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ON THE SPECTRUM OF SPECIAL CLASSES OF DOUBLY COMMUTING $n-$TUPLES OF OPERATORS

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Abstract

In this paper we study a spectral characterization of the Taylor-Browder spectrum for a double commuting $n-$tuple of totally $\theta-$operators, and study relation between the Taylor- Weyl spectrum, Taylar- Browder spectrum, joint-Weyl spectrum, and joint-Browder spectrum for commuting $n-$tuple of totally $\theta-$operators. Also we study a spectral characterization of the Taylor-Browder spectrum for a doubly commuting $n-$tuple of posinormal operators.

حول طيف الأصناف الخاصة لـ n من المؤثرات مضاعفة التبديل

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الخلاصة

في هذا البحث درس التمثيل الطيفي لطيف تيلر-براودر و ندرس العلاقة بين طيف تيلر- ويل و طيف تيلر- براودر وطيف ويل الرابط وطيف براودر الرابط لسلسلة محدودة ل n من المؤثرات من النمط $\theta-$ كليا الذي يكون تبادلها مضاعف وكذلك ندرس التمثيل الطيفي من طيف تيلر- براودر لسلسلة محدودة ل n من المؤثرات موجبة السوية الذي يكون تبادلها مضاعف

1. Introduction

Let $B(H)$ denote the algebra of all bounded linear operators on an infinite complex Hilbert space H. Recall **[1]** that an operator $T \in B(H)$ is said to be ***dominant*** if for each $\lambda \in C$ there exists a positive number M_λ such that

$(T-\lambda)(T-\lambda)^* \le M_\lambda (T-\lambda)^*(T-\lambda)$

If the constants M_λ are bounded by a positive number M, then T is said to be $M-$***hyponormal***. Also if $M=1$, then T is ***hyponormal***. It is well known that

Hyponormal operators $\Rightarrow$ $M-$*hyponormal operators* $\Rightarrow$ *Dominant operators*

An operator $T \in B(H)$ is called $\theta-$***operator*** if T^*T commutes with $T+T^*$, **[2]**.We say that an operator $T \in B(H)$ is ***totally*** $\theta-$***operator*** if $T-\lambda$ is $\theta-$operator for all $\lambda \in C$. Then we can notice that

Totally $\theta-$*operators* $\Rightarrow$ $\theta-$*operators*

It is well known, [3] that

$\theta-$*operators* $\Rightarrow$ *Dominant operators*

And [4] gave an example of a $M-$hyponormal which is not $\theta-$operator. An operator $T \in B(H)$ is called ***posinormal*** if there exists a positive operator $P \in B(H)$ such that $TT^* = T^*PT$, [5].

From, [5], it is well known that
Dominant operators $\Rightarrow$ *Posinormal operators*
Let T have the polar decomposition $T = U|T|$, where U is unitary and $|T| = (T^*T)^{\frac{1}{2}}$ and let $\tilde{T} = |T|^{1/2} U |T|^{1/2}$. If T is posinormal, then $\tilde{T}$ is hyponormal **([6])**.
Throughout this paper we let $\mathrm{T} = (T_1, T_2, \ldots, T_n)$ denote a commuting (that is $T_iT_j = T_jT_i$ for all $i, j = 1,2,\ldots,n$) n-tuple of operators on H. and denote $\mathrm{T}^* = (T_1^*, T_2^*, \ldots, T_n^*)$, $\tilde{\mathrm{T}} = (\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n)$. If $T_iT_j = T_jT_i$ and $T_i^*T_j = T_jT_i^*$ for every $i \neq j$, then $\mathrm{T} = (T_1, T_2, \ldots, T_n)$ is said to be a ***doubly commuting*** n***-tuple***, **[7]**.
Let us recall some basic definitions and notions

Definition 1.1. A ***cochain complex*** is a sequence of abelian groups $\ldots G_{-2}, G_{-1}, G_0, G_1, G_2, \ldots$ connected by boundary operators (homomorphisms) $T_n : G_n \to G_{n+1}$, such that the composition of any two consecutive maps is zero: $T_{n+1} \circ T_n = 0$ for all n:

$$\ldots \longrightarrow G_{n-1} \xrightarrow{T_{n-1}} G_n \xrightarrow{T_n} G_{n+1} \longrightarrow \ldots$$

, the index n in G_n is referred to as the *degree*.See **[8].**
Let $\Lambda[e] = \Lambda_n[e] = A\lg(e_1, e_2, \ldots, e_n)$ be the exterior algebra on n generators, that is, $\Lambda[e]$ is the complex algebra with identity e generated by indeterminates $e_1, e_2, \ldots, e_n$ such that $e_i \wedge e_j = -e_j \wedge e_i$, for all i, j, where $\wedge$ denotes multiplication. $\Lambda[e] = \oplus_{k=-\infty}^{k=\infty} \Lambda^k[e]$,with $\Lambda^k[e] \wedge \Lambda^l[e] \subset \Lambda^{k+l}[e]$. The elements $e_{j_1} \wedge \ldots\ldots \wedge e_{j_k}$, $1 \le j_1 < \ldots < j_k \le n$ form a basis for $\Lambda^k[e]$ $(k > 0)$, while $\Lambda^0[e] = Ce$ and $\Lambda^k[e] = (0)$ when $k > n, k < 0$. Also $\Lambda^n[e] = C(e_1 \wedge \ldots \wedge e_n)$ .Moreover, $\dim \Lambda^k[e] = \binom{n}{k}$, so that, as a vector space over C, $\Lambda^k[e]$ is isomorphic to $C^{\binom{n}{k}}$, **[9].**

Definition 1.2. Let H be a Hilbert space and $\mathrm{T} = (T_1, T_2, \ldots, T_n)$ be a commuting n-tuple of bounded linear operators on H. Let $\Lambda[e]$ be the exterior algebra on n generators , we consider $\Lambda^k(H) = \Lambda^k[e] \otimes H$ anddefine $\Lambda^k(\mathrm{T}) : \Lambda^k(H) \to \Lambda^{k+1}(H)$ for $k = 0,1,\ldots,n-1$ (where $\Lambda^0(H) = \Lambda^n(H) = H$) by $\Lambda^k(\mathrm{T})(x \otimes e_{j_1} \wedge \ldots \wedge e_{j_k}) = \sum_{i=1}^{n} T_i x \otimes e_i \wedge e_{j_1} \wedge \ldots \wedge e_{j_k}$ with these operators we can construct the following sequence

$$0 \longrightarrow \Lambda^0(H) \xrightarrow{\Lambda^0(\mathrm{T})} \Lambda^1(H) \xrightarrow{\Lambda^1(\mathrm{T})} \ldots \xrightarrow{\Lambda^{n-1}(\mathrm{T})} \Lambda^n(H) \longrightarrow 0,$$

[9] show that $\Lambda^{k+1}(\mathrm{T}) \circ \Lambda^k(\mathrm{T}) = 0$ for all k, i.e. that $\mathrm{Im}\Lambda^k(\mathrm{T}) \subseteq Ker\Lambda^{k+1}(\mathrm{T})$ for all k. So that $\{\Lambda^k(\mathrm{T}), \Lambda^k(H)\}_{k \in z}$ is a cochain complex, called the ***Koszul complex*** for $\mathrm{T} = (T_1, T_2, \ldots, T_n)$ and denoted $K(\mathrm{T}, H)$. Furthermore, all the operators $\Lambda^k(\mathrm{T})$ are bounded linear operators, [9].
Let's review definitions of joint spectra of a commuting $n-$tuple $\mathrm{T} = (T_1, T_2, \ldots, T_n)$ of operators in $B(H)$.

Definition 1.3. Let $\mathrm{T} = (T_1, T_2, \ldots, T_n)$ be a commuting n-tuple of bounded linear operators on H.

(1) $\mathrm{T} = (T_1, T_2, \ldots, T_n)$ is said to be ***Taylor invertible*** if the Koszul complex for T $K(\mathrm{T}, H)$ is exact, that is if $\mathrm{Im}\Lambda^k(\mathrm{T}) = Ker\Lambda^{k+1}(\mathrm{T})$ for $k = 0,1,\ldots,n-1$, **[**10].

(2) $\mathrm{T} = (T_1, T_2, \ldots, T_n)$ is said to be ***Taylor Fredholm*** if the all cohomologies of the Koszul complex for T $K(\mathrm{T}, H)$ are finite dimensional, that is if $Ker\Lambda^{k+1}(\mathrm{T}) / \mathrm{Im}^k \Lambda(\mathrm{T})$ for $k = 0,1,\ldots,n-1$, [10].

In this case the ***index*** of $\mathrm{T}-\lambda$, denoted by $ind(\mathrm{T})$, is defined as the Euler characteristic of $K(\mathrm{T},H)$, i.e., as the alternating sum of dimensions of all cohomology spaces of $K(\mathrm{T},H)$

$$ind(\mathrm{T}-\lambda)=\sum_{k=0}^{n-1}(-1)^k \dim(\ker\Lambda^{k+1}(\mathrm{T})/\operatorname{Im}\Lambda^k(\mathrm{T}))$$

, ([10], [11]).

(3) ***The Taylor spectrum,*** $\sigma_T(\mathrm{T})$, of T is defined by

$\sigma_T(\mathrm{T})=\{\lambda=(\lambda_1,...,\lambda_n)\in C^n:\mathrm{T}-\lambda$ is not invertible$\}$, **[10].**

(4) ***The Taylor essential spectrum***, $\sigma_{Te}(\mathrm{T})$ of T is defined as follows

$\sigma_{Te}(\mathrm{T})=\{\lambda=(\lambda_1,...,\lambda_n)\in C^n:\mathrm{T}-\lambda$ is not Fredholm$\}$, **[10].**

(5) $\mathrm{T}=(T_1,T_2,...,T_n)$ is said to be ***Taylor Weyl*** if T is (Taylor) Fredholm and $ind(\mathrm{T})=0$, **[10].**

(6) ***The Taylor-Weyl spectrum***, denoted by $\sigma_w^1(\mathrm{T})$, of T is defined by

$\sigma_w^1(\mathrm{T})=\sigma_{Te}(\mathrm{T})\bigcup\{\lambda\in C^n: ind(\mathrm{T}-\lambda)\neq 0\}$, **[10].**

(7) $\mathrm{T}=(T_1,T_2,...,T_n)$ is said to be ***Taylor Browder*** if T is Fredholm and there exists a deleted open neighborhood N_0 of $0\in C^n$ such that $\mathrm{T}-\lambda$ is invertible for all $\lambda\in N_0$, **([10], [12]).**

(8) ***The Taylor-Browder spectrum***, denoted by $\sigma_b^1(\mathrm{T})$, is defined by

$\sigma_b^1(\mathrm{T})=\sigma_{Te}(\mathrm{T})\bigcup acc\sigma_T(\mathrm{T})$

where $acc\sigma_T(\mathrm{T})$ denotes the set of ***accumulation points*** of the Taylor spectrum of T, **([10], [12]).**

Definition 1.4. Let $K(H)$ denote the set of all compact operators acting on H and let $\mathrm{K}=(K_1,...,K_n)\in K(H)^n$ denote an $n-$tuple of compact operators.

(1) ***The joint Weyl spectrum***, denoted by $\sigma_w^2(\mathrm{T})$, is defined by

$\sigma_w^2(\mathrm{T})=\bigcap_{\mathrm{K}\in K(H)^n}\{\sigma_T(\mathrm{T}+\mathrm{K})\}$, **[13]**.

(2) ***The joint Browder spectrum***, denoted by $\sigma_b^2(\mathrm{T})$, is defined by

$\sigma_b^2(\mathrm{T})=\bigcap_{\mathrm{K}\in K(H)^n}\{\sigma_T(\mathrm{T}\uplus\mathbf{K})\}$

where $\mathbf{T}\uplus\mathbf{K}$ means a commuting sum such that $\mathrm{T}+\mathrm{K}$ with $T_iK_j=K_jT_i$ for all i,j, [14].

Lay, D. C. **[15]** and Schechter, M. **[**16] proved the following results.

Theorem 1.5. *If* $T\in B(H)$ *is an arbitrary single operator, then*

$\sigma_w^1(T)=\sigma_w^2(T)\subset\sigma_b^2(T)=\sigma_b^2(T)$.

Theorem 1.6. *If* $T\in B(H)$ *is a normal operator*

$\sigma_w^1(T)=\sigma_w^2(T)=\sigma_b^1(T)=\sigma_b^2(T)$.

The situation for an $n-$tuple of operators is different in general. Kim, J. C. [14] proved the following results.

Theorem 1.7. *If* T *is a commuting* $n-$*tuple* T *of arbitrary operators on* H*, then*

$\sigma_w^1(\mathrm{T})\subset\sigma_w^2(\mathrm{T})\subset\sigma_b^2(\mathrm{T})\subset\sigma_b^1(\mathrm{T})$.

Theorem 1.8. *If* T *is a commuting* $n-$*tuple* T *of normal operators*

$\sigma_w^1(\mathrm{T})=\sigma_w^2(\mathrm{T})=\sigma_b^2(\mathrm{T})=\sigma_b^1(\mathrm{T})$.

We also review the definitions **[**17] of joint spectra of a commuting $n-$tuple $\mathrm{T}=(T_1,T_2,...,T_n)$ of operators in $B(H)$.

Definition 1.7.

(1) $\lambda=(\lambda_1,...,\lambda_n)\in C^n$ is called a ***joint eigenvalue*** of T if there exists a non- zero vector x such that

$(T_i-\lambda_i)x=0$ for all $i=1,...n$.

(2) ***The joint point spectrum***, denoted by $\sigma_p(\mathrm{T})$, of T is the set of all joint eigenvalues of T.

Let $\pi_0(\mathrm{T})$ denote the set of all joint eigenvalues of T of finite multiplicity and $\pi_{00}(\mathrm{T})$ denote the set of isolated eigenvalues of finite multiplicity.

Kim, J. C**.** [14**]** given a spectral characterization of $\sigma_b^1(\mathrm{T})$.

Theorem 1.8. *If* T *is a commuting* $n-$*tuple* T *of* $M-$*hyponormal operators, then*
$\sigma_b^1(\mathrm{T}) = \sigma_T(\mathrm{T}) \setminus \pi_{00}(\mathrm{T})$.

Let $D(z,r)$ is the open disc of center z and radius in the complex plane, then an open polydisc is a set of the form

$$D(z_1,r_1) \times ... \times D(z_n,r_n)$$

$(i.e., \{\lambda = (\lambda_1,...,\lambda_n) \in C^n : |z_k - \lambda_k| < r_k$, for all $k = 1,...,n\}$, [18].

Definition 1.9. A commuting $n-$tuple $\mathrm{T} = (T_1, T_2, ..., T_n)$ has the ***single valued extension property***, say **SVEP**, if for any open polydisk $D \subset C^n$, the Koszul complex $K(\mathrm{T}-\lambda, O(D,H))$ has vanishing homology in positive degrees (i.e., is exact in positive degrees). Here $O(D,H)$ denotes the Frechet space of $H-$valued analytic functions on D, **[19]**.

In **[20]**, Y. Y. Lee proved that

Theorem 1.10. *If* T *is a commuting* $n-$*tuple* T *of* $M-$*hyponormal operators* with SVEP

$$\sigma_w^1(\mathrm{T}) = \sigma_w^2(\mathrm{T}) = \sigma_b^2(\mathrm{T}) = \sigma_b^1(\mathrm{T}).$$

In this paper, we show that for a doubly commuting $n-$tuple T of totally $\theta-$operators in $B(H)$

$$\sigma_b^1(\mathrm{T}) = \sigma_T(\mathrm{T}) \setminus \pi_{00}(\mathrm{T})$$

and for a doubly commuting $n-$tuple of T totally $\theta-$operators in $B(H)$ with SVEP

$$\sigma_w^1(\mathrm{T}) = \sigma_w^2(\mathrm{T}) = \sigma_b^1(\mathrm{T}) = \sigma_b^2(\mathrm{T})$$

Also these results are proved for a doubly commuting $n-$tuple T of posinormal operators in $\mathcal{U}$, where $\mathcal{U}$ denote the class of operators $T \in B(H)$ that the partial isometry U in the polar decomposition $T = U|T|$ is unitary. Also we prove that

$$\sigma_b^1(\mathrm{T}) \setminus [0] = \{\sigma_T(\mathrm{T}) \setminus \pi_{00}(\mathrm{T})\} \setminus [0]$$

where $[0] = \{(\lambda_1,.......,\lambda_n) \in C^n : \lambda_i = 0$ for at least one $i \in I = \{1,.....,n\}\}$.

for doubly commuting $n-$tuple T of posinormal operators in $B(H)$.

2.Main Results

Recall **[**21**]** that the ***left (right) joint spectrum***, denoted by $\sigma_\ell(\mathrm{T})$ $(\sigma_r(\mathrm{T}))$, of T is defined by the set of all points $\lambda = (\lambda_1,...,\lambda_n) \in C^n$ such that $\{T_i - \lambda_i\}_{1 \le i \le n}$ generates a proper left (right) ideal in the algebra $B(H)$. Let $C(H) = B(H)/K(H)$ be the Calkin algebra with the canonical map $\pi : B(H) \to C(H)$. Then the ***left (right) joint essential spectrum***, denoted by $\sigma_{\ell e}(\mathrm{T})$ $(\sigma_{re}(\mathrm{T}))$, of T is defined by

$\sigma_{\ell e}(\mathrm{T}) = \sigma_\ell(\pi(\mathrm{T}))$ $(\sigma_{re}(\mathrm{T}) = \sigma_r(\pi(\mathrm{T})))$,

where $\pi(\mathrm{T}) = (\pi(T_1),...,\pi(T_n))$.

Following [20] we shall write $p_{00}(\mathrm{T}) := iso\sigma_T(\mathrm{T}) \setminus \sigma_{Te}(\mathrm{T})$ for the ***(joint) Riesz points*** of $\sigma_T(\mathbf{T})$. That is the set $p_{00}(\mathrm{T})$ consists of all isolated points that the associated spectral space is finite dimensional.

If M is a common invariant subspace of H for each $T_i \in B(H)$, then we let $\mathrm{T}|_M = (T_1|_M, T_2|_M, ..., T_n|_M)$ denote an $n-$tuple of compressions to M.

The following theorem was established by Kim, J. C. [14] for the case in which T is a doubly commuting $n-$tuple $M-$hyponormal. Here we replace the $M-$hyponormality assumption by totally $\theta-$operators.

Theorem 2.1. *Let* T *be a doubly commuting* $n-$*tuple totally* $\theta-$*operators. Then*
$\sigma_b^1(\mathrm{T}) = \sigma_T(\mathrm{T}) \setminus \pi_{00}(\mathrm{T})$

Proof. We first prove that $iso\sigma_T(\mathrm{T}) \subset \sigma_p(\mathrm{T})$, since **T** is a doubly commuting of $n-$tuple totally $\theta-$operators, then $\mathrm{T}-\lambda$ is a doubly commuting of $n-$tuple $\theta-$operators for all $\lambda = (\lambda_1,...,\lambda_n) \in C^n$. So, without loss of generality, we may assume that $0 \in iso\sigma_T(\mathrm{T})$**.** Then there exists a non-zero projection $P \in B(H)$ **[15, Corollary 4.10]** such that P commutes with T_i for all i, $\sigma_T(\mathbf{T}|_{PH}) = \{0\}$ and $0 \notin \sigma_T(\mathbf{T}|_{(I-P)H})$ with respect to the decomposition $H = PH \oplus (I-P)H$, thus each

T_i is quasi-nilpotent $\theta-$operator on PH , T_i is a zero operator on PH by **[22]**. Thus $PH \subset \ker T_i$ for all $i=1,...,n$, and so

$$\{0\} \neq PH \subset \bigcap_{i=1}^{n} \ker T_i$$

Hence $0 \in \sigma_p(\mathrm{T})$. Therefore $\sigma_p(\mathrm{T})^c \subset acc\sigma_T(\mathrm{T})$. On the other hand, since T is a doubly commuting of $n-$tuple totally $\theta-$operators, by **Lemma 2.1 in [23]** and **Theorem 2.8 in [21]** we have

$\sigma_T(\mathrm{T}) = \sigma_r(\mathrm{T})$ and $\sigma_{Te}(\mathrm{T}) = \sigma_{re}(\mathrm{T})$

and by **Theorem 2.10 in [21]**

$$\sigma_T(\mathrm{T}) = \sigma_{Te}(\mathrm{T}) \cup \overline{\pi_0(\mathrm{T}^*)}$$

Now since $\pi_0(\mathrm{T}) = \sigma_{Te}(\mathrm{T})^c \cap \sigma_p(\mathrm{T})$, and $\pi_{oo}(\mathrm{T}) = \pi_o(\mathrm{T}) \cap iso\sigma_T(\mathrm{T})$, then

$$\sigma_T(\mathrm{T}) \setminus \pi_{00}(\mathrm{T}) = \sigma_T(\mathrm{T}) \cap (\pi_0(\mathrm{T}) \cap iso\sigma_T(\mathrm{T}))^c$$
$$= \sigma_T(\mathrm{T}) \cap (\sigma_{Te}(\mathrm{T}) \cup \sigma_p(\mathrm{T})^c \cup acc\sigma_T(\mathrm{T}))$$
$$= \sigma_T(\mathrm{T}) \cap ((\sigma_{Te}(\mathrm{T}) \cup acc\sigma_T(\mathrm{T}))$$
$$= \sigma_{Te}(\mathrm{T}) \cup acc\sigma_T(\mathrm{T})$$
$$= \sigma_b^1(\mathrm{T}).$$ □

Theorem 2.2. *Let* T *be a doubly commuting* $n-$*tuple of totally* $\theta-$*operators with the SVEP. Then*

$$\sigma_w^1(\mathrm{T}) = \sigma_w^2(\mathrm{T}) = \sigma_b^2(\mathrm{T}) = \sigma_b^1(\mathrm{T})$$

Proof. It suffices show that

$$\sigma_w^1(\mathrm{T}) = \sigma_b^1(\mathrm{T})$$

We claim that

$$p_{00}(\mathrm{T}) = iso\sigma_T(\mathrm{T}) \setminus \sigma_w^1(\mathrm{T}) = \sigma_T(\mathrm{T}) \setminus \sigma_b^1(\mathrm{T})$$

Suppose $\lambda \in iso\sigma_T(\mathrm{T}) \setminus \sigma_w^1(\mathrm{T})$, then $\mathrm{T}-\lambda$ is Fredholm if and only if the spectral space corresponding to λ is finite dimensional by **[24]**. So $\lambda \in \pi_{00}(\mathrm{T})$. Since T is a doubly commuting of $n-$tuple of totally $\theta-$operators, then $\sigma_b^1(\mathrm{T}) = \sigma_T(\mathrm{T}) \setminus \pi_{00}(\mathrm{T})$ by **Theorem 2.1**. and hence $\lambda \notin \sigma_b^1(\mathrm{T})$. So $\lambda \in iso\sigma_T(\mathrm{T}) \setminus \sigma_b^1(\mathrm{T})$.

Conversly, suppose $\lambda \in \sigma_T(\mathrm{T}) \setminus \sigma_b^1(\mathrm{T})$, then $\lambda \in \sigma_T(\mathrm{T})$ and $\lambda \notin \sigma_b^1(\mathrm{T})$, since $\sigma_w^1(\mathrm{T}) \subset \sigma_b^1(\mathrm{T})$ by **Theorem 1.7**. we have $\lambda \notin \sigma_w^1(\mathrm{T})$. On the other hand since $\lambda \notin \sigma_b^1(\mathrm{T}) = \sigma_T(\mathrm{T}) \setminus \pi_{00}(\mathrm{T})$, then $\lambda \in \pi_{00}(\mathrm{T})$ and hence $\lambda \in iso\sigma_T(\mathrm{T})$. Thus $\lambda \in iso\sigma_T(\mathrm{T}) \setminus \sigma_w^1(\mathrm{T})$. Therefore

$$iso\sigma_T(\mathrm{T}) \setminus \sigma_w(\mathrm{T}) = \sigma_T(\mathrm{T}) \setminus \sigma_b^1(\mathrm{T})$$

Now suppose $\lambda \in p_{oo}(\mathrm{T})$, then $\lambda \in iso\sigma_T(\mathrm{T})$ but $\lambda \notin \sigma_{Te}(\mathrm{T})$. Since $\lambda \in iso\sigma_T(\mathrm{T})$, then $\lambda \in \sigma_T(\mathrm{T})$ but $\lambda \notin acc\sigma_T(\mathrm{T})$ and hence $\lambda \notin \sigma_b^1(\mathrm{T})$. So $\lambda \in \sigma_T(\mathrm{T}) \setminus \sigma_b^1(\mathrm{T})$.

Conversely, let $\lambda \in \sigma_T(\mathrm{T}) \setminus \sigma_b^1(\mathrm{T})$, then $\lambda \notin \sigma_b^1(\mathrm{T})$ implies that $\lambda \notin acc\sigma_T(\mathrm{T})$ and $\lambda \notin \sigma_{Te}(\mathrm{T})$. Therefore $\lambda \in iso\sigma_T(\mathrm{T})$, so $\lambda \in iso\sigma_T(\mathrm{T}) \setminus \sigma_{Te}^1(\mathrm{T}) = p_{00}(\mathrm{T})$. Thus

$$p_{00}(\mathrm{T}) = iso\sigma_T(\mathrm{T}) \setminus \sigma_w^1(\mathrm{T}) = \sigma_T(\mathrm{T}) \setminus \sigma_b^1(\mathrm{T})$$

On the other hand, since T has the SVEP from [25] we have

$$p_{00}(\mathrm{T}) = \sigma_T(\mathrm{T}) \setminus \sigma_w^1(\mathrm{T})$$

implies that

$$\sigma_T(\mathrm{T}) \setminus \sigma_w^1(\mathrm{T}) = \sigma_T(\mathrm{T}) \setminus \sigma_b^1(\mathrm{T})$$

So now from Theorem 1.7.

$$\sigma_w^1(\mathrm{T}) \subset \sigma_w^2(\mathrm{T}) \subset \sigma_b^2(\mathrm{T}) \subset \sigma_b^1(\mathrm{T})$$

We have

$$\sigma_w^1(\mathrm{T}) = \sigma_w^2(\mathrm{T}) = \sigma_b^2(\mathrm{T}) = \sigma_b^1(\mathrm{T}).$$ □

Let $\mathcal{U}$ denote the class of operators $T \in B(H)$ that the partial isometry U in the polar decomposition $T = U|T|$ is unitary.

Theorem 2.3. *Let* T *be a doubly commuting* $n-$*tuple of posinormal operators in* $\mathcal{U}$. *Then*

$$\sigma_b^1(\mathrm{T}) = \sigma_T(\mathrm{T}) \setminus \pi_{00}(\mathrm{T})$$

Proof. Since T is a doubly commuting $n-$tuple of posinormal operators. Then $\tilde{\mathrm{T}} = (\tilde{T}_1,.......,\tilde{T}_n)$ is a doubly commuting $n-$tuple of hyponormal operators, by **Theorem 2.8 in [21]** we have

$\sigma_T(\tilde{\mathrm{T}}) = \sigma_r(\tilde{\mathrm{T}})$ and $\sigma_{Te}(\tilde{\mathrm{T}}) = \sigma_{re}(\tilde{\mathrm{T}})$

and by **Theorem 2.10 in [21]**

$$\sigma_r(\tilde{\mathrm{T}}) = \sigma_{re}(\tilde{\mathrm{T}}) \cup \overline{\pi_0(\tilde{\mathrm{T}}^*)}$$

From **Theorem 1 in [26]** we have

$\sigma_T(\mathrm{T}) = \sigma_T(\widetilde{\mathrm{T}})$, where $\widetilde{\mathrm{T}} = (\widetilde{T}_1, \ldots\ldots, \widetilde{T}_n)$

By **Theorem 3 and Theorem 6 in [27]** we have

$$\pi_0(\widetilde{\mathrm{T}}) = \sigma_{Te}(\widetilde{\mathrm{T}})^c \cap \sigma_p(\widetilde{\mathrm{T}}) = \sigma_{Te}(\mathrm{T})^c \cap \sigma_p(\mathrm{T}) = \pi_0(\mathrm{T})$$

$$\sigma_b^1(\widetilde{\mathrm{T}}) = \sigma_{Te}(\widetilde{\mathrm{T}}) \cup acc\sigma_T(\widetilde{\mathrm{T}}) = \sigma_{Te}(\mathrm{T}) \cup acc\sigma_T(\mathrm{T}) = \sigma_b^1(\mathrm{T})$$

$$\pi_{oo}(\widetilde{\mathrm{T}}) = \pi_o(\widetilde{\mathrm{T}}) \cap iso\sigma_T(\widetilde{\mathrm{T}}) = \pi_o(\mathrm{T}) \cap iso\sigma_T(\mathrm{T}) = \pi_{oo}(\mathrm{T})$$

Since $\widetilde{\mathrm{T}} = (\widetilde{T}_1, \ldots\ldots, \widetilde{T}_n)$ is a doubly commuting $n-$tuple of hyponormal operators, by **Theorem 3 in [14]** we have

$$\sigma_b^1(\widetilde{\mathrm{T}}) = \sigma_T(\widetilde{\mathrm{T}}) \setminus \pi_{00}(\widetilde{\mathrm{T}})$$

Thus

$$\sigma_b^1(\mathrm{T}) = \sigma_b^1(\widetilde{\mathrm{T}}) = \sigma_T(\widetilde{\mathrm{T}}) \setminus \pi_{00}(\widetilde{\mathrm{T}}) = \sigma_T(\mathrm{T}) \setminus \pi_{00}(\mathrm{T})$$

□

For operators in $B(H)$ we have an improvement of **Theorem 2.3** as follows

Theorem 2.4. *Let* T *be a doubly commuting* $n-$*tuple of posinormal operators in* $B(H)$. *Then*

$$\sigma_b^1(\mathrm{T}) \setminus [0] = \{\sigma_T(\mathrm{T}) \setminus \pi_{00}(\mathrm{T})\} \setminus [0]$$

where $[0] = \{(\lambda_1, \ldots\ldots, \lambda_n) \in C^n : \lambda_i = 0$ *for at least one* $i \in I = \{1, \ldots\ldots, n\}\}$.

Proof. Using the same argue of previous then we can proof

$\sigma_T(\widetilde{\mathrm{T}}) = \sigma_r(\widetilde{\mathrm{T}})$, $\sigma_{Te}(\widetilde{\mathrm{T}}) = \sigma_{re}(\widetilde{\mathrm{T}})$

and $\sigma_T(\mathrm{T}) = \sigma_T(\widetilde{\mathrm{T}})$, where $\widetilde{\mathrm{T}} = (\widetilde{T}_1, \ldots\ldots, \widetilde{T}_n)$

By **Theorem 3 and Corollary 5 in [27]** we have

$$\pi_0(\widetilde{\mathrm{T}}) \setminus [0] = \{\sigma_{Te}(\widetilde{\mathrm{T}})^c \cap \sigma_p(\widetilde{\mathrm{T}})\} \setminus [0] = \{\sigma_{Te}(\mathrm{T})^c \cap \sigma_p(\mathrm{T})\} \setminus [0] = \pi_0(\mathrm{T}) \setminus [0]$$

$$\sigma_b^1(\widetilde{\mathrm{T}}) \setminus [0] = \{\sigma_{Te}(\widetilde{\mathrm{T}}) \cup acc\sigma_T(\widetilde{\mathrm{T}})\} \setminus [0] = \{\sigma_{Te}(\mathrm{T}) \cup acc\sigma_T(\mathrm{T})\} \setminus [0] = \sigma_b^1(\mathrm{T}) \setminus [0]$$

$$\pi_{oo}(\widetilde{\mathrm{T}}) \setminus [0] = \{\pi_o(\widetilde{\mathrm{T}}) \cap iso\sigma_T(\widetilde{\mathrm{T}})\} \setminus [0] = \{\pi_o(\mathrm{T}) \cap iso\sigma_T(\mathrm{T})\} \setminus [0] = \pi_{oo}(\mathrm{T}) \setminus [0]$$

Since $\widetilde{\mathrm{T}} = (\widetilde{T}_1, \ldots\ldots, \widetilde{T}_n)$ is a doubly commuting $n-$tuple of hyponormal operators, by **Theorem 3 in [14]** we have

$$\sigma_b^1(\widetilde{\mathrm{T}}) = \sigma_T(\widetilde{\mathrm{T}}) \setminus \pi_{00}(\widetilde{\mathrm{T}})$$

Thus

$$\sigma_b^1(\mathrm{T}) \setminus [0] = \sigma_b^1(\widetilde{\mathrm{T}}) \setminus [0] = \{\sigma_T(\widetilde{\mathrm{T}}) \setminus \pi_{00}(\widetilde{\mathrm{T}})\} \setminus [0] = \{\sigma_T(\mathrm{T}) \setminus \pi_{00}(\mathrm{T})\} \setminus [0]$$

□

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