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SOME REMARKS ON TWO MULLINEUX'S PARTITIONS

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Abstract

Mullineux in 1979 presented two researches taking in them some special methods of the partition theory, as he represented a kind of a special partition a mathematical sense without any proof of it. Fayers in 2009 presented it with its conditions adding another type pf partition and called the two partitions of Mullineux. The main aim of our work is to find a suitable partition for the equivalence of these two partitions, where this equivalence is not always occurred.

بعض الملاحظات حول تجزئتي مينولكس

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الخلاصة

قدم مينولكس في ١٩٧٩ بحثين تناول فيهما بعض المفاهيم الخاصة بنظرية التجزئة حيث عرض نوع مـن التجزئة الخاصة كحدس رياضي ومن دون أي برهان. إلا أن جاء فايرس في ٢٠٠٩ حيث عرضها بشـروطها مضيفا عليها نوع أخر من التجزئة أطلق عليهما تجزئات مينولكس. إن الهدف من هذا العمل هو إيجاد التجزئـة الملائمة لتساوي (أو تكافؤ) هاتين التجزئتين علما أنهما غير متساويين بشكل عام

1. Introduction

Let r be a non-negative integer. A composition $\mu = (\mu_1, \mu_2, ..., \mu_n)$ of r is a sequence of non-negative integers such that $|\mu| = \sum_{j=1}^{n} \mu_j = r$. For all $j \geq 1$, if $\mu_j \geq \mu_{j+1}$ the composition μ is called a partition. James [4] defined $\beta - numbers$ by : "Fix μ is a partition of r. Choose an integer b greater than or equal to the number of parts of μ and define: $\beta_i = \mu_i + b - i,\ 1 \leq i \leq b.$ The set $\{\beta_1, \beta_2, ..., \beta_b\}$ is said to be a set of $\beta - numbers$ for μ."

We can represent $\beta - numbers$ in many of runners (run.1, run.2, …, run.e) as the following:

$run.1$	$run.2$		$run.e$	
0	1	...	$e-1$	
e	$e+1$	...	$2e-1$	$diag.(A)$.
$2e$	$2e+1$	...	$3e-1$	
$\vdots$	$\vdots$		$\vdots$	

Where every β will be represented by a bead which takes its location in diag.(A). For example, if $\mu = (7,5,3,3,3,2,1,1)$ and if we take $b = 9,$ then $\beta - numbers$ are $\{15,12,9,8,7,5,3,2,0\}$ and if we choose $e = 3,$ then diag.(A) is

0	1	2		•	–	•
3	4	5		•	–	•
6	7	8		–	•	•
9	10	11	$\xrightarrow{b=9}$	•	–	–.
12	13	14		•	–	–
15	16	17		•	–	–

Fayers in [5] defined the following definitions: "Given any partition μ, the conjugate partition μ' is defined by $\mu_t' = \{j \geq 1 \mid \mu_j \geq t\}$, the partition μ is $e - regular$, if there no exist $j \geq 1$ such that $\mu_j = \mu_{j+e-1} > 0$, also μ is $e - restricted$ if $\mu_j - \mu_{j+1} < e, \forall j \geq 1$ or if μ' is $e - regular$."

2. Two Mullineux's Partitions

In 1979 Mullineux presented a sense, see [1, 2], of an existence of a type of the main partition without any proof, but Fayres in 2009 [3] could find the necessary conditions for this partition adding another type to it and called it the two partitions of Mullineux, here are these conditions:

Definition (2.1): [3, 6.1]

Suppose μ is an $e - regular$ partition, and take an abacus display for μ; (diag.(A)), with b beads, for some $b \geq \mu_1'$. Let α, γ be the positions of the last bead and the first empty space on the abacus, respectively; so α is the beta number $\beta_1 = \mu_1 + b - 1,$ while γ equals $b - \mu_1'$. Assuming $\mu \neq \phi,$ there is a unique sequence $a_1 > c_1 > ... > a_l > c_l$ of non-negative integers satisfying the following conditions:

1- For each $1 \leq i \leq l,$ position a_i is occupied and position c_i is empty.

2- $a_1 = \alpha$.

3-For $1 \leq i \leq l,$ we have

- $a_i \equiv c_i \pmod e$, and all the positions $a_i - e, a_i - 2e, ..., c_i + e$ are occupied;
- all the positions $c_i - 1, c_i - 2, ..., a_{i+1} + 1$ are empty.

4-Either:

- $a_l \equiv c_l \pmod e$, all the positions $a_l - e, ..., c_l + e$ are occupied, and all the positions $c_l - 1, c_l - 2, ..., \gamma$ are empty: or
- all the position $a_l - e, a_l - 2e, ...$ are occupied and $c_l = \gamma$.

We define μ^{Δ} to be the partition whose abacus display is obtained by moving the beads at positions $a_1, ..., a_l$ to position $c_1, ..., c_l$.

There exist another partition; "a conjugate" definition to (2.1); as the following:

Definition (2.2): [3, 6.2]

Suppose λ is an $e - restricted$ partition, and take an abacus display; (diag.(A)), for λ with b beads. Let δ, ε be the position of the last bead and the first empty space on the abacus, respectively. Assuming $\lambda \neq \phi$, there is a unique sequence $f_1 > g_1 > ... > f_u > g_u$ of non-negative integers satisfying the following conditions:

1- For each $1 \leq i \leq u,$ position f_i is occupied and position g_i is empty.

2- $g_u = \varepsilon.$

3- For $1 \leq i < u,$ we have

- $f_i \equiv g_i \pmod e$, and all the positions $f_i - e, f_i - 2e, ..., g_i + e$ are empty;

all the positions $f_i + 1, f_i + 2, ..., g_{i-1} - 1$ are occupied.

4- Either:

- $f_1 \equiv g_1 \pmod e$, and all positions $f_1 - e, ..., g_1 + e$ are empty, and all the positions $\delta, \delta - 1, ..., f_1 + 1$ are occupied; or

- all the positions $g_1+e, g_1+2e,...$ are empty and $f_1=\delta$.

We define λ^{∇} to be the partition whose abacus display is obtained by moving the beads at positions $f_1,...,f_u$ to positions $g_1,...,g_u$.

For example, let $\mu=(7,5^2,2^2,1)$ and $b=6$ then it's 3-regular and 4-restricted. Also we can easy to find μ^{Δ} and μ^{∇} by the following:

$$\mu^{\Delta} \qquad\qquad \mu^{\nabla}$$

$$\begin{array}{ccc} \bullet & \bullet & - \\ \bullet & - & - \\ \bullet & - & \bullet \\ \bullet & - & - \end{array} \qquad \begin{array}{cccc} \bullet & \bullet & - & \bullet \\ - & \bullet & - & - \\ \bullet & - & \bullet & - \\ - & - & - & - \end{array} .$$

Our main aim of this research is to find *a suitable partition* in which the two partitions of Mullineux are equal. We have to remained that this partition which we want to find, is mainly of:

$$\mu=(\mu_1,\mu_2,...,\mu_n)=(\nu_1^{\tau_1},\nu_2^{\tau_2},...,\nu_k^{\tau_k})$$

such that $|\mu|=\sum_{s=1}^{k}\nu_s^{\tau_s}=r$ and τ_s is a repetition of μ_s for all $1\le s\le k$.

We will use the mathematical induction to find the suitable partition for each case as the following:

Proposition (2.3):

Let $(\nu_y-\nu_{y+1}=e-1)$ and $(\tau_1=\tau_2=...=\tau_k=1)$ where $1\le y\le k-1$. Then there exist two cases of the suitable partition in each case there are many probabilities.

Proof: It is clear that when $e\ge 2$ then all the possible cases of the equivalence of two partitions of Mullineux will be under the following only two cases:

Case (M_{11}) There exists just one column full of beads, where all other columns are empty after drawing diag.(A). This full column is always in the runner (2), so this will be the first and unique partition in the case when $e=2$. Take $e=3$, then there exists another suitable partition when the full column in runner (2) will move to runner (3). Now for $e\ge 3$, we will see that there exists $(e-1)$ suitable partitions, always beginning by the existence of only one column full of beads in runner (2), then this one will move to runner (3), …, etc, till it will reach runner (e).

Case (M_{12}) When the column which full of beads will reach runner (e) in case M_{11}, then it is possible to move to runner (1), at this time we must leave all the first row of diag.(A) to be empty, then repeating the existence of the column full of beads in runner (1) •

Examples (2.4):

1- Let $e=2$, then

$$\begin{array}{cc} - & \bullet \\ - & \bullet \\ \vdots & \vdots \\ - & \bullet \end{array} \qquad \begin{array}{cc} - & - \\ \bullet & - \\ \bullet & - \\ \vdots & \vdots \\ \bullet & - \end{array} .$$

$$M_{11} \qquad\qquad M_{12}$$

2- Let $e=3$, then the number of cases M_{11} is

$$\begin{array}{ccc} - & \bullet & - \\ - & \bullet & - \\ \vdots & \vdots & \vdots \\ - & \bullet & - \end{array} \qquad \begin{array}{ccc} - & - & \bullet \\ - & - & \bullet \\ \vdots & \vdots & \vdots \\ - & - & \bullet \end{array}$$

and for M_{12} is

$$\begin{array}{ccc} - & - & - \\ \bullet & - & - \\ \bullet & - & - \\ \vdots & \vdots & \vdots \\ \bullet & - & - \end{array} .$$

Finding the value of the suitable partition of Proposition (2.3):

We introduced the explanation method of finding the suitable partition, where we must find its values in M_{11} as the following:

$$(\nu_1,(\nu_1-(e-1)),(\nu_1-2(e-1)),...,\pi_1,\pi_2) \quad ... \quad (2.5)$$

such that $\pi_1 = \pi_2 + (e-1)$ and $\pi_2 = 1,2,...,(e-1)$ respectively, and there is very clear the number of cases in M_{11} is $(e-1)$. Also, the unique case in M_{12} is the same (2.5) such that:

$\pi_1 = \pi_2 + (e-1)$ *and* $\pi_2 = e.$

Proposition (2.6): Let $(\nu_y - \nu_{y+1} = e-2)$, $(\tau_1, \tau_k = 1 \; or \; 2)$ and $(\tau_2 = \tau_3 = ... = \tau_{k-1} = 2)$ where $1 \le y \le k-1$. Then there exist three cases of the suitable partition in each case posses many probabilities.

Proof: It is clear that we must begin by $e \ge 3$. Were we will determine three cases as the following:

Case M_{21} : If we choose $e = 3$ then there is two types of the suitable partition in which the equivalence of the two partition of Mullineux will verified as the 1st existence of two columns full of beads exactly in runner (2) and runner (3), where runner (1) will be empty of any bead. The 2nd it is the same as the first one except the last bead in runner (3) which will be empty. Now, taking $e = 4$, we will obtain the same two types when $e = 3$ in M_{21} but we adding one step to the right for each type to the M_{21} i.e. all runner (2) will move to runner (3), and all runner (3) to runner (4). That is we have four probabilities here. If we continue for $e \ge 5$ in this case, then there will be $2(e-2)$ types.

Case M_{22} : For any $e \ge 3$ we have a unique type of suitable partition which will all the runners are empty except the runner $(e-1)$; (all the column in this runner is occupied by the beads minus the first position is empty) and the runner (e); (without any position is empty in this runner).

Case M_{23} : Because we reached to runner (e) in both cases M_{21} *and* M_{22} and as happened in case M_{12}, the movements will be in the same position of runner (1), so for any $e \ge 3$ we will have just four probabilities: (P1) each of runner (1) and runner (e) is full of beads except the other runners which are completely empty, (P2) each of runner (1) and runner (e) is full of beads except the last position of runner (e) which will be empty and all other runners will be empty too, (P3) each of runner (1) and runner (2) is full of beads except the last position of runner (2) which will be empty and all other runners will be empty too, and (P4) each of runner (1) and runner (2) is full of beads where all other runners will be empty always. Here we must appoint that in case M_{12} the first row of all probabilities will be always empty and then the existence of the beads will be repeat in the same order •

Examples (2.7):

1- Let $e = 3$, then the two Mullineux's partitions are equals about the Proposition (2.6) are

```
– • •    – • •          – – •
– • •    – • •          – • •
⋮ ⋮ ⋮    ⋮ ⋮ ⋮ ,        ⋮ ⋮ ⋮
– • •    – • •          – • •     and
– • •    – • –          – • •
     M21                  M22
```

```
– – –    – – –    – – –    – – –
• – •    • – •    • • –    • • –
⋮ ⋮ ⋮    ⋮ ⋮ ⋮    ⋮ ⋮ ⋮    ⋮ ⋮ ⋮
• – •    • – •    • • –    • • –
• – •    • – –    • – –    • • –
              M23
```

2- If we choose $e = 4$, then $(\# M_{21})$ are:

```
– • • –    – • • –
⋮ ⋮ ⋮ ⋮    ⋮ ⋮ ⋮ ⋮
– • • –    – • • –
– • • –    – • – –
```

```
– – • •  and  – – • •
⋮ ⋮ ⋮ ⋮       ⋮ ⋮ ⋮ ⋮ .
– – • •       – – • •
– – • •       – – • –
```

The unique diagram in M_{22} is

$$\begin{array}{cccc} - & - & - & \bullet \\ - & - & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ - & - & \bullet & \bullet \end{array}.$$

And $(\# M_{23})$ are

$$\begin{array}{cccc} - & - & - & - \\ \bullet & - & - & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & - & - & \bullet \\ \bullet & - & - & \bullet \end{array} \qquad \begin{array}{cccc} - & - & - & - \\ \bullet & - & - & \bullet \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & - & - & \bullet \\ \bullet & - & - & - \end{array}$$

$$\begin{array}{cccc} - & - & - & - \\ \bullet & \bullet & - & - \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & - & - \\ \bullet & - & - & - \end{array} \quad and \quad \begin{array}{cccc} - & - & - & - \\ \bullet & \bullet & - & - \\ \vdots & \vdots & \vdots & \vdots \\ \bullet & \bullet & - & - \\ \bullet & \bullet & - & - \end{array}$$

Finding the value of the suitable partition of Proposition (2.6):

The partition in (2.5) will be generalize and will be as:

$$(v_1^{\tau_1}, (v_1-(e-2))^{\tau_2}, (v_1-2(e-2))^{\tau_3}, \ldots, \pi_1^{\tau_{k-1}}, \pi_2^{\tau_k}) \ \ldots \qquad (2.7)$$

Such that:

1- For M_{21}:
$\tau_1 = 1,2 \qquad \tau_k = 2$
$\pi_1 = \pi_2 + (e-2)$ *and*
$\pi_2 = 1,2,\ldots,(e-2)$ *respectivly*.

2- For M_{22}:
$\tau_1 = 2 \qquad \tau_k = 1$
$\pi_1 = \pi_2 + (e-2)$ *and*
$\pi_2 = (e-1)$.

3- For M_{23}:
$\tau_1 = 1,2 \qquad \tau_k = 1,2$
$\pi_1 = \pi_2 + (e-2)$ *and*
$\pi_2 = e$.

Proposition (2.8): Let $(v_y - v_{y+1} = e-3)$, $(\tau_1, \tau_k = 1, 2 \text{ or } 3)$ and $(\tau_2 = \tau_3 = \ldots = \tau_{k-1} = 3)$ where $1 \le y \le k-1$. Then there exist four cases of the suitable partition in each case posses many probabilities.

Proof: Depending on Propositions (2.3) and (2.6), then we have the following main cases:

$M_{31}: \ \tau_1 = 1,2,3 \qquad \tau_k = 3$
1- $\pi_1 = \pi_2 + (e-3)$ *and*
$\pi_2 = 1,2,\ldots,(e-3)$ *respictivly*.

$M_{32}: \ \tau_1 = 2,3 \qquad \tau_k = 2$
2- $\pi_1 = \pi_2 + (e-3)$ *and*
$\pi_2 = (e-2)$.

$M_{33}: \ \tau_1 = 3 \qquad \tau_k = 1$
3- $\pi_1 = \pi_2 + (e-3)$ *and*
$\pi_2 = (e-1)$.

$M_{34}: \ \tau_1 = 1,2,3 \qquad \tau_k = 1,2,3$
4- $\pi_1 = \pi_2 + (e-3)$ *and*
$\pi_2 = e \bullet$

It is possible to generalize the Propositions (2.3), (2.6) and (2.8) and the relations (2.5) and (2.7) in the following theorem:

Theorem (2.9): Let $(v_y - v_{y+1} = e - e_1)$, $(\tau_1, \tau_k = 1, 2, \ldots or\ e_1)$ and $(\tau_2 = \tau_3 = \ldots = \tau_{k-1} = e_1)$ where $1 \le y \le k-1$ *and* $1 \le e_1 < e$. Then there exist $(e_1 + 1)$ of possible cases of the suitable partition in each case there is many probabilities.

Proof: Depending on all the propositions and the relations in this work, then the possible cases are:

$M_{e_1 1}$	$\tau_1 = 1,2,...,e_1$ $\tau_k = e_1$	$\pi_1 = \pi_2 + (e - e_1)$ $\pi_2 = 1,2,...,(e - e_1)$ *respectivly*
$M_{e_2 2}$	$\tau_1 = 2,3,...,e_1$ $\tau_k = e_1 - 1$	$\pi_1 = \pi_2 + (e - e_1)$ $\pi_2 = (e - e_1) + 1$
$M_{e_1 3}$	$\tau_1 = 3,4,...,e_1$ $\tau_k = e_1 - 2$	$\pi_1 = \pi_2 + (e - e_1)$ $\pi_2 = (e - e_1) + 2$
$\vdots$	$\vdots$	$\vdots$
$M_{e_1(e_1-1)}$	$\tau_1 = (e_1 - 1), e_1$ $\tau_k = 2$	$\pi_1 = \pi_2 + (e - e_1)$ $\pi_2 = e - 2$
$M_{e_1(e_1)}$	$\tau_1 = e_1$ $\tau_k = 1$	$\pi_1 = \pi_2 + (e - e_1)$ $\pi_2 = e - 1$
$M_{e_1(e_1+1)}$	$\tau_1 = 1,2,...,e_1$ $\tau_k = 1,2,...,e_1$	$\pi_1 = \pi_2 + (e - e_1)$ $\pi_2 = e$

Theorem (2.10): The number of the probabilities in every cases of $(e_1 + 1)$ in Theorem (2.9) will be

$$(e_1(e - e_1) + \sum_{z=1}^{(e_1-1)} (e_1 - z) + (e_1)^2).$$

Proof: It is easy to calculate the number of the probabilities in all cases of $(e_1 + 1)$ in Theorem (2.9), so will put the cases in three sorts. The 1st sort in the first case, i.e. $M_{e_1 1}$ it is clear that the number of this case is $e_1(e - e_1)$ directly. Where the 2nd sort in all cases beginning from $M_{e_2 2}$ till $M_{e_1(e_1)}$, and the number of the possible probabilities is

$$((e_1 - 2) + 1) + ((e_1 - 3) + 1) + ... +$$
$$(((e_1 - (e_1 - 1)) + 1) + ((e_1 - e_1) + 1)$$
$$= (e_1 - 1) + (e_1 - 2) + ... + 2 + 1$$
$$= \sum_{z=1}^{e_1-1} e_1 - z.$$

It remaining to us the 3rd and the final sort, i.e. $\# M_{e_1(e_1-1)} = (e_1)^2$ •

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