ON CENTRALIZERS OF PRIME GAMMA RINGS

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Abstract

In this paper , we introduce some results in a prime Γ-ring M with centralizer which is related to the quotient Γ-ring Q of M, and prove our main result; Let M be a prime Γ-ring with char M ≠ 2, U a non-zero right ideal of M and T_1 and T_2 two non-zero centralizers of M. If $T_1T_2(U) = (0)$, then there exists two elements p, q of Q such that qΓU = (0) and pΓU = (0).

تمركزات حلقات كاما الاولية

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الخلاصة

في هذا البحث قدمنا بعض النتائج عن حلقات كاما الاولية M مع داله التمركز ، والتي ترتبط بدالـة القسمة Q على حلقة كاما M وبرهنا نتيجتنا الرئيسية والتي تنص: لتكن M حلقة كاما الاولية ممثلها ≠ ٢، U مثالي يميني غير صفري في M،ولتكن T_1وT_2 تمركات غير صفرية علـى M، اذا كـان T_1 T_2 (U)=0 فأن يوجد عنصرين q, p من Q بحيث ان (0) = qΓU و (0) = pΓU

1.Introduction

Nobusawa [1] introduced the notion of a Γ-ring, an object more general than a ring. Barnes [2] slightly weakened the conditions in the definition of Γ-ring in the sense of Nobusawa. Öztürk et al. [3,4] studied extended centroid of prime Γ-rings. In this paper, we consider the main results as follows.1) Let M be a prime Γ-ring of characteristic 2, U a non-zero ideal of M, and T_1 and T_2 two non-zero centralizers of M. If $T_1T_2(U) = (0)$, there exists $\lambda \in C_\Gamma$ such that $T_2 = \lambda\alpha T_1$ for all $\alpha \in \Gamma$ where C_Γ is the extended centroid of M. (2) Let M be a prime Γ-ring, U a non-zero right ideal of M and T a non-zero centralizer of M. If $T(U)\Gamma a = (0)$ where a is a fixed element of M, then there exists an element q of Q such that $q\alpha a = 0$ and $q\beta u = 0$ for all $u \in U$ and $\alpha,\beta \in \Gamma$. (3) Let M be a prime Γ-ring with char $M \neq 2$, U a non-zero right ideal of M and T_1 and T_2 two non-zero centralizers of M. If $T_1T_2(U) = (0)$, then there exists two elements p, q of Q such that $q\Gamma U = (0)$ and $p\Gamma U = (0)$.

1. Preliminaries

Let M and Γ be (additive) abelian groups. If for all a, b, c $\in$ M and $\alpha, \beta \in \Gamma$ the conditions:

(1) $a\alpha b \in M$,

(2) $(a + b)\alpha c = a\alpha b + a\alpha c$,

$a(\alpha + \beta)b = a\alpha b + a\beta b$,

$a\alpha(b + c) = a\alpha b + a\alpha c$

(3) $(a\alpha b)\beta c = a\alpha(b\beta c)$.

are satisfied, then we call M a Γ-ring. Let M be a Γ-ring. The subset

$Z=\{x \in M \mid x\gamma m = m\gamma x \text{ for all } m\in M \text{ and } \gamma\in \Gamma\}$

is called the *center* of M. By a *right* (resp. *left*) *ideal* of a Γ-ring M we mean an additive subgroup U of M such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). If U is both a right and a left *ideal*, then we say that U is an *ideal* of M. For each a of a Γ-ring M the smallest right ideal *containing* a is called the *principal right ideal generated* by a and is denoted by $\langle a\rangle_r$. Similarly we define $\langle a\rangle_l$ (resp. $\langle a\rangle$),the *principal left* (resp. *two sided*) *ideal generated* by a. An ideal P of a Γ-ring M is said to be *prime* if for any ideals U and V of M, $U\Gamma V \subseteq P$ implies $U \subseteq P$ or $V \subseteq P$. A Γ-ring M is said to be *prime* if the zero ideal is prime.

Theorem 2.1 ([5, Theorem 4]). If M is a Γ-ring, the following conditions are equivalent:

(i) M is a prime Γ-ring.

(ii) If a, b $\in$ M and $a\Gamma M\Gamma b = (0)$, then a = 0 or b = 0.

(iii) If $\langle a\rangle$ and $\langle b\rangle$ are principal ideals of M such that $\langle a\rangle\Gamma\langle b\rangle = (0)$, then a = 0 or b = 0.

(iv) If U and V are right ideals of M such that $U\Gamma V = (0)$, then U = (0) or V = (0).

(v) If U and V are left ideals of M such that $U\Gamma V = (0)$, then U = (0) or V = (0).

Let M be a prime Γ-ring such that $M\Gamma M \neq M$. Denote

$M := \{(U, f) \mid U(\neq 0)$ is an ideal of M and $f : U\rightarrow M$ is a right M-module homomorphism$\}$.

Define a relation ~ on M by

$(U, f) \sim (V, g) \Leftrightarrow \exists W(\neq 0) \subset U \cap V$ such that f = g on W.

Since M is a prime Γ-ring, it is possible to find a non-zero W and so "~" is an equivalence relation. This gives a chance for us to get a partition of M. We then denote the equivalence class by $Cl(U, f) = \hat{f}$, where

$\hat{f} := \{g : V \rightarrow M \mid (U, f) \sim (V, g)\}$,

and denote by Q the set of all equivalence classes. Then Q is a Γ-ring, which is called the quotient Γ-ring of M (see [4]). The set

$C_\Gamma := \{g \in Q \mid g\gamma f = f\gamma g \text{ for all } f \in Q \text{ and } \gamma \in \Gamma\}$

is called the extended centroid of M (See [4]).

Lemma 2.2. Let M be a prime Γ-ring, U a non-zero ideal of M, and T a centralizer of M. If $a\Gamma T(U) = (0)$ $(T(U)\Gamma a = (0))$ for all $a \in M$, then a = 0 or T = 0.

Proof: clear by the primness condition on M.

Theorem 2.3 ([6, Theorem 3.5]). The Γ-ring Q satisfies the following properties:

(i) For any element $q \in Q$, there exists an ideal $U_q \in F$ such that $q(U_q) \subseteq M$ (or $q\gamma U_q \subseteq M$ for all $\gamma \in \Gamma$).

(ii) If $q \in Q$ and $q(U) = (0)$ for some $U \in F$ (or $q\gamma U_q = (0)$ for some $U\in F$ and for all $\gamma\in\Gamma$), then q = 0.

(iii) If $U \in F$ and $\Psi : U \rightarrow M$ is a right M-module homomorphism, then there exists an element $q \in Q$ such that $\Psi(u) = q(u)$ for all $u \in U$ (or $\Psi(u) = q\gamma u$ for all $u \in U$ and $\gamma \in \Gamma$).

(iv) Let W be a submodule (an(M,M)-subbimodule[7]) in Q and $\Psi : W \rightarrow Q$ a right M-module homomorphism. If W contains the ideal U of the Γ-ring M such that $\Psi(U) \subseteq M$

and AnnU = $Ann_r W$, then there is an element q $\in$ Q such that $\Psi(b) = q(b)$ for any b $\in$ W (or $\Psi(b) = q\gamma b$ for any b $\in$ W and $\gamma \in \Gamma$) and q(a) = 0 for any a $\in$ $Ann_r W$ (or $q\gamma a = 0$ for any a $\in$ $Ann_r W$ and $\gamma \in \Gamma$).

Let M be a Γ-ring. A map T : M → M is called a centralizer if

$T(x+y)=T(x)+T(y)$ and $T(x\gamma y)=T(x)\gamma y = x\gamma T(y)$ for all x, y $\in$ M and $\gamma \in \Gamma$.

Lemma 2.4 :A) Let M be a 2-torsion free prime Γ-ring, T_1 and T_2 the symmetric centralizers of M. If

$T_1(x)\gamma T_2(y) = T_2(x)\gamma T_1(y)$ (2.1)

for all x, y $\in$M and $\gamma \in\Gamma$ and $T_1 \neq 0$, then there exists $\lambda\in C_\Gamma$ such that $T_2(x)= \lambda\alpha T_1(x)$ for $\alpha\in\Gamma$, where C_Γ is the extended centroid of M.

B) Let M be a 2-torsion free prime Γ-ring, T_1, T_2, T_3 and T_4 the symmetric centralizers of M. If

$T_1(x)\gamma T_2(y)=T_3(x)\gamma T_4(y)$ (2.2)

for all x, y$\in$M and $\gamma\in\Gamma$ and $T_1\neq 0\neq T_4$,then there exists $\lambda\in C_\Gamma$ such that $T_2(x)= \lambda\alpha T_4(x)$ and $T_3(x) = \lambda\alpha T_1(x)$ for $\alpha \in \Gamma$ where C_Γ is the extended centroid of M

Proof (A):- Let x, y, z $\in$ M and $\beta, \gamma \in \Gamma$. Substituting y + z for y in (2.1), we have

$T_1(x)\gamma DT_2(y, z) =T_2(x)\gamma T_1(y, z)$. (2.3)

Replacing z by zβy in (2.3), we have

$T_1(x)\gamma z\beta T_2(y) = T_2(x)\gamma z\beta T_1(y)$. (2.4)

Now if we replace y by x in (2.4), then

$T_1(x)\gamma z\beta T_2(x) = T_2(x)\gamma z\beta T_1(x)$. (2.5)

If $T_1(x) \neq 0$ then $T_2(x) = \lambda(x)\alpha T_1(x)$ for all $\alpha\in\Gamma$ and for some $\lambda(x)\in C_\Gamma$. Thus if $T_1(x) \neq 0 \neq T_1(y)$, then it follows from (2.4) that

$(\lambda(y) - \lambda(x))\alpha T_1(x)\gamma z\beta T_1(y) = 0$.
(2.6)

Since M is a prime Γ-ring, by using Lemma 2.2 we conclude that $\lambda(x) = \lambda(y)$. Hence we have proved that there exists $\lambda \in C_\Gamma$ such that $T_2(x) = \lambda\alpha T_1(x)$ for all $\alpha \in \Gamma$ and x $\in$ M with $T_1(x) \neq 0$. On the other hand, if $T_1(x) = 0$ then $T_2(x) = 0$ as well. Therefore $T_2(x) = \lambda\alpha T_1 Z$

Proof (B): Let x, y, z, w $\in$M and $\alpha, \beta, \gamma \in \Gamma$. Replacing y by y + z in (2.2), we get

$T_1(x)\gamma T_2(y+ z) = T_3(x)\gamma T_4(y+ z)$. (2.7)

If we substitute zβx for z in (2.7), then

$T_1(x)\gamma z\beta T_2(y) = T_3(x)\gamma z\beta T_4(y)$. (2.8)

Substituting $z\alpha T_4(w)$ for z in (2.8), we have

$T_1(x)\gamma z\alpha T_4(w)\beta T_2(y)=T_3(x)\gamma z\alpha T_4(w)\beta T_4(y)$.
(2.9)

By (2.8), we know that

$T_1(x)\gamma z\alpha T_2(w) = T_3(x)\gamma z\alpha T_4(w)$ and so

$T_1(x)\gamma z\alpha(T_4(w)\beta T_2(y) - T_2(w)\beta T_4(y)) = 0$

which implies that $T_4(w)\beta T_2(y) = T_2(w)\beta T_4(y)$ since $T_1 \neq 0$ and M is a prime Γ-ring. It follows from $T_4\neq 0$ and Lemma 3.6 that $T_2(y)= \lambda\alpha T_4(y)$ for some $\lambda \in C_\Gamma$. Hence, by (2.8), we conclude that

$(\lambda\alpha T_1(x) - T_3(x))\gamma z\beta T_4(y) = 0$,

and so $T_3(x) = \lambda\alpha T_1(x)$. This completes the proof. □

Lemma 2.5 :([8, Lemma 1]). Let M be a prime Γ-ring and Z the center of M.

1. If a, b, c $\in$ M and $\beta, \gamma \in \Gamma$, then
$[a\gamma b, c]_\beta = a\gamma[b, c]_\beta + [a, c]_\beta\gamma b + a\gamma(c\beta b) - a\beta(c\gamma b)$
where $[a, b]_\gamma$ is $a\gamma b - b\gamma a$ for all a, b $\in$ M and $\gamma \in \Gamma$.
2. If a $\in$ Z, then $[a\gamma b, c]_\beta = a\gamma[b, c]_\beta$ where $[a,b]_\gamma$ is $a\gamma b - b\gamma a$ for all a, b $\in$ M and $\gamma \in \Gamma$.

Lemma 2.6 ([9, Lemma 2]). Let M be a prime Γ-ring, U a non-zero right (resp. left) ideal of M and a $\in$ M. If UΓa = (0) (resp. aΓU = (0)), then a = 0.

2. Main results

In what follows, let M denote a prime Γ-ring such that MΓM ≠ M, Z is the center of M, C_Γ is the extended centroid of M and $[a, b]_\gamma = a\gamma b - b\gamma a$ for all a, b $\in$ M and $\gamma \in \Gamma$.

Lemma 3.1. Let M be a prime Γ-ring of characteristic 2. Let T_1 and T_2 two non-zero centralizers of M and right M-module homeomorphisms. If

$T_1T_2(x) = 0$ for all x $\in$ M, (3.1)

then there exists $\lambda \in C_\Gamma$ such that $T_2(x) = \lambda\alpha T_1(x)$ for all $\alpha \in \Gamma$ and x $\in$ M.

Proof. Let x, y $\in$ M and $\alpha \in \Gamma$. Replacing x by xγy in (3.1), it follows from charM=2 that for all x, y $\in$ M and $\gamma \in \Gamma$

$T_1(x)\gamma T_2(y) = 0$. (3.2)

Replacing y by $T_1(y)$ in (3.2), we get

$T_1(x)\gamma T_2(T_1(y)) = 0$. (3.3)

for all x, y $\in$ M and $\gamma \in \Gamma$. Now, if we replace y by zγy in (3.3), then we obtain

$T_1(x)\gamma T_2(y)\, \gamma\, T_1(z) = 0$. (3.4)

for all x $\in$ M and $\gamma \in \Gamma$.Now replace y by zβy in (3.4),then we obtain

$T_1(x)\gamma z\beta T_2(y)\gamma T_1(z) = 0$

Then

$T_2(y)\gamma T_1(z)=0$ (3.5)
for all y,z $\in$ M and $\gamma \in \Gamma$.since M is a prime Γ-ring, then from(3.2)and(3.5)we obtain
$T_1(x)\gamma T_2(y) = T_2(y)\gamma T_1(x)$ (3.6)
If $T_1(x) \neq 0$, then there exists $\lambda(x) \in C_\Gamma$ such that $T_2(x) = \lambda(x)\alpha T_1(x)$ for all x $\in$ M and $\alpha \in \Gamma$ by Lemma 2.4. Thus, if $T_1(x) \neq 0 \neq T_1(y)$, then (3.3) implies that
$(\lambda(y) - \lambda(x))\alpha T_1(x)\beta z\gamma T_2(x) = 0.$ (3.7)
Since M is a prime Γ-ring, we conclude by using Lemma 2.2 that $\lambda(y) = \lambda(x)$ for all x, y $\in$ M. Hence we proved that there exists $\lambda \in C_\Gamma$ such that $T_2(x) = \lambda\alpha T_1(x)$ for all x $\in$ M and $\alpha \in \Gamma$ with $T_1(x) \neq 0$. On the other hand, if $T_1(x) = 0$, then $T_2(x) = 0$ as well. Therefore, $T_2(x) = \lambda\alpha T_1(x)$ for all x $\in$ M and $\alpha \in \Gamma$. This completes the proof. □

Proposition 3.2. Let M be a prime Γ-ring of characteristic 2 and T a non-zero centralizer of M. If $T(x) \in Z$ for all x $\in$ M, (3.8)
then there exists $\lambda(m) \in C_\Gamma$ such that $T(m) = \lambda(m)\alpha T(z)$ for all m, z $\in$ M and $\alpha \in \Gamma$ or M is commutative.

Proof. From (3.8), we have
$[T(x), y]_\beta = 0$ for all x, y $\in$ M and $\beta \in \Gamma$. (3.9)
Replacing x by xγz in (3.9), it follows from Lemma 2.5 that
$T(x)\gamma[z, y]_\beta = 0$ (3.10)
for all x, y, z $\in$ M and $\gamma, \beta \in \Gamma$. Replacing x by T(x) in (3.10), we obtain
$T^2(x)\gamma[z, y]_\beta =0$ (3.11)
for all x, y, z $\in$ M and $\gamma,\beta \in \Gamma$.
Now, substituting zαm for z in (3.11) it follows from char M = 2 that
$T^2(x)\alpha m\gamma[z, y]_\beta = 0.$ (3.12)
for all x, y, z, m $\in$ M and $\gamma, \beta, \alpha \in \Gamma$. Since M is a prime Γ-ring, we obtain
$T^2(x) = 0 \ \forall x \in M$ or $[z, y]_\beta = 0$ (3.13)
$\forall$z,y $\in$ M and $\forall\beta \in \Gamma$.
From (3.13), if $T^2(x) = 0$ for all x $\in$ M, then replacing x by xγm in this last relation, it follows from $T(x) \in Z$ that
$T(x)\gamma T(m)=T(m)\gamma T(x).$ (3.14)
for all x,m$\in$M and$\gamma\in \Gamma$.
Replacing x by xαn in (3.14), it follows from (3.8) that for all x, m, n $\in$ M and$\gamma,\alpha\in \Gamma$
$T(x)\alpha n\gamma T(m) = T(m)\alpha n\gamma T(x).$ (3.15)
If $T(x) \neq 0$, then there exists $\lambda(m) \in C_\Gamma$ such that $T(m) = \lambda(m)\alpha T(x)$ for all z, m $\in$ M and $\alpha \in \Gamma$ by Lemma 2.4. On the other hand, it follows from (3.13) that if $[z, y]_\beta = 0$ for all z, y $\in$ M and $\beta \in \Gamma$, then M is commutative. This completes the proof. □

Theorem 3.3. Let M be a prime Γ-ring of characteristic 2, T_1 and T_2 two non-zero centralizers of M and U a non-zero ideal of M. If
$T_1T_2(u) = 0$ for all u $\in$ U (3.16)
then there exists $\lambda\in C_\Gamma$ such that $T_2(x)=\lambda\alpha T_1(x)$ for all $\alpha \in \Gamma$ and x $\in$ M.

Proof. Let u, v $\in$ U and $\gamma \in \Gamma$. Replacing u by $T_2(u)\gamma v$ in (3.16), we get
$T_1T_2 (T_2(u)\gamma v) = 0$, (3.17)
for all u, v $\in$ U and $\gamma \in \Gamma$.Then $T^2_2(u)\gamma T_1(v) = 0$
Since $T_1 \neq 0$, it follows from Lemma 2.2 that $T^2_2 (u) = 0$ for all u $\in$ U, so from char M = 2 that $T^2_2 = 0$. Now, substituting u$\gamma T_2(x)$ for u in (3.16), we get
$T_1(T_2(u\gamma T_2(x)))=0,$ (3.18)
for all u$\in$U,x$\in$M and $\gamma \in \Gamma$.
Then $T_2(u)\gamma(T_1(T_2(x))) = 0$
Since $T_2 \neq 0$, we get $T_1(T_2(x)) = 0$ for all x$\in$ M by Lemma 2.2.Hence there exists $\lambda\in C_\Gamma$ such that $T_2 = \lambda\alpha T_1$ for all $\alpha \in \Gamma$ by Lemma 3.1.

Theorem 3.4. Let M be a prime Γ-ring, U a non-zero right ideal of M and T a non-zero centralizer of M. If
$T(u)\gamma a = 0$ for all u $\in$ U and $\gamma \in \Gamma$ (3.19)
Where a is a fixed element of M, then there exists an element q of Q such that qαa=0 and qβu = 0 for all u $\in$ U and $\gamma \in \Gamma$

Proof. Let u $\in$ U, x $\in$ M and $\beta \in \Gamma$. Since U is a right ideal of M, we have uβx $\in$ U. Replacing u by uβx in (3.19), we get
$T(u\beta x)\gamma a = 0$ for all u $\in$ U, x$\in$M and $\gamma,\beta\in\Gamma$,then $T(u)\beta x\gamma a=0$,Hence $T(u)\beta x\gamma a\alpha m=0$ for any m$\in$M and $\alpha\in\Gamma$,and so $T(u)\beta(\sum x\gamma a\alpha m)=0$ .Therefore, for any u$\in$V = MΓaΓM which is a non-zero ideal of M, we have
$T(u)\beta f(v)= 0$ (3.20)
for all u$\in$U. f(v) is independent of u but it isdependent on v. Since M is a prime Γ-ring,f(v) is well-defined and uni que for all v$\in$V. Note that $T(u)\beta f(v)\alpha y = 0,$ (3.21)

For any y$\in$M, and $\alpha\in\Gamma$. Now since vαy$\in$V for any y$\in$M, v$\in$V. Replacing v by vαy in (3.20) we get
$T(u)\beta f(v\alpha y) = 0$ for all y$\in$M, (3.22)
and so by using (3.21) and (3.22), we have

T(u)β(f(vαy)- f(v)αy) =0.which implies from Lemma 2.6 that

f(vαy) = f(v)αy , (3.23)

for all y∈M,v∈V and α∈Γ.It follows from (3.23) that f : V → M is a right M-module homomorphism. In this case, q=Cl(V, f)∈Q. Moreover, f(v) = qβv for all v ∈ V and α ∈ Γ by Theorem 2.3. Let x ∈ M, v ∈ V, u ∈ U and γ, β ∈ Γ. Replacing v by xγv in (3.20), we get T(u)β f(xγv) = 0,and

T(u)β qβxγv= 0 (3.24)

Also, replacing u by uγx in (3.20), we get T(uγx)βf(v) = 0, we get T(u)γxβf(v)=0 , and

T(u)γxβqβv = (3.25)

Now, replacing β by γ and replacing γ by β in (3.25), we get

T(u) βxγqγv =0. (3.26)

Thus, from (3.24) and (3.26) we obtain

T(u)β(xγq - qβx)γv = 0 . (3.27)

for all x∈M,v∈V,u∈U and γ,β ∈ Γ.then by primness of Γ-ring we get T(u)β(xγq-qβx)=0 for all x ∈ M, u∈U and γ, β∈Γ,thus T(u)βxγq −T(u)βqβx =0,for all x ∈ M and γ, β∈Γ, since T is centralizer then uβT(x)γq − uβqβT(x) =0,replace x by uβx in last equation we get uβT(uβx)γq − uβqβT(uβx) =0, then we have uβuβT(x)γq − uβqβuβT(x) =0,and so since M is prime Γ-ring we get uβT(x)γq − qβuβT(x) =0, then we have uβT(x)γqαa − qβuβT(x) αa =0,then we get

uβT(x)γqαa = qβuβT(x)αa by Lemma 2.6. Now, we shall prove that q can be chosen in Q such that qαa = 0 and qβu = 0 for all u ∈ U and γ ∈ Γ. If qαa = 0, then qβuβT(x) αa = 0, then qβu =0 and so since M is prime Γ-ring, we get qΓU = (0). On the other hand, if qαa ≠ 0, then qβu≠ 0. In fact, if qβu = 0, then qαa = 0 since uβT(x)γqαa = qβuβT(x)αa. Thus, we may suppose that qαa ≠ 0 and qβu ≠ 0 for all u ∈ U and α,β ∈ Γ. In this case, we get

uβT(x)γqαa = qβuβT(x)αa

for all x ∈ M, u ∈ U and γ, β, α ∈ Γ. It follows from Lemma 2.4 that there exists $\lambda \in C_\Gamma$ such that qαa = λδa and qβu = λδu for all u ∈ U and γ, δ, α,β ∈ Γ. Hence, if q′=q−λ, then q′Γa = 0 and q′ΓU = (0). This completes the proof.

Theorem 3.5. Let M be a prime Γ-ring with char M ≠ 2, U a non-zero right ideal of M and T a non-zero centralizer of M. Then the subring of M generated by T(U) contains no non-zero right ideals of M if and only if T(U)ΓU = (0).

Proof. Let A be the subring generated by T(U). Let S = A ∩ U, u ∈ U, s∈S and γ∈Γ. Then T(sγu) = T(s) γu ∈ A, and so we have T(s) γu ∈ S. Thus T(S)ΓU is a right ideal of M. In this case, T(S)ΓU = (0) by hypothesis. T(uγa) =uγT(a) ∈ S. Therefore, T(uγT(a))βu = 0, then T(u)T(a)u=0. Since M is a prime Γ-ring then

T(u)T(a) =0 (3.28)

Replacing u by uβv where v ∈ U, β ∈ Γ in (3.28),we get, for all u, v ∈U, β, γ∈Γ and a∈ A

T(u)βvγT(a) = 0. (3.29)

Since M is a prime Γ-ring, we get T(U)ΓU = (0) or T(A)ΓU = (0). If T(A)ΓU = (0), then T^2(U)ΓU = (0), so T^2(U)=0. Let u, v ∈ U and β ∈ Γ. Then

0 = T^2(uβv)) =T(T(uβv))=T(u)βT(v),for all u, v ∈ U and β ∈ Γ by char M ≠ 2. Replacing u by uγw where w ∈ U, γ ∈ Γ in last relation, we have T(u)γwβT(v) = 0 which yields T(u)γv = 0 for all u, v ∈ U and γ ∈ Γ.

Conversely assume that T(U)ΓU = (0). Then AΓU = (0). Since M is a prime Γ-ring, A contains no non-zero right ideals.

Theorem 3.6. Let M be a prime Γ-ring with char M ≠ 2, U a non-zero right ideal of M and T_1 and T_2 two non-zero centralizers of M. If T_1T_2(U) = (0), then there exists two elements p, q of Q such that qΓU = (0) and pΓU = (0).

Proof. If T_1T_2(U) = (0), then T_1(A) = (0) where A is a subring generated by T_2(U). Since $T_1 \neq 0$, A contains no non-zero right ideals of M. Thus, from Theorem 3.5, we have T_2(u)γv = 0 for all u, v ∈ U and γ ∈ Γ. Also, there exists q ∈ Q such that qΓU = (0) by Theorem 3.4. Therefore T_2(uγv) = uγT_2(v) for all u, v ∈ U and γ ∈ Γ. In this case, 0 = T_1T_2(uγv) = T_1(uγT_2(v)) = T_1(u)γT_2(v), and since M is a prime Γ-ring, we get T_2(u)γv = 0 for all u, v ∈ U and γ ∈ Γ. Again, by Theorem 3.4, there exists p ∈ Q such that pΓU = (0). This completes the proof.

Remark 3.7. Consider the following example. Let R be a ring. A centralizer T : R → R is called an inner centralizer if there exists a ∈ R such that T(x) = ax for all x ∈ R. Let S be the 2 × 2 matrix ring over Galois field {0, 1, w, w^2}, with inner centralizer T_1 and T_2 defined by

$$T_1(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} , \qquad T_2(x) = \begin{pmatrix} 0 & wx \\ 0 & 0 \end{pmatrix}$$

for all $x \in S$. Then the characteristic of S is 2 and we have $T_1 \neq 0$, $T_2 \neq 0$, $T_1T_2 = 0$ and $T^2_2=0$. Also, if we take
$M := M_{1\times 2}(S) = \{(a, b) \mid a, b \in S\}$ and $\Gamma := \left\{ \begin{pmatrix} n \\ 0 \end{pmatrix} \mid \text{n is an integer} \right\}$, then M is a prime Γ-ring of characteristic 2. Define an additive map $H_1:M \to M$ by $H_1(x, y) = (T_1(x), T_1(y))$.Sinc $(x,y)\begin{pmatrix} n \\ 0 \end{pmatrix}(a,b) = (nxa, nxb)$, therefore T_1 is a centralizer on M. Similarly $H_2 : M \to M$ given by $H_2(x, y) = (T_2(x), T_2(y))$ is a centralizer. In this case, we have $H_1 \neq 0$, $H_2 \neq 0$, $H_1H_2 = 0$ and $H^2_2=0$ (see [9]). Thus we know that there exist two centralizers H_1, H_2 of M such that $H_1H_2(M) = (0)$ but $H_1(M)\Gamma M \neq (0)$ and $H_2(M)\Gamma M \neq (0)$. Therefore the condition of char $M \neq 2$ in Theorem 3.5 and 3.6 is necessary.

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