MODULES WITH CHAIN CONDITIONS ON SEMISMALL SUBMODULES

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Abstract

Let R be an associative ring with identity and M be unital non zero R-module. A submodule N of a module M is called a semismall submodule of M (briefly $N \ll_S M$) if N = 0 or for each nonzeror submodule K of N, $N / K \ll M / K$. In this work,we study this kind of submodule of M and the modules which is satisfies the ascending chain condition (a. c. c.) and descending chain condition (d. c. c.) on semismall submodules .Then we generalize the Rad(M) into s- Rad(M) ,It is equale to the sum of all semismall submodule of M . We show that if N not semismall submodule of M.Then s-Rad (N) = N∩s-Rad (M)and we discuss some of the basic properties of this types of submodules.

المقاسات التي تحقق خاصية السلسة للمقاسات الجزئية شبه الصغيرة

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الخلاصة

لتكن R حلقة تجميعية ذات عنصر محايد وليكن M مقاسا احاديا غير صفري ايمن معرفـا علـى R ألمقاس ألجزئيN من M يقال بأنه شبه صغير أذا كان N = 0 أولكل مقاس جزئي غير صـفري K مـن N يكون M / K >> N / K في هذا البحث سنقوم بدراسة هذا النوع من المقاسات الجزئية والمقاسات التي تحقق خاصيتي السلسلة على المقاسات الجزئية الشبه صغيرة . كذالك قمنا بتعميم مفهؤم جذر جاكوبسون بالاعتمـاد على هذا المفهوم الى مفهوم(s- Rad(M بأنه يساوي مجموع كل المقاسات الجزئية الشبه صغيرة .فقد بينـا أذا كان مقاس جزئي ليس شبه صغير فأن(s-Rad (N) = N∩s-Rad (M مع دراسة الخواص الرئيسية لهـذا النـوع مـن الجـذور

Introduction

Let R be an associative ring with identity and M is a non zero unital right R-module. A submodule N of a module M is called a small submodule of M , denoted by $N \ll M$, if $N + L \neq M$ for any proper submodule L of M .
In [2] Inaam and Layla introuced the definition of the concept of semismall submodule that A submodule N of a module M is called a semismall submodule of M (briefly $N \ll_S M$) if N = 0 or for each nonzeror submodule K of N, $N / K \ll M / K$.In section one, we review the concept of semismall submodule and we discuss some of the basic properties of this types of submodules.
In this section ,we introduce the definition of module which is satisfies the ascending chain condition (a. c. c.) and descending chain condition (d. c. c.) on semismall submodules as a

generalization of chain condition (a. c. c.) and descending chain condition (d. c. c.) on small submodules [3] and we study the relation btween the ring that satisfies (a. c. c.) and descending chain condition (d. c. c.) on semismall ideals.

It is known that Rad(M) is the sum of all small submodules of M. In section Two we generaliz Rad(M) into s- Rad(M) as the sum of all semismall submodules of M ,

s-Rad(M) =∑{L≤ M ,L semismall submodule of M}.and we will study this class of submodules.

Let N and L be submodules of a module M. N is called a supplement of L in M if M=N +L and N ∩ L<<M.[4]

In [2] If N and L be submodules of a module M.N is called a semi-supplement of L in M if M=N +L and N ∩ L$<<_S$M.

Then it is clear that every supplement submodule is semi-supplement but the converse is not true.

We show that if N not semismall or semi-supplement submodule of M.Then s-Rad (N) = N∩s-Rad (M).

1.Modules with chain conditions on semismall submodules

In this section , we review the concept of semismall submodule and we discuss some of the basic properties of this types of submodules.

Definition (1.1): [2] . A submodule N of a module M is called a semismall submodule of M (briefly N $<<_S$M) if N = 0 or for each nonzeror submodule K of N, N / K << M / K.

It is clear that every small submodule is semismall submodule,but the converse is not true in general for example In Z-module Z_{12} the submodulo N =< $\bar{4}$ > is not small but it is semismall [2] .

Lemma 1.2: [2] let M be an R- module then

1- If N$<<_S$ M and K <N then K$<<_S$ M
2- If K,N are submodules of M such that K $<<_S$ N then K$<<_S$ M
3- If K$<<_S$ M and f : M→N is a homomorphism then f (K) $<<_S$N .
4- Let K <N < M .If K$<<_S$ M and N direct summand of M ,then K$<<_S$ N.
5- Let M = $M_1 \oplus M_2$ and let K<M such that K=$K_1 \oplus K_2$ then $K_1 <<_S M_1$ and $K_2 <<_S M_2$.
6- Let N <K if N$<<_S$ M and K is direct summand then N$<<_S$k.

The following is characterization of semismall submodule.

Proposition(1.3): [2] A submodule N of a module M is semismall submodule of M (N $<<_S$)M iff N +L =M for someL≤ M implies that K+L=M for all K≤ N,K≠ (0).

An R-module M is said to satisfy the ascending chain condition (a.c.c.) on small submodules. respectively descending chin condition (d.c.c.) on small submodules if every ascending (descending) chain of small submodules $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_{n \ldots}$ respectively $K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq$ ..Terminates[3].

In the following we introduce the definition of module which satisfies the ascending chain condition (a.c.c.) and descending chain condition (d. c. c.) on semismall submodules

Definition (1.4): An R-module M is said to satisfy the ascending chain condition (a.c.c.) on semismall submodules. respectively descending chin condition (d.c.c.) on semismall submodules if every ascending (descending) chain of semismall submodules $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_{n \ldots}$ respectively $K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq$ ..Terminates.

Since every small submodule is semismall then the following remark is clear.

Remark (1.5): If M satisfy the a.c.c.(d.c.c.) on semismall submodules then M satisfy the a.c.c.(d.c.c.) on small submodules.

Proposition (1.6): Let M_1 and M_2 be two modules.Then $M_1 \oplus M_2$ satisfies a.c.c.(d.c.c.) on semismall submodules iff M_1 and M_2 satisfies a.c.c.(d.c.c.) on semismall submodules.

Proof : Is clear by lemma (1.2) .

Remark (1.7): An ideal I of a ring R is semismall ideal if we consider R as R –module.

The following proposition is appered in [2] , here we give it with another proof.

Proposition (1.8): Let M be a finitely generated faithful multiplication R- module, and let N= M I, for some ideal I of R then N is semismall submodule in M iff I is semismall ideal in R .

Proof : Assume N is semismall in M , and N = M I , let I + J = R , for some J of R . then M I + M J = M R = M . since N is semismall in M , then M J+K= M, K≤N,K K≠ (0).
Since M multiplication R- module , then K=MT , for some ideal T of R ,T ≤ I , therfore M J+MT= M then J +T = R hence I is semismall in R.
Conversely , let N + K = M , for some submodule K of M . since M is a multiplication R- module , then K = M J, for some ideal J of R , [5]
Hence N + K = M I + M J =M (I + J) = M But M is finitely generated faithful multiplication R-module . then I + J = R, since I is semismall in R , then T + J = R, T ≤ I , T≠ (0). and hence MT + M J = M R = M .let A = MT thusA≤Nand A+K=M, A≠ (0).then N is semismall submodule in M.
The following results are sequences of this proposition.

Corollary (1.9): Let M be a finitely generated faithful multiplication R-module, then R satisfies a. c. c. on semismall ideal if and only if M satisfies a. c. c. on semismall submodules .

Proof : Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots$ be an ascending chain of semismall submodule of M . Since M is a multiplication R-module, then $N_i = I_i M$, for some ideal I_i of R for all i [6] . Hence $M I_1 \subseteq M I_2 \subseteq M I_3 \subseteq \ldots \subseteq M I_k \subseteq \ldots$ But M is finitely generated then by proposition (1.8) $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots$ is an ascending chain of semismall ideals in R . Since R satisfies a.c.c on semismall ideal , then $\exists$ K $\in$ N , such that $I_k = I_{k+1} = \ldots$, hence $M I_k = M I_{k+1} = \ldots$which implies $N_k = N_{k+1} = \ldots$, that is M satisfies a. c. c. on semismall submodules. Conversely , let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq \ldots$ be an ascending chain of small ideals in R , then by proposition (1.8) $M I_1 \subseteq M I_2 \subseteq M I_3 \subseteq \ldots \subseteq M I_k \subseteq \ldots$ is an ascending chain of semismall submodule of M .
Since M satisfies a. c. c. on semismall submodules then then $\exists$ K $\in$ N , , such that $M I_k = M I_{k+1} = \ldots$ But M is a finitely generated faithful module , then $I_k = I_{k+1} = ..$ [6] . Thus R satisfies a. c. c. on semismall ideals of R .

By a similar proof of cor . 1.9 ,we have .

Corollary(1.10): Let M be a finitely generated faithful multiplication R-module, then M satisfies d.c. c. on semismal1 submodules, if and only if R satisfies d. c. c. on semismall ideal.
Hence we have the following result :

Theorem(1.11) : Let M be a finitely generated faithful multiplication R-module, then the following are equivalent.
1) M satisfies a.c.c (d.c.c) on semismall submodules
2) R satisfies a.c.c (d.c.c) on semismall ideals .
3) S =End_R(M) satisfies a.c.c (d.c.c) on semismall ideals .
4) M satisfies a.c.c (d.c.c) on semismall submodules as S- module.

Proof : (1) $\Rightarrow$ (2) By cor (1.6)
(2) $\Rightarrow$ (3)since M is a finitely generated faithful multiplication R-module, then $R \approx S$ hence R satisfies a.c.c(d.c.c) S =End_R(M) satisfies a.c.c (d.c.c) on semismall ideals .
(3) $\Rightarrow$ (4) By cor (1.9)
(4) $\Rightarrow$ (1) By cor (1.9) S satisfies a.c.c (d.c.c) on semismall ideals . $R \approx S$[6] hence R satisfies a.c.c(d.c.c) on semismall ideals and by cor (1.9) M satisfies a.c.c (d.c.c) on semismall submodules

2. Semi radical of R-module M.

It is known that Rad(M) is the sum of all small submodules of M . In this section we introduce the semi- Rad(M) as a generalization of Rad(M).
s-Rad(M) =$\sum$\{L≤ M ,L semismall submodule of M\}
Since every small submodule is semismall submodules then it is clear that Rad(M)≤s-Rad(M). The converse is not true,for example .In Z-module Z_{12} ,the submodules,$<\bar{4}>$and $<\bar{6}>$ are semismall submodule , Rad(Z_{12}) =$<\bar{6}>$,s-Rad(Z_{12})=$<\bar{2}>$.
The following lemmas give some properties of The s-Rad(M).

Lemma(2.1):
1-Let M be R-modul and let m $\in$ M if $Rm \ll_S M$ then , m $\in$ s-Rad(M).
2-If m $\in$ s-Rad(M) then ther existe submodule nR≤mR such that $nR \ll_S M$.

3- Let R be a ring and let $\varphi : M \to M'$ be a homomorphism of R- modules M , M′. Then φ(s-Rad(M)) ≤ s-Rad(M′).
4- Let $M_{i(i\in I)}$ be any collection of R-modules and let $M = \oplus_{i\in I}M_i$. Then s-Rad(M) = $\oplus_{i\in I}$ s- Rad(M_i).
5-M(s-Rad(R)) ≤ s-Rad(M),R as an R-module.

Proof :

1) mR$<<_S$ M then , m ∈mR≤ s-Rad(M).

2) m ∈ s-Rad(M) then m=$\sum n_i$,n_i ∈ $N_k <<_S$ M ,thus by(1.2) $Rn_i <<_S M\ \forall$ i .

3)From s- Rad(M) =∑{L≤ M ,L semismall submodule of M } .it follows that φ(s-Rad(M))= $\sum_{L\le M}$ φ(L),by Lemma 1.2 , φ(L) $<<_S$ M′. thus φ(s-Rad(M)) ≤ s-Rad(M′).

4) From (2) Then s- Rad(M_i) ≤ s-Rad(M) then $\sum_{i\in I}$s-Rad(M_i)=$\oplus_{i\in I}$s-Rad(M_i)≤s-Rad(M) .Let now m= $\sum m_i$ ∈ s-Rad(M) and let β_i :M→ M_i be the ith projection then β_i(m)= m_i ∈s-Rad(M_i) by(2).And so m ∈ ⊕s-Rad(M_i) hence s-Rad(M) ≤ ⊕s-Rad(M_i).

5) Let m ∈M ,then φ_m :R→ M defined by φ_m(r)= mr is a homomorphisim thus by (3) we have m(s-Rad(R))=φ(s-Rad(R)) ≤ s-Rad(M) then, $\sum_{m\in M}$ m(s-Rad(R)= M(s-Rad(R)) ≤ s-Rad(M).

Remark (2.2): Let M be an R-module then s-Rad(M)=M iff all finitly generated submodules are semismall submodule of M .

Proof : Suppose s-Rad(M)=M and N a f.g submodules of M with N+L=M then
$N=Rx_1+ Rx_2+\ldots\ldots.+ Rx_n$,x_i ∈ M = s-Rad(M) then $Rx_i <<_S$ M thus K_i+L=M ∀ i
K_i ≤ N , K≠ (0).
Conversely , let m ∈ M then Rm is finitly generated then Rm$<<_S$ M , m ∈ s-Rad(M).

Remark (2.3): Let M be R-modul if s-Rad(M) $<<_S$ M then M / s-Rad(M). has no non-zero semismall submodule .

Proof : Let L /s-Rad(M) be semismall submodule of M /s-Rad(M), then L$<<_S$ M To show that ,let L+K=M then (L/s-Rad(M)+((K+ s-Rad(M) /s-Rad(M))=M /s-Rad(M) then
A+ K+ s-Rad(M) =M ,A≤ L, A≠ (0).Then L$<<_S$M thus L≤ s-Rad(M) therefore
L= s-Rad(M).Then L/s-Rad(M) is zero semismall submodule .

Proposition (2.4): Let M be a module then s-Rad(M) is Noetherian if and only if M satisfies a. c. c. on semismall modules .

Proof :(⇒) is Clear.
(⇐) suppose that s-Rad(M) is not Noetherian .Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq\ldots.\subseteq N_k \subseteq\ldots.$ be an infinte ascending chain of semismall submodule of s-Rad(M) .Let a_1 ∈ $N_1$and a_j ∈ N_j - N_{J-1} for each j>1 . For any K≥1,let $N_k=\sum_{j=1}^{k} a_j R$.Then N_k is finitely generated and N_k ≤ s-Rad(M) then $N_k <<_S$ s-Rad(M) by Lemm(1.2) and $N_1 \subseteq N_2 \subseteq N_3 \subseteq\ldots\ldots$ thus M not satisfy a. c. c. on semismall modules.
In[2] the notion of semi-supplement was introduced , and show that every supplement submodule is semi-supplement submodule.

Definition (2.5): [2] Let N and L be submodules of a module M.N is called a semi-supplement of L in M ifM=N +L and N ∩ L$<<_S$M.

Proposition(2.6): [2]Let N be submodules of a module M consider the following statementes;
1) N is semi-supplement submodule in M .
2)N is not semismall submodule in M.
3)For all x ≤ N,X$<<_S$M then x$<<_S$ N.Then (1)⇒(2)⇒(3) and(3)⇒(1) if M is weakly supplemented.

Proposition(2.7): Let N be not semismall submodule of M.Then s-Rad (N) = N∩s-Rad (M).

Proof: By lemma 2.1 it is clear that
s-Rad(N) ≤ N ∩ s-Rad(M) . Now, we have to show that N ∩ s-Rad(M) ≤ s-Rad(N) . Let x ∈ N ∩ s-Rad(M) then x ∈ N implies that xR ≤ N and x∈s- Rad(M) then by lemma (2.1) ther existe submodule Rn≤Rm such that Rn$<<_S$M then by prop.(2.6) Rn$<<_S$N Therefore x ∈s-ad(N) i.e. N ∩ s-Rad(M) ≤ s-Rad(N) .Thus s-Rad(N) = N ∩ s-Rad(M) .

Corollary(2.8): Let M be an R-module and let N be semi-supplement submodule then
Then s-Rad (N) = N∩s-Rad(M).

Proof: By prop.(2.6) and prop.(2.7).

Corollary(2.9): Let M be an R-module and let N be direct summand of M Then
s-Rad (N) = N∩s-Rad(M).

Proof: Sinnce every direct summand of M is supplement submodule[7] and every supplement submodule is semi-supplement submodule then by corollary(2.8), s-Rad (N) = N∩s-Rad(M).

Lemma (2.10): Let M be a module such that s-Rad(M)$\leq_e$ M. Let M_1 and M_2 be direct summands of M with $M_1 \leq M_2$ then
s-Rad(M_1) = s-Rad(M_2)if and only if $M_1 = M_2$

Proof: Let M= $M_1 \oplus L$.Then by moduler low
$M_2 = M_1 \oplus (M_2 \cap L)$ and by (prop. 2.1) s-Rad(M_2)= s-Rad(M_1) ⊕ s-Rad ($M_2 \cap L$). Thes if s-Rad(M_1) = s-Rad(M_2) then
0=s-Rad(M_2∩L)=s-Rad(M)∩(M_2∩L)by (cor.2.9) Thus,since s-Rad(M)$\leq_e$M. we get $M_2 \cap L=0$ and hence $M_1 = M_2$.

Refrences

1. Kasch,F.**1982** , Modules and Rings , Academic Press Ins, London .

2. Hadi,I and Mohmood,L.**2009**.,semismall submodules and semilifting modules,proceeding of 3rd scientific conference of the college of science ,univ.of Baghdad, p . 385-393.
3. A.G. Naoum,A. and I.M.Hadi **1996** Module with ascending(descending) chain conditions on small submodules , Iragi Journal of science ,v(**37)**,no.3, p . 1085-1102 .
4. Wisbauer,R.**1991** Foundations of Module and Ring Theory, Gordon and Breach, Reading.
5. El.Bast Z.A. and Smith,P.F.**1988**,multiplication modules ,Comm. In Algebra ,v(**16**) 755-779.
6. Naoum A.G**.1984,** on the ring of finitily generated multiplication modules,Periodica Math. Hungarica ,29,277-284.