ON GENERALIZED (θ,ϕ)-HIGHER DERIVATIONS AND GENERALIZED $(U,R)-(\theta,\phi)$-HIGHER DERIVATIONS OF PRIME RINGS

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Abstract

Let U be a Lie ideal of a 2-torsion free prime ring R and θ,ϕ be commuting endomorphisms of R. In this paper we generalize the main result of M. Ashraf, A. Khan and C. Heatinger on (θ,ϕ)-higher derivation of prime ring R to generalized (θ,ϕ)-higher derivation of Lie ideal by introducing the concept of generalized (θ,ϕ)-higher derivation. Under some conditions we prove that a Jordan generalized (θ,ϕ) -higher derivation of U is either a generalized (θ,ϕ)-higher derivation of U or $U \subseteq Z(R)$ and every Jordan generalized (θ,θ)-higher derivation of R is a generalized (θ,θ)-higher derivation of R. Also, we generalize this result to generalized $(U,R)-(\theta,\theta)$-higher derivation by introducing the concepts of $(U,R)-(\theta,\phi)$-higher derivation and generalized $(U,R)-(\theta,\phi)$-higher derivation. Under some conditions we prove that if $F=(f_i)_{i\in N}$ is a generalized $(U,R)-(\theta,\theta)$-higher derivation of R, then
$f_n(ur)=\sum_{i+j=n} f_i(\theta^{n-i}(u))d_j(\theta^{n-j}(r))$, for all $u\in U,\ r\in R, n\in N$.

Key words: (θ,ϕ)-derivation, generalized (θ,ϕ)-derivation, higher derivation, (U,R)-derivation, Lie ideal, prime ring.

حول تعميم مشتقات - (θ,ϕ) العاليه و تعميم مشتقات $(U,R)-(\theta,\phi)$ العاليه للحلقات الاوليه

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الخلاصه

ليكن U مثالي لي للحلقه R طلقية الالتواء من النمط ٢ و θ,ϕ تشاكلات تجميعيه للحلقه R. في هذا البحث عممنا النتيجه الاساسيه لكل من A. Ashraf, A. Khan, C. Heatinger لمشتقات - (θ,ϕ) العاليه للحلقات الاوليه الى تعميم مشتقات - (θ,ϕ) العاليه لمثالي لي وذلك بتقديم مفهوم تعميم مشتقة- (θ,ϕ) العاليه. تحت بعض الشروط برهنا ان تعميم مشتقة جوردان - (θ,ϕ) العاليه لمثالي لي تكون اما تعميم مشتقة - (θ,ϕ) العاليه على U او $U \subseteq Z(R)$ و ان تعميم مشتقة جوردان - (θ,θ) العاليه على R يكون مشتقة - (θ,θ) العاليه على R. كذلك ، عممنا هذه النتيجه الى تعميم مشتقة- $(U,R)-(\theta,\theta)$ العاليه بتقديم المفاهيم مشتقة $(U,R)-(\theta,\phi)$ العاليه و تعميم مشتقة $(U,R)-(\theta,\phi)$ العاليه. تحت بعض الشروط برهنا اذا كانت $F=(f_i)_{i\in N}$ تعميم مشتقة - $(U,R)-(\theta,\theta)$ العاليه للحلقه R فان $f_n(ur)=\sum_{i+j=n} f_i(\theta^{n-i}(u))d_j(\theta^{n-j}(r))$ لكل $u\in U, r\in R, n\in N$.

1. Introduction:

Throughout this paper, R will denote an associative ring with center $Z(R)$, not necessarily with an identity element and θ,ϕ be endomorphisms of R. Endomorphisms θ,ϕ are said to be commuting endomorphisms if $\theta\phi=\phi\theta$, [1]. The set of natural numbers including 0 is denoeted by N. A ring R is said to be prime (resp. semiprime) if $xRy=0$ (resp. $xRx=0$) implies either $x=0$ or $y=0$ (resp. $x=0$), [2]. A ring R is 2-torsion free if $2x=0$, for all $x\in R$ implies $x=0$, [2]. A Lie ideal of R is any additive subgroup U of R with $[u,r]=ur-ru$, for all $u\in U, r\in R$, [2]. A Lie ideal U of R with $u^2\in U$, for all $u\in U$ is called square closed Lie ideal, [2]. A square closed Lie ideal which is not contained in $Z(R)$ is called admissible Lie ideal, [2]. A derivation (resp. Jordan derivation) of R is an additive mapping $d:R\to R$ such that $d(ab)=d(a)b+ad(b)$ (resp. $d(a^2)=d(a)a+ad(a)$), for all $a,b\in R$, [1]. For a fixed $a\in R$, define $d:R\to R$ by $d_a(x)=[a,x]$, for all $x\in R$, is called an inner derivation, [3]. Every derivation is a Jordan derivation but the converse is not true in general. In 1957, Herstein proved that if R is a prime ring of characteristic different of 2, then every Jordan derivation of R is a derivation, ([4], Theorem 3.1). This result was extended by several authors([5],[6]). R. Awater extended Hersetine's result to Lie ideals, ([7], Theorem). An additive mapping $F:R\to R$ is called generalized derivation (resp. Jordan generalized derivation) if there exists a derivation (resp. Jordan derivation) $d:R\to R$ such that $F(ab)=F(a)b+ad(b)$ (resp. $d(a^2)=F(a)a+ad(a)$, for all $a,b\in R$, [8]. Clearly, every generalized derivation is Jordan generalized derivation, but the converse is not true in general. It is shown in [8] that if U is a square closed Lie ideal of a 2 -torsion free prime ring R then every Jordan generalized derivation on U is a generalized derivation on U. An additive mapping $d:R\to R$ is called (θ,ϕ)-derivation (resp.Jordan (θ,ϕ)-derivation)if $d(ab)=d(a)\theta(b)+\phi(a)d(b)$ (resp $d(a^2)=d(a)\theta(a)+\phi(a)d(a)$),for all , [3]. M. Bresar and J. Vukman [9] extended Hersetine's result to (θ,ϕ)-derivation. Inspired by definition of (θ,ϕ)-derivation the notation of generalized (θ,ϕ)-derivation was extended as follows: An additive mapping $F:R\to R$ is called generalized (θ,ϕ)-derivation (resp. Jordan generalized (θ,ϕ)-derivation) on R if there exist

a (θ,ϕ)-derivation (resp. Jordan (θ,ϕ)-derivation) $d: R \to R$ such that $F(ab) = F(a)\phi(b) + \theta(a)d(b)$ (resp. $F(a^2) = F(a)\phi(a) + \theta(a)d(a)$), for all $a,b \in R$, [1]. In [3], S. Ali and C. Haetinger proved that every Jordan generalized (θ,ϕ)-derivation on a 2-torsion free semiprime ring is a generalized (θ,ϕ)-derivation. Let $D = (d_i)_{i\in N}$ be a family of additive mappings of R such that $d_0 = id_R$. D is said to be a higher derivation (resp. Jordan higher derivation)
If $d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b)$ (resp. $d_n(a^2) = \sum_{i+j=n} d_i(a)d_j(a)$), for all $a,b \in R$, $n \in N$, [2]. M, Ferrero and C. Haetinger in [2], extended Hersetine's result to higher derivations, they proved that every Jordan higher derivation of 2-torsion free semiprime ring is a higher derivation. A. K. Faraj, C. Haetinger and A. H. Majeed in [11], extended this result to generalized (U,R)-higher derivation.

In 2010, M. Ashraf, A. Khan and C. Haetinger introduced the concept of (θ,ϕ)-higher derivation and they extended the above result to (θ,ϕ)-higher derivation, [1]. In this paper we extend this result to generalized (θ,ϕ)-higher derivations and generalized $(U,R)-(\theta,\phi)$-higher derivations by introducing the concepts of generalized (θ,ϕ)-higher derivation, $(U,R)-(\theta,\phi)$-higher derivation, generalized $(U,R)-(\theta,\phi)$-higher derivation. Throughout this paper, consider θ,ϕ are commuting endomorphisms of R.

2. Prelimineries

Now we will introduce the definition of generalized (θ,ϕ)-higher derivations and some basic results which extensively to prove our main results.

2.1 Definition:[1]

Let $D = (d_i)_{i\in N}$ be a family of additive mappings of R such that $d_0 = id_R$. D is said to be a (θ,ϕ)-higher derivation (resp. Jordan (θ,ϕ)-higher derivation) if

2.2 Definition:

$F = (f_i)_{i\in N}$ be a family of additive mappings of R such that $f_0 = id_R$. F is said to be a generalized (θ,ϕ)-higher derivation (resp. Jordan generalized Jordan (θ,ϕ)-higher derivation) if there exists a (θ,ϕ)-higher derivation $D = (d_i)_{i\in N}$ (resp. Jordan (θ,ϕ)-higher derivation) of R such that $f_n(ab) = \sum_{i+j=n} f_i(\theta^{n-i}(a))d_j(\phi^{n-j}(b))$ (resp. $f_n(a^2) = \sum_{i+j=n} f_i(\theta^{n-i}(a))d_j(\phi^{n-j}(a))$), for all $a,b \in R, n \in N$.

If U is a Lie ideal of R, then D is said to be a (θ,ϕ)-higher derivation (resp. Jordan (θ,ϕ)-higher derivation) of U into R and F is said to be a generalized (θ,ϕ)-higher derivation (resp. Jordan generalized (θ,ϕ)-higher derivation) of U into R in case that the above corresponding conditions are satisfied for all $a,b \in U$.

2.3 Example:

$d_n(ab) = \sum_{i+j=n} d_i(\theta^{n-i}(a))d_j(\phi^{n-j}(b))$ (resp. $d_n(a^2) = \sum_{i+j=n} d_i(\theta^{n-i}(a))d_j(\phi^{n-j}(a))$), for all $a,b \in R, n \in N$.

Now, we introduce the generalization of definition (2.1).

Consider the ring $R = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a,b \in Z \right\}$, where Z denotes the set of integer numbers. Let $F = (f_i)_{i\in N}$ be a family of mappings of R into R defined by

$$f_n\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{cases} id_R & n = 0 \\ \begin{bmatrix} -na & 0 \\ 0 & 0 \end{bmatrix} & n \geq 1 \end{cases},$$ for all $\begin{bmatrix} a & o \\ o & b \end{bmatrix} \in R$. Then there exists a (θ,ϕ)-higher derivation $D = (d_i)_{i\in N}$ of R which is defined by

$$d_n\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{cases} id_R & n=0 \\ \begin{bmatrix} na & 0 \\ 0 & 0 \end{bmatrix} & n \geq 1 \end{cases},$$ where

$\theta\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ and $\phi\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, for all $\begin{bmatrix} a & o \\ o & b \end{bmatrix} \in R$. Hence by definition (2.2), F is generalized (θ,ϕ)-higher derivation.

Using similar techniques as used to prove Lemma (2.6) and Lemma (2.7) in [1], one can prove the following lemmas which are generalization of these lemmas to generalized (θ,ϕ)-higher derivation on Lie ideals.

2.4 Lemma:

Let U be a Lie ideal of R and $F=(f_i)_{i\in N}$ be generalized Jordan (θ,ϕ)-higher derivation of U into R. Then for all $a,b\in U, n\in N$ we have

(i) $f_n(ab+ba) = \sum_{i+j=n} f_i(\theta^{n-i}(a))d_j(\phi^{n-j}(b)) + f_i(\theta^{n-i}(b))d_j(\phi^{n-j}(a))$.

If R is a 2- torsion free ring and U is a square closed Lie ideal, then

(ii)
$$f_n(aba) = \sum_{i+j+k=n} f_i(\theta^{n-i}(a))d_j(\theta^k\phi^i(b))d_k(\theta^{n-k}(a))$$

(iii)
$$f_n(abc+cba) = \sum_{i+j+k=n} f_i(\theta^{n-i}(a))d_j(\theta^k\phi^i(b))$$
$$d_k(\phi^{n-k}((c)) + f_i(\theta^{n-i}(c))d_j(\theta^k\phi^i(b))d_k(\phi^{n-k}(a))$$

2.5 Remark:

Let $F=(f_i)_{i\in N}$ be a generalized (θ,ϕ)-higher derivation of U into R. For all $n\in N, a,b\in U$ we denote by $\psi_n(a,b)$ (resp. $\varphi_n(a,b)$) the element of R defined by
$$\psi_n(a,b) = f_n(ab) - \sum_{i+j=n} f_i(\theta^{n-i}(a))d_j(\phi^{n-j}(b))$$
(resp.
$$\varphi_n(a,b) = d_n(ab) - \sum_{i+j=n} d_i(\theta^{n-i}(a))d_j(\phi^{n-j}(b)))$$

2.6 Lemma:

Let U be a square closed Lie ideal of a 2 -torsion free ring R and $F=(f_i)_{i\in N}$ be a Jordan generalized (θ,ϕ)-higher derivation of U into R such that $\psi_m(a,b) = \varphi_m(a,b)=0$, for all $a,b,c\in U, m,n\in N$ such that $m<n$, then

(i) $\psi_n(a,b)\phi^n[a,b]=0$.

(ii)
$$\psi_n(a,b)\phi^n(c)\phi^n[a,b] + \theta^n[a,b]\theta^n(c)\varphi_n(a,b)=0$$

2.7 Theorem:

Let U be a square closed Lie ideal of a 2 -torsion free prime ring R and $F=(f_i)_{i\in N}$ be a Jordan generalized (θ,ϕ)-higher derivation of U into R where ϕ is an automorphism of R. Then either F is a generalized (θ,ϕ)-higher derivation of U into R or $U\subseteq Z(R)$.

Proof:

We'll proceed by induction on N. We know that for $n=0$, $\psi_0(a,b) = \varphi_0(a,b)=0$ for all $a,b\in U$. Hence we may assume that $\psi_m(a,b) = \varphi_m(a,b)=0$, for all $a,b\in U$, $m,n\in N$ such that $m<n$.

By Lemma (2.6,(ii)), we have $\psi_n(a,b)\phi^n(c)\phi^n[a,b] + \theta^n[a,b]\theta^n(c)\varphi_n(a,b)=0$, for all $a,b,c\in U, n\in N$.

Now multiplying the above equation by $\phi^n[a,b]$ from right and using Lemma ((2.7,(i)),[1]), we have
$$\psi_n(a,b)\phi^n(c)\phi^n[a,b]\phi^n[a,b]$$
$$=\psi_n(a,b)\phi^n(c)\phi^n[a,b]^2=0,$$ for all $a,b,c\in U, n\in N$. Replace c by $[[a,b],r]$, the last equation becomes
$$0=\psi_n(a,b)\phi^n([a,b])\phi^n(r)\phi^n([a,b]^2)$$
$$-\psi_n(a,b)\phi^n(r)\phi^n([a,b]^3)$$
$$=\psi_n(a,b)\phi^n(r)\phi^n([a,b]^3),$$ for all $a,b\in U$, $r\in R, n\in N$.

Since R is prime and ϕ is automorphism, either $\psi_n(a,b)=0$ or $[a,b]^3=0$, for all $a,b\in U$. If $[a,b]^3=0$, then $(I_a(b))^3=0$, for all $b\in U$, where $I_a(b)$ is the inner derivation such that $I_a(b)=[a,b]$. Thus by ([12],Theorem 4), we find that $[a,b]=0$, for all $a,b\in U$ and this implies that U is commutative Lie ideal of R. Hence by ([13],Lemma 2) $U\subseteq Z(R)$.

2.8 Corollary:

Let U be an admissible Lie ideal of a 2-torsion free prime ring R. Then every Jordan generalized (θ,ϕ)-higher derivation of U into R where ϕ is an automorphism of R is a generalized (θ,ϕ)-higher derivation of U into R.

If $f_i = d_i$, for all $i \in N$ we get

2.9 Corollary:

Let U be an admissible Lie ideal of a 2-torsion free prime ring R. Then every Jordan (θ,ϕ)-higher derivation of U into R where ϕ is an automorphism of R is a (θ,ϕ)-higher derivation of U into R.

If $\theta = \phi = id_R$, we get

2.10 Corollary: (Corollary 1.4, [4])

Let U be an admissible Lie ideal of a 2-torsion free prime ring R. Then every Jordan higher derivation of U into R is a higher derivation of U into R.

If $U = R$, we get

2.11 Corollary: (Theorem 2.3, [1])

Let R be a non commutative 2-torsion free prime ring R. Then every Jordan (θ,ϕ)-higher derivation on R where ϕ is an automorphism of R is a (θ,ϕ)-higher derivation on R.

The following theorem is generalization of Theorem (2.10, [1]):

2.12 Theorem:

Let R be a 2-torsion free prime ring and θ be an automorphism on R. Then every Jordan generalized (θ,θ)-higher derivation of R is a generalized (θ,θ)-higher derivation of R.

Proof:

Let F be a Jordan generalized (θ,θ)-higher derivation of R, then exists a Jordan (θ,θ)-higher derivation D of R. By ([1], Theorem 2.10), D is (θ,θ)-higher derivation of R.

We'll proceed by induction on N. We know that for $n=0$, $\psi_0(a,b)=0$, then we can assume that $\psi_m(a,b)=0$, for all $a,b \in R, m,n \in N$ such that $m<n$. By Theorem (2.7), we get either F is generalized (θ,θ)-higher derivation of R or R is commutative.

If R is commutative. Then by Lemma (2.4, i), we get

$$\begin{aligned}
f_n(abc+cba) &= f_n((ab)c+c(ab)) \\
&= \sum_{i+j=n} f_i(\theta^{n-i}(ab))d_j(\theta^{n-j}(c)) + f_i(\theta^{n-i}(c))d_j(\phi^{n-j}(ab)) \\
&= f_n(ab)\theta^n(c)+\theta^n(ab)d_n(c) + \sum_{\substack{0\langle i,j\langle n\\ i+j=n}} f_i(\theta^{n-i}(ab))d_j(\theta^{n-j}(c)) \\
&\quad + f_n(c)\theta^n(ab)+\theta^n(c)d_n(ab) + \sum_{\substack{0\langle i,j\langle n\\ i+j=n}} f_i(\theta^{n-i}(c))d_j(\theta^{n-j}(ba)) \\
&= f_n(ab)\theta^n(c)+\theta^n(ab)d_n(c) + \sum_{\substack{0\langle l+t,j\langle n\\ l+t+j=n}} f_l(\theta^{n-l}(a))d_t(\theta^{n-t}(b))d_j(\theta^{n-j}(c)) \\
&\quad + f_n(c)\theta^n(ab)+\theta^n(c)d_n(ab) \\
&\quad + \sum_{\substack{0\langle i,p+q\langle n\\ i+p+q=n}} f_i(\theta^{n-i}(c))d_p(\theta^{n-p}(b))d_q(\theta^{n-q}(a)) \qquad (1)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
f_n(abc+cba) &= \sum_{i+j+k=n} f_i(\theta^{n-i}(a))d_j(\theta^{k+i}(b))d_k(\theta^{n-k}(c)) \\
&\quad + f_i(\theta^{n-i}(c))d_j(\theta^{k+i}(b))d_k(\theta^{n-k}(a)) \\
&= \sum_{i+j=n} f_i(\theta^{n-i}(a))d_j(\theta^{i}(b))\theta^n(c)+\theta^n(ab)d_n(c) \\
&\quad + \sum_{\substack{0\langle i+j,k\langle n\\ i+j+k=n}} f_i(\theta^{n-i}(a))d_j(\theta^{k+i}(b))d_k(\theta^{n-k}(c)) \\
&\quad + f_n(c)\theta^n(ba) \\
&\quad +\theta^n(c)\sum_{j+k=n} d_j(\theta^{n-k}(b))d_k(\theta^{n-k}(a)) \\
&\quad + \sum_{\substack{0\langle i,j+k\langle n\\ i+j+k=n}} f_i(\theta^{n-i}(c))d_j(\theta^{k+i}(b))d_k(\theta^{n-k}(a)) \qquad (2)
\end{aligned}$$

Compare equation (1) and equation (2), we get $\psi_n(a,b)\theta^n(c)+\varphi_n(b,a)\theta^n(c)=0$, for all $a,b,c \in R, n \in N$. By ([1], Lemma 2.6,i), $\varphi_n(a,b) = -\varphi_n(b,a)$. Hence, we get $(\psi_n(a,b)-\varphi_n(a,b))\theta^n(c)=0$, for all $a,b,c \in R, n \in N$. Since θ is automorphism and R is prime, then $\psi_n(a,b)=\varphi_n(a,b)$, for all $a,b,c\in R, n\in N$. (3) since D is (θ,θ)-higher derivation of R, then equation (3) becomes $\psi_n(a,b)=\varphi_n(a,b)=0$ and this means F is a generalized (θ,θ)-higher derivation of R.

2.13 Corollary: (Theorem 2.10, [1])

Let R be a 2-torsion free prime ring and θ be an automorphism on R. Then every Jordan (θ,θ)-higher derivation of R is a (θ,θ)-higher derivation of R.

3.Generalized $(U,R)-(\theta,\phi)$ -Higher Derivations

We generalize the previous results by introducing the concepts of $(U,R)-(\theta,\phi)$ -higher derivation and $(U,R)-(\theta,\phi)$ -higher derivation.

3.1 Definition:

Let U be a Lie ideal of a ring R and $D=(d_i)_{i\in N}$ be a family of additive mappings of R such that $d_0=id_R$. D is said to be a $(U,R)-(\theta,\phi)$-higher derivation of R, if for every $n\in N$

$$d_n(ur+su)=\sum_{i+j=n} d_i(\theta^{n-i}(u))d_j(\phi^{n-j}(r)) + d_i(\theta^{n-i}(s))d_j(\phi^{n-j}(u))$$

, for all $u\in U, r,s\in R$.

3.2 Definition:

Let U be a Lie ideal of a ring R and $F=(f_i)_{i\in N}$ be a family of additive mappings of R such that $f_0=id_R$. F is said to be a generalized $(U,R)-(\theta,\phi)$-higher derivation of R, if there exists a $(U,R)-(\theta,\phi)$-higher derivation $D=(d_i)_{i\in N}$ of R such that for every $n\in N$ we have

$$f_n(ur+su)=\sum_{i+j=n} f_i(\theta^{n-i}(u))d_j(\phi^{n-j}(r)) + f_i(\theta^{n-i}(s))d_j(\phi^{n-j}(u)),$$

for all $u\in U, r,s\in R$.

If $f_i=d_i$, for all $i\in N$, then F becomes $(U,R)-(\theta,\phi)$-higher derivation.

3.3 Example:

Consider the ring $R=\left\{\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix} : a,b\in Z\right\}$, where Z denotes the set of integer numbers. Then $U=\left\{\begin{bmatrix} a & 0\\ 0 & 0\end{bmatrix} : a\in Z\right\}$ is a Lie ideal of R.

Then the family $F=(f_i)_{i\in N}$ of additive mappings which is defined in Example (2.3) is generalized $(U,R)-(\theta,\phi)$-higher derivation of U.

3.4 Lemma:

Let U be a Lie ideal of a 2-torsion free ring R and $F=(f_i)_{i\in N}$ be a generalized $(U,R)-(\theta,\phi)$-higher derivation of R. Then for every $u,v\in U, r\in R$ $n\in N$ the following statements hold:

(i)

$$f_n(uru)=\sum_{i+j+k=n} f_i(\theta^{n-i}(u))d_j(\theta^k\phi^i(r))d_k(\phi^{n-k}(u))$$

(ii) $f_n(urv+vru)$

$$=\sum_{i+j+k=n} f_i(\theta^{n-i}(u))d_j(\theta^k\phi^i(r))d_k(\phi^{n-k}(v)) + \sum_{i+j+k=n} f_i(\theta^{n-i}(v))d_j(\theta^k\phi^i(r))d_k(\phi^{n-k}(u))$$

Proof:

(i) Replace r and s by $w=(2u)r+r(2u)$ in definition (3.8),

$$f_n(w)=f_n(u((2u)r+r(2u))+((2u)r+r(2u))u)$$

$$=2\sum_{i+j=n} f_i(\theta^{n-i}(u))d_j(\phi^{n-j}(ur+ru)) + d_i(\theta^{n-i}(ur+ru))d_j(\phi^{n-j}(u))$$

$$=2\left\{\sum_{i+j=n} f_i(\theta^{n-i}(u))\sum_{l+t=j} d_l(\theta^{j-l}\phi^{n-j}(u)) d_t(\phi^{j-t}\phi^{n-j}(r)) + d_l(\theta^{j-l}\phi^{n-j}(r))d_t(\phi^{j-t}\phi^{n-j}(u)) + \sum_{i+j=n}\left(\sum_{p+q=i} f_p(\theta^{i-p}\theta^{n-i}(u))d_q(\phi^{i-q}\theta^{n-i}(r)) + f_p(\theta^{i-p}\theta^{n-i}(r))d_q(\phi^{i-q}\theta^{n-i}(u))\right)d_j(\phi^{n-j}(u))\right\}$$

$$=2\left\{\sum_{i+l+t=n} f_i(\theta^{n-i}(u))d_l(\theta^t\phi^{n-j}(u))d_t(\phi^{n-t}(r)) + f_i(\theta^{n-i}(u))d_l(\theta^t\phi^{n-j}(r))d_t(\phi^{n-t}(u))\right.$$

$$+ \sum_{p+q+j=n} f_p(\theta^{n-p}(u))d_q(\theta^j\phi^p(r))d_j(\phi^{n-j}(u))$$
$$+ f_p(\theta^{n-p}(r))d_q(\theta^j\phi^p(u)))d_j(\phi^{n-j}(u))\}$$

Using,

$$2\sum_{i+l+t=n} f_i(\theta^{n-i}(u))d_l(\theta^t\phi^{n-j}(r))d_t(\phi^{n-t}(u))$$
$$+2\sum_{p+q+j=n} f_p(\theta^{n-p}(u))d_q(\theta^j\phi^p(r))d_j(\phi^{n-j}(u))$$
$$=4\sum_{i+j+k=n} f_i(\theta^{n-i}(u))d_j(\theta^k\phi^i(r))d_k(\phi^{n-k}(u))$$

Then

$$f_n(w)$$
$$=2\left\{\sum_{i+l+t=n} f_i(\theta^{n-i}(u))d_l(\theta^t\phi^{n-j}(u))d_t(\phi^{n-t}(r)) + \sum_{p+q+j=n} f_p(\theta^{n-p}(r))d_q(\theta^j\phi^p(u))d_j(\phi^{n-j}(u))\right\}$$
$$+4\sum_{i+j+k=n} f_i(\theta^{n-i}(u))d_j(\theta^k\phi^i(r))d_k(\phi^{n-k}(u)) \quad (4)$$

On the other hand,

$$f_n(w) = f_n((2u^2)r + r(2u^2)) + 4f_n(uru)$$
$$=\sum_{i+j=n} f_i(\theta^{n-i}(2u^2))d_j(\phi^{n-j}(r)$$
$$+ f_i(\theta^{n-i}(r))d_j(\phi^{n-j}(2u^2)) + 4f_n(uru)$$
$$=2\left\{\sum_{i+j=n}\sum_{p+q=i} f_p(\theta^{i-p}\theta^{n-i}(u))d_q(\phi^{i-q}\theta^{n-i}(u))\right.$$
$$d_j(\phi^{n-j}(r))$$
$$\left.+\sum_{i+j=n} f_i(\theta^{n-i}(r))\sum_{l+t=j} d_l(\theta^{j-l}\phi^{n-j}(u)d_t(\phi^{j-t}\phi^{n-j}(u))\right\}$$
$$+4f_n(uru)$$

$$=2\left\{\sum_{p+q+j=n} f_p(\theta^{n-p}(u))d_q(\theta^j\phi^p(u))d_j(\phi^{n-j}(r)) + \sum_{i+l+t=n} f_i(\theta^{n-i}(r))d_l(\theta^t\phi^i(u))d_t(\phi^{n-t}(u))\right\}$$
$$+4f_n(uru) \quad (5)$$

Compare equation (4) and equation (5) and since R is 2-torsion free, we get

$$f_n(uru) = \sum_{i+j+k=n} f_i(\theta^{n-i}(u))d_j(\theta^k\phi^i(r))d_k(\phi^{n-k}(u))$$

.

(ii) Replace u by $u+v$ in (i), we get the required result.

Let $F=(f_i)_{i\in N}$ be a generalized $(U,R)-(\theta,\phi)$-higher derivation of R. For all $n\in N, u\in U, r\in R$ we denote by $\Gamma_n(u,r)$ (resp. $\Theta_n(u,r)$) the element of R defined by

$$\Gamma_n(u,r) = f_n(ur) - \sum_{i+j=n} f_i(\theta^{n-i}(u))d_j(\phi^{n-j}(r))$$

(resp.

$$\Theta_n(u,r) = d_n(ur) - \sum_{i+j=n} d_i(\theta^{n-i}(u))d_j(\phi^{n-j}(r))$$

)

3.5 Lemma:

Let U be an admissible Lie ideal of a 2 -torsion free prime ring R and F be a generalized $(U,R)-(\theta,\phi)$-higher derivation of R such that where ϕ is an automorphism on R. Then $\Gamma_n(u^2,r)=0$, for all $u\in U, r\in R, n\in N$.

Proof:

Replace r and s by u in definition (3.2), then we get F is Jordan generalized (θ,ϕ)-higher derivation of U into R. By Corollary (2.8), F is generalized (θ,ϕ)-higher derivation of U into R and this means $\Gamma_n(u,v)=0$, for all $u,v\in U$. (6)

Replace v by $ur-ru$ in equation (6), we get

$$0=\Gamma_n(u,ur-ru) = f_n(u^2r) - f_n(uru)$$
$$-\sum_{i+j=n} f_i(\theta^{n-i}(u))d_j(\phi^{n-j}(ur-ru))$$

Since D is $(U,R)-(\theta,\phi)$-higher derivation, the last equation becomes

$$0 = f_n(u^2r) - f_n(uru)$$
$$-\sum_{i+j=n} f_i(\theta^{n-i}(u))\sum_{l+t=j} d_l(\theta^{j-l}\phi^{n-j}(u))$$
$$d_t(\phi^{j-t}\phi^{n-j}(r)))$$
$$+\sum_{i+j=n} f_i(\theta^{n-i}(u))\sum_{l+t=j} d_l(\theta^{j-l}\phi^{n-j}(r))$$
$$d_t(\phi^{j-t}\phi^{n-j}(u)))$$
$$= f_n(u^2r) - f_n(uru)$$

$$- \sum_{i+l+t=n} f_i(\theta^{n-i}(u))d_l(\theta^t\phi^i(u))d_t(\phi^{n-t}(r))$$

$$+ \sum_{i+l+t=n} f_i(\theta^{n-i}(u))d_l(\theta^t\phi^i(r))d_t(\phi^{n-t}(u))$$

$$= f_n(u^2r) - \sum_{s+t=n}\sum_{i+l=s} d_i(\theta^{s-i}\theta^{n-s}(u)) d_l(\phi^{s-l}\theta^{n-s}(u))d_t(\phi^{n-t}(r))$$

$= \Gamma_n(u^2, r)$, for all $u \in U, r \in R, n \in N$.

3.6 Theorem:

Let U be an admissible Lie ideal of a 2 -torsion free prime ring R and $F = (f_i)_{i\in N}$ be a generalized $(U,R)-(\theta,\theta)$-higher derivation of R such that θ is an automorphism on R. Then $\Gamma_n(u,r) = 0$, for all $u \in U, r \in R, n \in N$.

Proof:

We prove the theorem by induction on $n \in N$. For any $u \in U, r \in R$, $\Gamma_0(u,r) = 0$. Hence we can assume that $\Gamma_m(u,r) = 0$, for all $u \in U, r \in R, m,n \in N$ such thatt $m < n$. Since F is generalized $(U,R)-(\theta,\theta)$-higher derivation, then

$$f_n(uur + uru)$$

$$= \sum_{i+j=n} f_i(\theta^{n-i}(u))d_j(\theta^{n-j}(ur)) + f_i(\theta^{n-i}(ur))d_j(\theta^{n-j}(u))$$

$$= f_n(u)\theta^n(ur) + \theta^n(u)d_n(ur) + \sum_{\substack{i+j=n\\ 0\langle i,j\langle n}} f_i(\theta^{n-i}(u))d_j(\theta^{n-j}(ur))$$

$$+ f_n(ur)\theta^n(u) + \theta^n(ur)d_n(u) + \sum_{\substack{i+j=n\\ 0\langle i,j\langle n}} f_i(\theta^{n-i}(ur))d_j(\theta^{n-j}(u))$$

Since D is $(U,R)-(\theta,\theta)$-higher derivation and $\psi_m(u,r) = 0$, for all $u \in U, r \in R, m,n \in N$ such that $m < n$, the last equation becomes

$$f_n(uur + uru) = f_n(u)\theta^n(ur) + \theta^n(u)d_n(ur)$$

$$+ \sum_{\substack{i+j=n\\ 0\langle i,j\langle n}} f_i(\theta^{n-i}(u))\sum_{l+t=j} d_l(\theta^{j-l}\theta^{n-j}(u))d_t(\theta^{j-t}\theta^{n-j}(r))$$

$$+ f_n(ur)\theta^n(u) + \theta^n(ur)d_n(u)$$

$$+ \sum_{\substack{i+j=n\\ 0\langle i,j\langle n}} \sum_{p+q=i} f_p(\theta^{i-p}\theta^{n-i}(u))d_q(\theta^{i-q}\theta^{n-i}(r))d_j(\theta^{n-j}(u))$$

$$= f_n(u)\theta^n(ur) + \theta^n(u)d_n(ur) + \sum_{\substack{i+l+t=n\\ 0\langle i,l+t\langle n}} f_i(\theta^{n-i}(u))d_l(\theta^t\theta^i(u))d_t(\theta^{n-t}(r))$$

$$+ f_n(ur)\theta^n(u) + \theta^n(ur)d_n(u) + \sum_{\substack{p+q+j=n\\ 0\langle p+q,j\langle n}} f_p(\theta^{n-p}(u))d_q(\theta^p\theta^j(r))d_j(\theta^{n-j}(u)) \quad (7)$$

On the other hand, by Lemma (3.5) and Lemma (3.4,i) we get

$$f_n(uur + uru) = f_n(u^2r) + f_n(uru)$$

$$= \sum_{i+j=n} f_i(\theta^{n-i}(u^2))d_j(\theta^{n-j}(r)) + \sum_{i+j+k=n} f_i(\theta^{n-i}(u))d_j(\theta^{k+i}(r))d_k(\theta^{n-k}(u))$$

$$= \sum_{l+t+j=n} f_l(\theta^{i-l}\theta^{n-i}(u))d_t(\theta^{i-t}\theta^{n-i}(u))d_j(\theta^{n-j}(r)) + \sum_{i+j+k=n} f_i(\theta^{n-i}(u))d_j(\theta^{k+i}(r))d_k(\theta^{n-k}(u))$$

$$= \sum_{l+t+j=n} f_l(\theta^{n-l}(u))d_t(\theta^{j+l}(u))d_j(\theta^{n-j}(r)) + \sum_{i+j+k=n} f_i(\theta^{n-i}(u))d_j(\theta^{k+i}(r))d_k(\theta^{n-k}(u))$$

$$= f_n(u)\theta^n(ur) + \theta^n(u)\sum_{t+j=n} d_t(\theta^j(u))d_j(\theta^{n-j}(r))$$

$$+ \sum_{\substack{l+t+j=n\\ 0\langle l,t+j\langle n}} f_l(\theta^{n-l}(u))d_t(\theta^{j+l}\theta^i(u))d_j(\theta^{n-j}(r))$$

$$+ \sum_{i+j=n} f_i(\theta^{n-i}(u))d_j(\theta^i(r))\theta^n(u)$$

$$+ \theta^n(ur)d_n(u)$$

$$+ \sum_{\substack{i+j+k=n\\ 0\langle i+j,k\langle n}} f_i(\theta^{n-i}(u))d_j(\theta^{k+i}(r))d_k(\theta^{n-k}(u))$$

Compare equation (7) and equation (8) to get

$\theta^n(u)\Gamma_n(u,r) + \Gamma_n(u,r)\theta^n(u) = 0$, for all $u \in U, r \in R, n \in N$. (9)

Linearize equation (9) on u and use equation (9), we get

$$\theta^n(u)\Gamma_n(v,r) + \theta^n(v)\Gamma_n(u,r) + \Gamma_n(v,r)\theta^n(u)$$

$+\Gamma_n(u,r)\theta^n(v)=0$, for all $u,v\in U, r\in R$, $n\in N$. Replace v by v^2 in the last equation and by Lemma (3.5), we get

$\theta^n(v^2)\Gamma_n(u,r)+\Gamma_n(u,r)\theta^n(v^2)=0$, for all $u,v\in U, r\in R$, $n\in N$.

Since θ is automorphism, then $v^2\theta^{-n}(\Gamma_n(u,r))+\theta^{-n}(\Gamma_n(u,r))v^2=0$, for all $u,v\in U, r\in R$, $n\in N$. since U is an admissible Lie ideal, then by using Lemma (2.7) in [11], we get $\Gamma_n(u,r)=0$, for all $u\in U, r\in R, n\in N$.

3.8 Corollary:

Let U be an admissible Lie ideal of a 2-torsion free prime ring R and F be a Jordan generalized (θ,θ)-higher derivation of U into R such that θ is an automorphism on R. Then F is generalized (θ,θ)-higher derivation of U into R.

3.9 Corollary:

Let R be a non commutative 2-torsion free prime ring and F be a Jordan generalized (θ,θ)-higher derivation of R such that θ is an automorphism on R. Then F be a generalized (θ,θ)-higher derivation of R .

If $\theta=id_R$, we have

3.10 Corollary:(Theorem 5.1,[11])

Let U be an admissible Lie ideal of a 2-torsion free prime ring R and $F=(f_i)_{i\in N}$ be a generalized (U,R)-higher derivation of R. Then $\Gamma_n(u,r)=0$, for all $u\in U, r\in R, n\in N$.

3.11 Corollary:(Corollary 1.4,[2])

Let U be an admissible Lie ideal of a 2-torsion free prime ring R and D be a Jordan higher derivation of U into R. Then D is a higher derivation of U into R.

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