion are controlled by Reynolds number and non-Newtonian parameter. The motion equations are solved by an explicit method namely MAC. Our study is ended with studying the effect of Reynolds number and non-Newtonian parameter on the fluid flow.

Keywords:-Finite difference, naveir stock equation

مسألة مائع لانيوتيني غير مستقر في المستوي محلولة باستخدام خوارزمية MAC

محمد صباح حسين
قسم الرياضيات ، كلية العلوم ، جامعة بغداد. بغداد-العراق

الخلاصة

يقدم هذه البحث دراسةِ جريان لمائع لانيوتوني ،لزج، غير قابل للانضغاط في مقطع عرضي مربّع تحت تأثير الضغطِ. وبشكل خاص أعتبر المائع من الرتبة الثانية الذي يُمْكِنُ أَنْ يُمثّلَ بمعادلةِ حالةِ من النوع:

$$T_{ij} = 2\eta\, e_{ij} + 4\zeta e_{ik} e_{kj} \quad \text{i,j = 1,2}$$

حيث η هو ثابت للمائع و T_{ij} و e_{ij} هما مركبات الاجهاد و مركبات معدل المرونة على التوالي. نظام الإحداثيات المتعامدة تم استخدامه لوصف حركة المائع وقد وجد أن معادلات الحركة مُسيطر عليها من قبل وسيطين عديمة الأبعاد وهما عدد رينولدز والوسيط اللا نيوتيني. و إنّ معادلاتَ الحركةَ مَحْلُولة بطريقة MAC حيث أنَّها طريقة صريحة .

الكلمات المفتاحية:- الفروقات المنتهية ومعادلات نايفير ، ستوك

1-Introduction

There was a prolific development of computational fluid dynamics (CFD) methods in the fluid dynamics group at Los Alamos Laboratories in the year's from1958 to the late 1960s. This development was largely due to the energy, creativity and leadership of Francis Harlow.

The MAC method first appeared in 1965. It was developed by Harlow and Welch [1] specifically for free surface flows, and this method is a finite difference solution technique for investigating the dynamic of an incompressible viscose fluid, it employs the primitive variables of pressure and velocity.

In 1970 Amsden and Harlow [2] subsequently developed a simplified MAC method (SMAC) which circumvented difficulties with the original method by splitting the calculation cycle into two parts, namely: a provisional velocity field calculation followed by a velocity revision employing an auxiliary potential function to ensure incompressibility throughout.

Miyata [3], in 1986 used SMAC for the simulation of both water waves generated by ships and breaking waves over circular and elliptical bodies. In the 1990s many authors considered different, but related methods, like volume of fluid (VOF), [4] for example, in [5] developed an Improvement version for general regions called GENSMAC, an adaptation for generalized Newtonian flow.

More recently, the MAC method has been extended to cope the generalized Newtonian flows in both two and three dimensions by [6], In 2004, Oishi CM. et.al, they are studied two dimensional time dependent incompressible fluid flow problem by GENSMAC.[7]

In 2008, McKee S. et.al,[8] they study the MAC method and it will be applied to several problems such as free surface, hydraulic jump, rising bubbles and jet buckling.

In this paper I will steady the MAC method with a non-Newtonian fluid and the effect of each of Reynolds number and non-Newtonian parameter on the flow with square cross section.

2-A Mathematical Formulation

Unsteady flow of fluid in the xy- plane is considered. The non-Newtonian fluid is characterized by equation of state of the form:

$$T_{ij} = 2\eta\, e_{ij} + 4\zeta \sum_{k=1}^{2} e_{ik} e_{kj} \quad \text{i,j=1,2} \qquad ...(1)$$

Where T_{ij} , e_{ij} and η , ζ are stress , rate of strain and viscosity coefficient and normal stress coefficient respectively, where the strain and the stress components are;

$$e_{x_1x_1} = \frac{\partial U}{\partial x_1}\ ,\ e_{y_1y_1} = \frac{\partial V}{\partial y_1}\ ,\ e_{x_1y_1} = e_{y_1x_1} = \frac{1}{2}\left(\frac{\partial V}{\partial x_1} + \frac{\partial U}{\partial y_1}\right)$$

$$T_{x_1x_1} = 2\eta \frac{\partial U}{\partial x_1} + 4\zeta\left(\left(\frac{\partial U}{\partial x_1}\right)^2 + \frac{1}{4}\left(\frac{\partial V}{\partial x_1} + \frac{\partial U}{\partial y_1}\right)^2\right)$$

$$T_{y_1y_1} = 2\eta \frac{\partial V}{\partial y_1} + 4\zeta\left(\frac{1}{4}\left(\frac{\partial V}{\partial x_1} + \frac{\partial U}{\partial y_1}\right)^2 + \left(\frac{\partial V}{\partial y_1}\right)^2\right)$$

$$T_{x_1y_1} = T_{y_1x_1} = \eta\left(\frac{\partial V}{\partial x_1} + \frac{\partial U}{\partial y_1}\right) + 2\zeta\left(\frac{\partial U}{\partial x_1}\left(\frac{\partial V}{\partial x_1} + \frac{\partial U}{\partial y_1}\right) + \left(\frac{\partial V}{\partial x_1} + \frac{\partial U}{\partial y_1}\right)\frac{\partial V}{\partial y_1}\right)$$

Where U and V are the velocity component in the direction coordinates x_1 and y_1 respectively

3-The Motion Equations and Continuity Equation in stress form

The motion equations for two dimensional flow in Cartesian coordinates my be written as:

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x_1} + V\frac{\partial U}{\partial y_1}\right) = -\frac{\partial P^*}{\partial x_1} + \frac{\partial T_{x_1x_1}}{\partial x_1} + \frac{\partial T_{y_1x_1}}{\partial y_1} \qquad (2)$$

$$\rho\left(\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x_1} + V\frac{\partial V}{\partial y_1}\right) = -\frac{\partial P^*}{\partial y_1} + \frac{\partial T_{y_1y_1}}{\partial y_1} + \frac{\partial T_{x_1y_1}}{\partial x_1} \qquad .(3)$$

And the continuity equation

$$\frac{\partial U}{\partial x_1} + \frac{\partial V}{\partial y_1} = 0 \qquad . (4)$$

Where U,V is the dimensional velocity components in x_1,y_1 directions respectively and ρ is density of the fluid and P^* is the dimensional pressure and the terms $T_{x_1x_1}$, $T_{y_1y_1}$ are the normal stress in the directions x_1,y_1 and $T_{x_1y_1}$ $T_{y_1x_1}$ are the shear stress in the direction x_1,y_1 and y_1,x_1 respectively. In the above equations, we assume that the fluid is incompressible (i.e.,ρ = constant), and the above equations is called Navier-Stokes equations.

4-Naiver-Stokes equations in Non-Dimensional Form

We can write down the motion and continuity equation (2)-(4) in non-dimensional form through using scaling and order of magnitude analysis.

This is can be done by introducing the following new quantities;

$$x=\frac{x_1}{a}\ ,\ y=\frac{y_1}{a}\ ,\tau=\frac{V_0 t}{a}, u=\frac{U}{V_0}\ ,v=\frac{V}{V_0}\ ,\mathrm{P}=\frac{P^*}{\rho V_0^2}$$

Where a, V_0 are the diameter of pipe and free stream velocity respectively. The substitution of these quantities into equations (2, 3 and 4) gives the motion and continuity equations in dimensionless form which are:

$$\frac{\partial u}{\partial \tau}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial P}{\partial x}=\frac{1}{\mathrm{Re}}\left(\nabla^2 u\right)+ 8\beta\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+2\beta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right) \quad ..(5)$$

$$\frac{\partial v}{\partial \tau}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial P}{\partial y}=\frac{1}{\mathrm{Re}}\left(\nabla^2 v\right)+ 8\beta\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}+2\beta\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}\right) \quad ...(6)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad ..(7)$$

The above equations are controlled by two parameters namely the Reynold's Number $\mathrm{Re}=\frac{aV_0}{\nu}$, and non-Newtonian $\beta=\frac{\zeta}{\rho a^2}$ where ν,ρ is kinematics viscosity and density of the fluid respectively.

5- Naiver-Stokes equations in conservative form:

In the last equations (5,6) the left hand side of convective term are in the non-conservative form but to apply the MAC formulation we need to transform the convective term to conservative form which can be do this with the help of continuity equation (7) as ;

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}\ ,\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{\partial uv}{\partial x}+\frac{\partial v^2}{\partial y}$$

Hence the equations (5,6) have the form;

$$\frac{\partial u}{\partial \tau}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial P}{\partial x}=\frac{1}{\mathrm{Re}}\left(\nabla^2 u\right)+ 8\beta\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+2\beta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right) \quad (8)$$

$$\frac{\partial v}{\partial \tau}+\frac{\partial uv}{\partial x}+\frac{\partial v^2}{\partial y}+\frac{\partial P}{\partial y}=\frac{1}{\mathrm{Re}}\left(\nabla^2 v\right)+ 8\beta\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}+2\beta\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}\right) \quad ...(9)$$

6-MAC Formulation

One of the earliest, and most widely used method for solving (8,9) is the MAC method which is due to Harlaw and Welch (1965) [1].The method is characterized by use the staggered grid and the solution of a Poisson equation for pressure at every time-step.

The MAC method was initially devised to solve problems with free surfaces, but it can be applied to any incompressible fluid flow.

6-1 Staggered Grid [9]

Computational solution of equations (8)-(9) are often obtain on a staggered grid, this implies that different dependent variables are evaluated at different grid point , It can be seen that pressures are defined at the center of each cell and that velocity components are defined at the cell faces, which is the prototype of MAC mesh distribution.

6-2 Discretizations of MAC [9]

The spatial discretization makes use of the staggered grid (MAC mesh). We consider a very simple explicit discretization in time. We choose the conservative form of Navier-Stokes equations as in (sec.5). In discrediting (8), the finite difference expressions centered at grid point (j+1/2, k) are used. This allows $\partial P/\partial x$ to be discredited as $(P_{j+1,k}-P_{j,k})/\Delta x$ which is a second-order discretization about grid point (j+1/2,k).Similarly equation (9) is discredited with finite difference expressions centered at grid point (j,k+1/2)and $\partial P/\partial x$ is represented as $(P_{j,k+1}-P_{j,k})/\Delta y$ The use of the staggered grid primates coupling of the u,v and P solutions at adjacent grid points. This in turn prevents the appearance of oscillatory solutions, particularly for P, that can occur if centered differences are used to discretize all derivatives on non-staggered grid. The oscillatory behavior is usually worse at high Reynolds number where the dissipative terms which do introduce adjacent grid point coupling for u and v, are small.

The following finite differences expressions are utilized:

$$\left.\begin{aligned}
\left[\frac{\partial u}{\partial \tau}\right]_{j+1/2,k} &= \frac{\left(u^{n+1}_{j+1/2,k}-u^{n}_{j+1/2,k}\right)}{\Delta\tau}+O(\Delta t)\\
\left[\frac{\partial u^2}{\partial x}\right]_{j+1/2,k} &= \frac{\left(u^{2}_{j+1/2,k}-u^{2}_{j,k}\right)}{\Delta x}+O(\Delta x^2)\\
\left[\frac{\partial uv}{\partial y}\right]_{j+1/2,k} &= \frac{(uv)_{j+1/2,k+1/2}-(uv)_{j+1/2,k-1/2}}{\Delta y}+O(\Delta y^2)\\
\left[\frac{\partial^2 u}{\partial x^2}\right]_{j+1/2,k} &= \frac{\left(u_{j+3/2,k}-2u_{j+1/2,k}+u_{j-1/2,k}\right)}{\Delta x^2}+O(\Delta x^2)\\
\left[\frac{\partial^2 u}{\partial y^2}\right]_{j+1/2,k} &= \frac{\left(u_{j+1/2,k-1}-u_{j+1/2,k}+u_{j+1/2,k+1}\right)}{\Delta y^2}+O(\Delta y^2)\\
\left[\frac{\partial P}{\partial x}\right]_{j+1/2,k} &= \frac{\left(P_{j+1,k}-P_{j,k}\right)}{\Delta x}+O(\Delta x^2)\\
\left[\frac{\partial^2 u}{\partial x\partial y}\right]_{j+1/2,k} &= \frac{\left(u_{j+3/2,k+1}-u_{j+3/2,k-1}-u_{j-1/2,k+1}+u_{j-1/2,k-1}\right)}{\Delta x\Delta y}+O(\Delta y^2)
\end{aligned}\right\} \quad (10)$$

Where Δx, Δy are the step size in the x and y axes, respectively. In the above expressions terms like $u_{j+1,k}$ appears. To evaluate such terms averaging is employed, i.e., $u_{j+1,k}= 0.5(u_{j+1/2,k}+u_{j+3/2,k})$. Similarly $(uv)_{j+1/2,k+1/2}$ is evaluated as $(uv)_{j+1/2,k+1/2}=0.25(u_{j+1/2,k}+u_{j+1/2,k+1})(v_{j+1,k+1/2}+v_{j,k+1/2})$. In the MAC formulation the discretizations (10) allow the following explicit algorithm to be generated from (8) - (9);

$$u^{n+1}_{j+1/2,k}=F^{n}_{j+1/2,k}-\frac{\Delta\tau}{\Delta x}\left[P^{n+1}_{j+1,k}-P^{n+1}_{j,k}\right] \quad (11)$$

Where;

$$F^{n}_{j+1/2,k}=u^{n}_{j+1/2,k}+$$
$$\Delta\tau\left[\begin{aligned}
&\frac{(u)^n{}_{j+3/2,k}-2(u)^n{}_{j+1/2,k}+(u)^n{}_{j-1/2,k}}{(\mathrm{Re})(\Delta x^2)}-\frac{(u)^n{}_{j+1/2,k-1}-2(u)^n{}_{j+1/2,k}+(u)^n{}_{j+1/2,k+1}}{(\mathrm{Re})(\Delta y^2)}-\\
&\frac{(u^2)^n_{j+1,k}-(u^2)^n_{j,k}}{\Delta x}-\frac{(uv)^n_{j+1/2,k+1/2}-(uv)^n_{j+1/2,k-1/2}}{\Delta y}\\
&+8\beta\left(\frac{(u)^n{}_{j+1,k}-(u)^n{}_{j,k}}{\Delta x}\right)\left(\frac{(u)^n{}_{j+3/2,k}-2(u)^n{}_{j+1/2,k}+(u)^n{}_{j-1/2,k}}{\Delta x^2}\right)\\
&+2\beta\left(\begin{aligned}&\left(\frac{(u)^n{}_{j,k+1}-(u)^n{}_{j,k}}{\Delta y}\right)+\\&\left(\frac{(v)^n{}_{j+1,k}-(v)^n{}_{j,k}}{\Delta x}\right)\end{aligned}\right)\left(\begin{aligned}&\left(\frac{(u)^n{}_{j+3/2,k+1}-(u)^n{}_{j+1/2,k-1}-(u)^n{}_{j-1/2,k+1}+(u)^n{}_{j-1/2,k-1}}{\Delta x\Delta y}\right)+\\&\left(\frac{(v)^n{}_{j+3/2,k}-2(v)^n{}_{j+1/2,k}+(v)^n{}_{j-1/2,k}}{\Delta x^2}\right)\end{aligned}\right)
\end{aligned}\right] \quad \ldots (12)$$

Similarly the discredited form of equation (9) can be written as

$$v^{n+1}_{j,k+1/2}=G^{n}_{j,k+1/2}-\frac{\Delta\tau}{\Delta y}\left[P^{n+1}_{j,k+1}-P^{n+1}_{j,k}\right] \quad \ldots (13)$$

Where;

$$G^{n}_{j,k+1/2}=v^{n}_{j,k+1/2}+$$
$$\Delta\tau\left[\begin{aligned}
&\frac{(v)^n_{j+1,k+1/2}-2(v)^n_{j,k+1/2}+(v)^n_{j-1,k+1/2}}{(\mathrm{Re})(\Delta x^2)}+\frac{(v)^n_{j,k+3/2}-2(v)^n_{j,k+1/2}+(v)^n_{j,k-1/2}}{(\mathrm{Re})(\Delta y^2)}-\\
&\frac{(uv)^n_{j+1/2,k+1/2}-(uv)^n_{j-1/2,k+1/2}}{\Delta x}-\frac{(v^2)^n_{j,k+1}-(v^2)^n_{j,k}}{\Delta y}\\
&+8\beta\left(\frac{(v)^n{}_{j,k+1}-(v)^n{}_{j,k}}{\Delta y}\right)\left(\frac{(v)^n{}_{j,k+3/2}-2(v)^n{}_{j,k+1/2}+(v)^n{}_{j,k-1/2}}{\Delta y^2}\right)\\
&+2\beta\left(\begin{aligned}&\left(\frac{(u)^n{}_{j,k+1}-(u)^n{}_{j,k}}{\Delta y}\right)+\\&\left(\frac{(v)^n{}_{j+1,k}-(v)^n{}_{j,k}}{\Delta x}\right)\end{aligned}\right)\left(\begin{aligned}&\left(\frac{(v)^n{}_{j+1,k+3/2}-(v)^n{}_{j+1,k-1/2}-(v)^n{}_{j-1,k+3/2}+(v)^n{}_{j-1,k-1/2}}{\Delta y\Delta x}\right)+\\&\left(\frac{(u)^n{}_{j,k+3/2}-2(u)^n{}_{j,k+1/2}+(u)^n{}_{j,k-1/2}}{\Delta y^2}\right)\end{aligned}\right)
\end{aligned}\right] \quad (14)$$

In equations (11) and (13) the pressure appears implicitly; however, P^{n+1} is obtained before equations (11) and (13) are used, as follows.

The continuity equation (7) is discredited as;

$$\frac{\left(u^{n+1}_{j+1/2,k}-u^{n+1}_{j-1/2,k}\right)}{\Delta x}+\frac{\left(v^{n+1}_{j,k+1/2}-v^{n+1}_{j,k-1/2}\right)}{\Delta y}=0 \quad \ldots (15)$$

The substitution $u^{n+1}_{j+1/2,k}$, $v^{n+1}_{j,k+1/2}$ From (11),(13) allows the equations (7-9) to be rewritten as a discrete Poisson equation for pressure, i.e.

$$\left[\frac{\left(P_{j-1,k}-2P_{j,k}+P_{j+1,k}\right)}{\Delta x^2}+\frac{\left(P_{j,k-1}-2P_{j,k}+P_{j,k+1}\right)}{\Delta y^2}\right]^{n+1}=\frac{1}{\Delta\tau}\left[\frac{\left\{F^{n}_{j+1/2,k}-F^{n}_{j-1/2,k}\right\}}{\Delta x}+\frac{\left\{G^{n}_{j,k+1/2}-G^{n}_{j,k-1/2}\right\}}{\Delta y}\right] \quad \ldots (16)$$

Equation (16) is solved at each time step, either using iterative techniques or the direct Poisson solvers [5]. For us we will use the Gauss-sidel method to get the solution for P^{n+1} which has been obtained from (16), substitution into equations (11), (13) permits $u^{n+1}_{j+1/2,k}, v^{n+1}_{j,k+1/2}$ to be computed.

7-Treatment of Boundary Conditions for velocities and pressure [10]

Let Γ be the boundary of the computational domain and assume that the velocity V is given on Γ; i.e., $V_\Gamma = (u_\Gamma, v_\Gamma)$ there is no condition for the pressure P. But in our problem there is boundary conditions for velocity which is Dirichelt boundary condition, i.e., u=v=0 on Γ.

Hence we try to find a formulation for the pressure P on the boundary Γ,where the computational domain is a square-cross section named as ABCD. The grid is arranged so that boundaries pass through velocity points but not pressure points.

$v_{1,1/2} = v_{2,1/2} = . \; . \; . = 0$, since BC is a solid wall and also $u_{1/2,1} = u_{1/2,2} = . \; . \; . = 0$, since AB is a solid wall or in general form:

$$v_{j,1/2} = 0 \text{ for each } j = 1,2,3,\ldots,n \text{ on BC}$$
$$u_{1/2,k} = 0 \text{ for each } k = 1,2,3,\ldots,m \text{ on AB}$$
$$v_{j,m+1/2} = 0 \text{ for each } j = 1,2,3,\ldots,n \text{ on AD}$$
$$u_{n+1/2,k} = 0 \text{ for each } k = 1,2,3,\ldots,m \text{ on CD}$$

. (17)

The evaluation of the Poisson equation for pressure (16) requires values of the pressure outside of the domain, when (16) is evaluated centered at node (2,1) values of $P_{2,0}$ and $v_{2,-1/2}$ are required.

The $P_{2,0}$ can be calculated by expand equation (9) at the center of the wall, since V at the boundary is not a function of time that implies ∂v/∂τ = 0 and also $\partial v^2/\partial y = 0$ (by using boundary conditions (17)). And $\partial uv/\partial x$, $\partial^2 v/\partial x^2$ will be vanished at the wall, hence the equation (9) will be

$$\frac{\partial P}{\partial y} = \frac{1}{\text{Re}}\frac{\partial^2 v}{\partial y^2} \quad .. (18)$$

In discredited form this becomes;

$$\frac{P_{j,k} - P_{j,k-1}}{\Delta y} = \frac{1}{\text{Re}}\frac{v_{j,k+1} - 2v_{j,k} + v_{j,k-1}}{(\Delta y)^2} \quad ..(19)$$

We have

$$P_{j,k-1} = P_{j,k} - \frac{v_{j,k+1} - 2v_{j,k} + v_{j,k-1}}{\text{Re}(\Delta y)} \quad . (20)$$

We apply equation (26) at the node (2, 1)

$$P_{2,0} = P_{2,1} - \frac{v_{2,3/2} - 2v_{2,1/2} + v_{2,-1/2}}{\text{Re}(\Delta y)} \quad (21)$$

For equation (21), we put $v_{2,1/2} = 0$ by using boundary conditions (17) , but we need the value of $v_{2,-1/2}$, the continuity equation (7) is satisfied at boundary, this implies that $\partial v/\partial y = 0$ (Since $\partial u/\partial x = 0$) which may be written in difference form as:

$$\frac{v_{j,k+1} - v_{j,k-1}}{\Delta y} = 0$$

From which, we obtain $v_{j,k+1} = v_{j,k-1}$; we have $v_{2,3/2} = v_{2,-1/2}$ ... (22)

The substitution of equation (22) in to (21) gives

$$P_{2,0} = P_{2,1} - \frac{2v_{2,3/2}}{\text{Re}(\Delta y)} \quad ... (23)$$

In general we have

$$P_{j,k-1} = P_{j,k} - \frac{2v_{j,k+1/2}}{\text{Re}(\Delta y)} \quad ... (24)$$

This is the pressure formulation at the boundary i.e., at the wall BC.

By similar technique we can find respectively the pressure formulation at the walls BC, CD and AD which are:-

$$P_{j-1,k} = P_{j,k} - \frac{2u_{j+1/2,k}}{\text{Re}(\Delta x)} \quad j=1,n \text{ and } k=1 \quad . \; . (25)$$

$$P_{j+1,k} = P_{j,k} + \frac{2u_{j-1/2,k}}{\text{Re}(\Delta x)} \quad k=1,\ n \text{ and } j=m \quad .. . (26)$$

$$P_{j,k+1} = P_{j,k} + \frac{2v_{j,k-1/2}}{\text{Re}(\Delta y)} \quad j=1,n \text{ and } k=m \quad .. \; . (27)$$

Where *m* and *n* are the number of discretizations on *x* and *y* direction.

9- Stability Conditions for Time [8]

A time-stepping procedure for computing the appropriate time-step size for every cycle is employed. It is based on the stability conditions (written in non-dimensional form)

$$\Delta t < \frac{\Delta x}{\|\mathbf{u}\|} \quad ... (28)$$

$$\Delta t < \frac{\Delta x^2 \Delta y^2 \Delta z^2}{\Delta x^2 \Delta y^2 + \Delta x^2 \Delta z^2 + \Delta z^2 \Delta y^2}\frac{\text{Re}}{2} \quad ... (29)$$

Where the first inequality is understood component wise. The restriction (28) requires that no particles should cross more than one cell boundary in a given time interval; this is an accuracy requirement. The second restriction (29) comes from the explicit discretization of the Navier–Stokes equations and is essentially a local von Neumann stability requirement. Since low Reynolds number flows ($0 \leq Re \leq 10$) are the primary concern, it is anticipated that (29) is generally the more restrictive condition.

10- Discussion the Results

In this section I will analyze the result that obtain from solution equations (7)-(9) with different values for Reynolds number and non-Newtonian parameter which is for the first one I take some values Re= 10,50,100,250 and 300, [8] and for the other I choose β = 0.01, 0.1 and 0.4; and also study all these values in separate cases, and it noted that all figure for cross section there is multiple vortices appear in the diagram of cross section but in different intensity which observe that appear one vortex with strong intensity located in the left side of cross section and there are many other vortices have different intensities and gradually decay with it 's intensity from strong to moderate to weak.

Clearly that all vortices has symmetric and parallel to y-axis. For (figure 1-5) we note that as Re increase from 10 to 300 and β = 0.01 the main vortex have intensity with the range (9e-006 - 5e-007) and we see there are many new vortex appear when the values of Reynolds number are increase with β still constant.

For the (figures 6-15) we see that the flow has the same behavior as before but with β= 0.1 and 0.4 respectively. But for Re=300 and the value of β increase from 0.01 to 0.4 we see the behavior of flow in cross section has change because in (fig.6,12) we see that there is five vortices that appear but in (fig.15) we see only four, that mean when β increase the number of vortices decrease with Re have constant value.

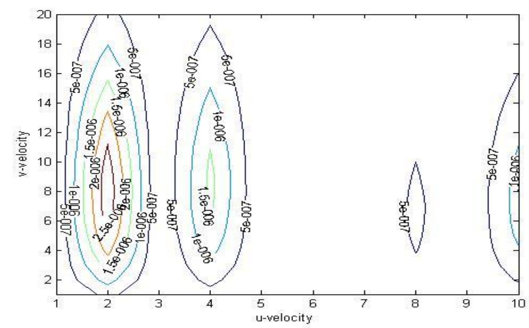

Fig 1:The axial velocity for Re=10,β=0.01

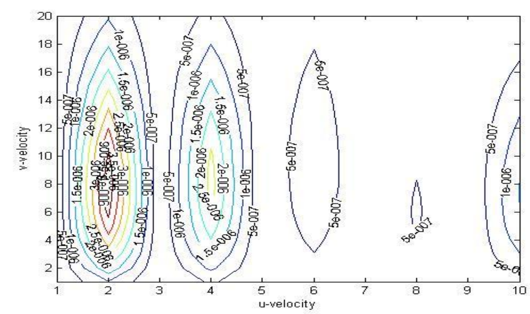

Fig 2: The axial velocity for Re=50,β=0.01

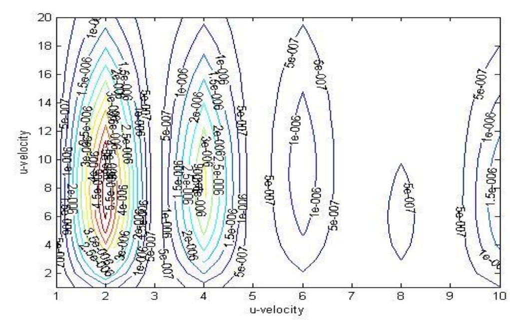

Fig 3: The axial velocity for Re=100,β=0.01

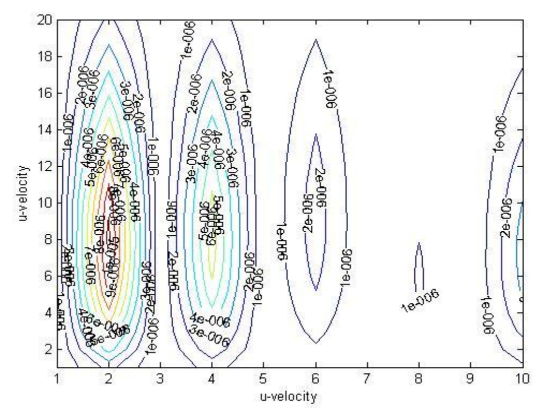

Fig 4: The axial velocity for Re=250,β=0.01

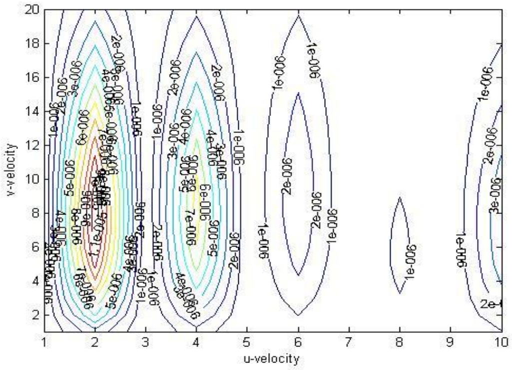

Fig 5: The axial velocity for Re=300, β=0.01

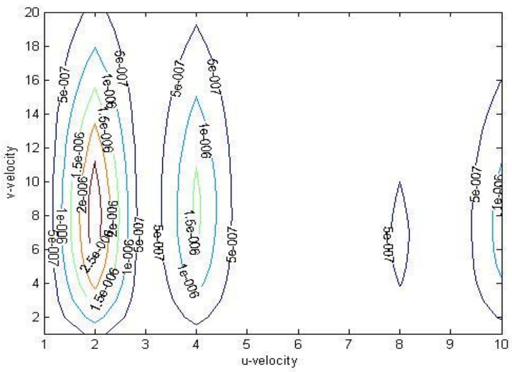

Fig 6: The axial velocity for Re=10, β=0.1

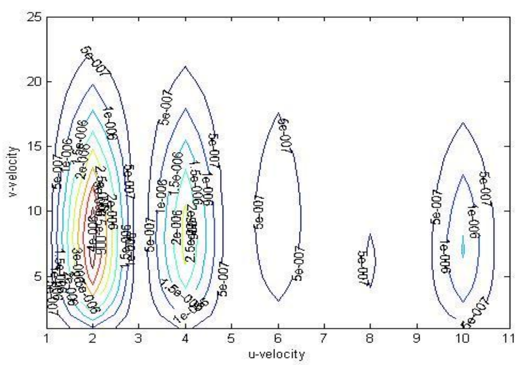

Fig 7: The axial velocity for Re=50, β=0.1

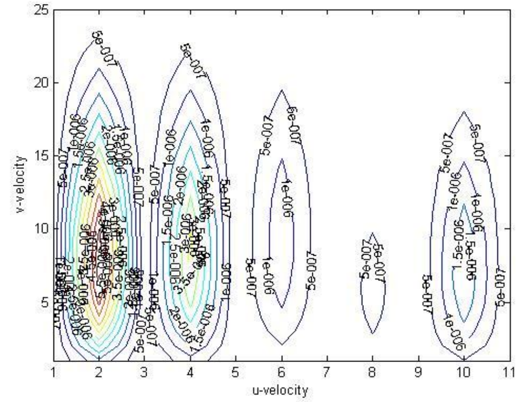

Fig 8: The axial velocity for Re=100, β=0.1

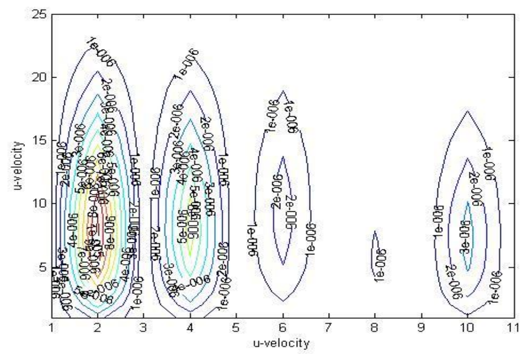

Fig 9: The axial velocity for Re=250, β=0.1

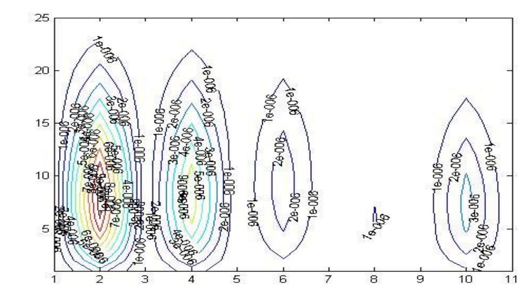

Fig 10: The axial velocity for Re=300, β=0.1

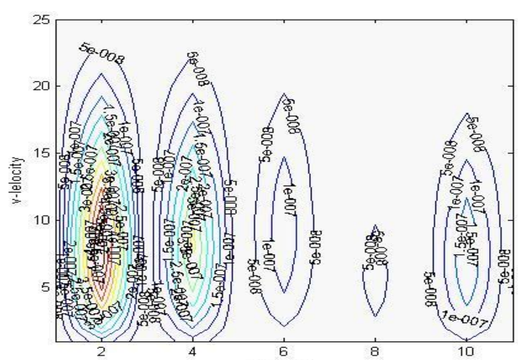

Fig 11: The axial velocity for Re=10, β=0.4

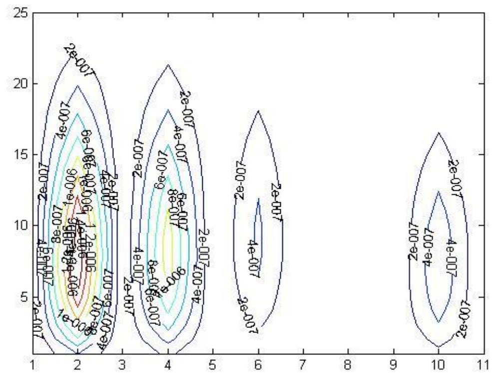

Fig 12: The axial velocity for Re=50, β=0.4

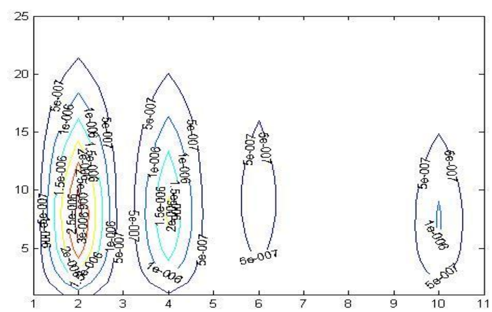

Fig 13: The axial velocity for Re=100, β=0.4

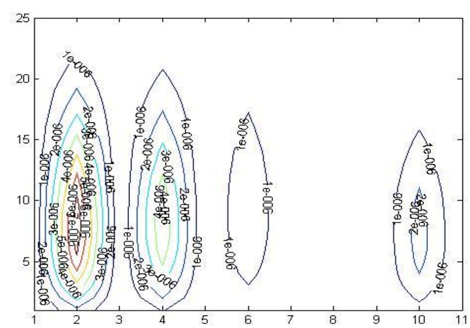

Fig 14: The axial velocity for Re=250, β=0.4

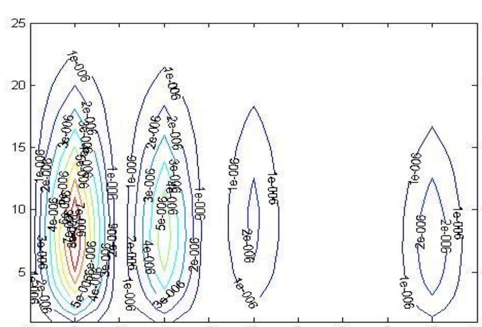

Fig 15: The axial velocity for Re=300, β=0.4

References:

1. Harlow F.H, and Welsh J.E.**1965**. Numerical Calculation of Time-Dependent Viscous Incompressible Flow of Fluid with Free Surface, *J. Phys. Fluids* **8**(12):2128-2189.
2. Amsden A, Harlow F.**1970**. The SMAC method: a numerical technique for calculating incompressible fluid flows**.** *Technical Report* LA**-4370**, Los Alamos National Laboratory.
3. Miyata H.**1986**. Finite-difference simulation of breaking waves. *J. Comput. Phys*,**65**:179–214.
4. Tome´ MF, McKee S. GENSMAC.**1994**. a computational marker-and cell method for free surface flows in general domains. *J. Comput.Phys*, **110**:171–86.
5. Tome´ MF, McKee S. **1999**. Numerical simulation of viscous flow: buckling of planar jets. *Int. J. Numer*. *Methods Fluids*,**29**:705–18.
6. Tome´ MF, Grossi L, Castelo A, Cuminato JA, Mangiavacchi N,Ferreira VG, et al. **2004**. A numerical method for solving three-dimensional generalized Newtonian free surface flows. *J Non-Newton Fluid Mech*.**123**:85–103.
7. Oishi CM, Ferreira VG, Cuminato JA, Castelo A, Tome´ MF,Mangiavacchi N. **2004**. Implementing implicit procedures in GENSMAC.*TEMA (Tend Encias Mat. Apl. Comput.),* **5**:257–66.
8. S. McKee , M.F. Tome´, V.G. Ferreira , J.A. Cuminato ,A. Castelo , F.S. Sousa , N. Mangiavacchi.**2008**. The MAC method. *Computers & Fluids*, **37** : 907–930 .
9. Fletcher C.A.J.**1988**. *Computational Techniques for Fluid Dynamics 2*, Springer – Verlag.pp.330-337
10. Peyret R. and Taylor .D.**1983**. *Computational Methods for Fluid Flow*, Springer Series in Computational Physics.pp.146-160.