COMPARISON OF SOME BAYES' ESTIMATORS FOR THE WEIBULL RELIABILITY FUNCTION UNDER DIFFERENT LOSS FUNCTIONS

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Abstract

Weibull distribution is one of the most widely used distributions of lifetime in reliability engineering. It has the ability to provide reasonably accurate failure analysis and failure forecasts especially with extremely small samples. Bayesian approach has received a lot of attention along with the traditional methods of estimation such as maximum likelihood and moment estimation methods. The object of the present paper is to compare some Bayes' estimators for the scale Parameter λ of the Weibull reliability function using different loss functions, based on Jeffrey prior information for estimating the scale parameter of Weibull distribution. The comparison was based on a Monte Carlo study. Through the simulation study comparison was made on the performance of these estimators with respect to the mean square error (MSE) and the mean percentage error (MPE). The results of comparison by MSE showed that the Weibull reliability function based upon squared error loss function was the best followed by the modified El- Sayyad's loss function. While comparison by MPE showed the reverse.

مقارنة بعض مقدرات بيز لدالة معولية نمزذج ويبل عند دوال خسارة مختلفة

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الخلاصة

يعد توزيع ويبل واحدا من توزيعات البقاء الاكثر استخداما في مجالات المعولية الهندسية. فهـو يزودنا بتقديرات وتنبؤات دقيقة لحالات الفشل حتى عند حجوم العينات الصـغيرة. اسـتحوذ اسـلوب التقدير البيزي اهتماما واسعا الى جانب طرق التقدير التقليدية الاخرى كطريقة الامكان الاعظم وطريقة العزوم. يهدف هذا البحث الى مقارنة بعض مقدرات بيز لمعلمة المقياس λ لدالة المعولية لنموذج ويبل باستخدام دوال خسارة مختلفة والمستندة على دالة جيفري للمعلومات الاولية. جرت الدراسة بواسـطة طريقة مونت كارلو للمحاكاة وذلك بمقارنة اداء تلك المقدرات باستخدام مقياس متوسط مربعات الخطأ (MSE) ومقياس متوسط الخطأ النسبي (MPE). وقد أظهرت نتائج المقارنة ان مقدر بيز لدالة معولية

نموذج ويبل المستخرج وفقا لدالة الخسارة التربيعية كانت الافضل بمقياس *(MSE)* يليها مقدر بيـ ز
المستخرج وفقا لدالة خسارة السيد المعدلة. بينما أظهرت نتائج المقارنة بمقياس *(MPE)* خلاف ذلك

Introduction

In Bayesian analysis the unknown parameter λ is regarded as being the value of a random variable from a given probability distribution, with the knowledge of some information about the value of λ prior to observing the data $t_1, t_2 \ldots t_n$[6].Many literatures have dealt with the aspect of the Bayesian inference of the Weibull distribution [1,2]. The object of the present paper is to obtain Bayesian estimates of the parameter λ of the weibull reliability function, using Jeffrey prior under four loss functions .The comparison was based on a Monte Carlo study. The efficiency of the four estimators was compared according to the mean square error (MSE) and the mean percentage error (MPE).

The Bayesian Approach

Suppose that n items have an independent and identically distributed Weibull lifetimes each having parameters (*p*, λ), Where λ is assumed to be unknown. The probability density function *f (t,p, λ)* of the lifetime is given by[7]:

$$f(t,p,\lambda) = \frac{p}{\lambda}\, t^{p-1}\, e^{-\frac{t^p}{\lambda}} \qquad (1)$$

Then with Jeffrey prior information given by:

$$g(\lambda) = \frac{1}{\lambda^c} \text{ , with c a constant,} \qquad (2)$$

The posterior distribution for the parameter λgiven the data ($t_1, t_2 \ldots t_n$) is:

$$h(\lambda|\boldsymbol{t}) = \frac{f(\boldsymbol{t}/\lambda)\, g(\lambda)}{\int f(\boldsymbol{t}/\lambda)\, g(\lambda)\, d\lambda}$$

Substituting the density function given in (1) and the Jeffrey prior given in (2), the simplification leads us to the following form:

$$h(\lambda|\boldsymbol{t}) = \frac{-\left(\sum_{i=1}^{n} t_i^p\right)^{n+c-1} e^{-\frac{\sum_{i=1}^{n} t_i^p}{\lambda}}}{\lambda^{n+c}\,\Gamma(n+c-1)} \qquad (3)$$

The Weibull Reliability Function

The complementary cumulative distribution function (*CCDF*) is called reliability function. The two parameter Weibull reliability function is given by [4]:

$$R(t;p,\lambda) = e^{-\left(\frac{t^p}{\lambda}\right)}, \qquad (4)$$

and *p* are the scale and shape parameters where λ respectively. We will assume that the shape parameter *p* is known a *priori* from the pastexperience. The advantage of doing this is that data sets with few or no failures can be analyzed. The Bayesian point estimator of the Weibull reliability function is the mean of the posterior distribution of R (t) under some loss functions [5].

Loss Functions

We consider four different loss functions [3]:

(i)Squared-Error Loss Function:

$$L_1(\hat{\lambda},\lambda) = (\hat{\lambda} \quad \lambda)^2$$

(ii) Modified Squared-Error Loss Function:

$$L_2(\hat{\lambda},\lambda) = \left[\lambda^r(\hat{\lambda}-\lambda)^2\right]$$

where *r* is an integer constant.

(iii) El-Sayyad's Loss Function:

$$L_3(\hat{\lambda},\lambda) = \left[\lambda^l(\hat{\lambda}^r-\lambda^r)^2\right]$$

where *l* is a positive integer.

(iv) Modified El-Sayyad's Loss Function:

$$L_4(\ddot{\lambda},\lambda) = \left(\frac{\hat{\lambda}}{\lambda}\right)^r - r\,ln\left(\frac{\hat{\lambda}}{\lambda}\right) - 1$$

Bayes Estimators

According to the above mentioned loss functions, we found the corresponding Bayes' estimators forλ:

(i) With squared error loss function, the corresponding Bayes estimator for λ with posterior distribution (3) comes out as:

$$E[L_1(\hat{\lambda},\lambda)|\boldsymbol{t}] = \int_0^\infty (\hat{\lambda}-\lambda)^2\, h(\lambda|\boldsymbol{t})\, d\lambda$$

Let:

$$\frac{\partial L_1(\ddot{\lambda},\lambda)}{\partial\hat{\lambda}} = 0$$

Then,

$$\hat{\lambda} = E[\lambda|\boldsymbol{t}] = \int_0^\infty \lambda h(\lambda|\boldsymbol{t})\, d\lambda$$

$$\hat{\lambda} = \int_0^\infty \lambda \frac{-\left(\sum_{i=1}^{n} t_i^p\right)^{n+c-1} e^{-\left(\frac{\sum_{i=1}^{n} t_i^p}{\lambda}\right)}}{\lambda^{n+c}\,\Gamma(n+c-1)} d\lambda$$

Let

$$y = \frac{\sum_{i=1}^{n} t_i^p}{\lambda}$$

Then after substitution we find that:

$$\hat{\lambda} = \frac{-\left(\sum_{i=1}^{n} t_i^p\right)^{n+c-1}}{\Gamma(n+c-1)} \int_0^\infty e^{-y} \frac{y^{n+c-1}}{\left(\sum_{i=1}^{n} t_i^p\right)^{n+c-1}} \frac{-\left(\sum_{i=1}^{n} t_i^p\right)}{y^2} dy$$

Hence

$$\hat{\lambda}_1 = \frac{\sum_{i=1}^{n} t_i^p}{n+c-2} \qquad (5)$$

(ii) With modified squared error loss function, the corresponding Bayes estimator for λ with posterior distribution (3) comes out as:

$$E[L_2(\hat{\lambda},\lambda)|t] = \int_0^{\infty} \lambda^r(\hat{\lambda}-\lambda)^2\ h(\lambda|t)\,d\lambda$$

By letting
$\frac{\partial L_2(\hat{\lambda},\lambda)}{\partial \hat{\lambda}} = 0$, we find that:

$$\hat{\lambda} = \frac{E[\lambda^{r+1}|t]}{E[\lambda^r|t]}$$

where, $E[\lambda^r|t] = \int_0^{\infty} \lambda^r h(\lambda|t)\,d\lambda$

By acting in a similar manner as in (i) and after few steps we get:

$$E[\lambda^r|t] = \frac{\left(\sum_{i=1}^{n} t_i^p\right)^r \Gamma(n+c-r-1)}{\Gamma(n+c-1)} \qquad (6)$$

Similarly

$$E[\lambda^{r+1}|t] = \frac{\left(\sum_{i=1}^{n} t_i^p\right)^{r+1} \Gamma(n+c-r-2)}{\Gamma(n+c-1)} \qquad (7)$$

From (6) and (7), we find that:

$$\hat{\lambda}_2 = \frac{\sum_{i=1}^{n} t_i^p}{(n+c-r-2)} \qquad (8)$$

(iii) With El- Sayyad loss function, the corresponding Bayes estimators for λ with posterior distribution (3) come out as:

$$E[L_3(\hat{\lambda},\lambda)|t] = \int_0^{\infty} \lambda^l(\hat{\lambda}^r-\lambda^r)^2\ h(\lambda|t)\,d\lambda$$

Assuming ***l*** = *1* and by letting $\frac{\partial L_3(\hat{\lambda},\lambda)}{\partial \hat{\lambda}} = 0$, we find that:

$$\hat{\lambda} = \left(\frac{E[\lambda^{r+1}|t]}{E[\lambda|t]}\right)^{1/r}$$

from (5) and (7) we find:

$$\hat{\lambda}_3 = \left[\frac{\left(\sum_{i=1}^{n} t_i^p\right)^r \Gamma(n+c-r-2)}{\Gamma(n+c-2)}\right]^{1/r} = \left(\sum_{i=1}^{n} t_i^p\right)\left[\frac{\Gamma(n+c-r-2)}{\Gamma(n+c-2)}\right]^{1/r} \qquad (9)$$

(iv) With modified El- Sayyad loss function, the corresponding Bayes estimator for λ with posterior distribution (3) comes out as:

$$E[L_4(\hat{\lambda},\lambda)|t] = \int_0^{\infty}\left[\left(\frac{\hat{\lambda}}{\lambda}\right)^r - r\ln\left(\frac{\hat{\lambda}}{\lambda}\right) - 1\right] h(\lambda|t)\,d\lambda$$

By letting:

$\frac{\partial L_4(\hat{\lambda},\lambda)}{\partial \hat{\lambda}} = 0$, we find that

$$\hat{\lambda}^r E\left[\frac{1}{\lambda^r}|t\right] = 1$$

where
$E\left[\frac{1}{\lambda^r}|t\right] = \int_0^{\infty} \frac{1}{\lambda^r}\ h(\lambda|t)\,d\lambda$

and by making similar substitution as above we get:

$$E\left[\frac{1}{\lambda^r}|t\right] = \frac{\Gamma(n+c+r-1)}{\left(\sum_{i=1}^{n} t_i^p\right)^r \Gamma(n+c-1)}$$

Hence

$$\hat{\lambda}_4 = \left(\sum_{i=1}^{n} t_i^p\right)\left[\frac{\Gamma(n+c-1)}{\Gamma(n+c+r-1)}\right]^{1/r} \qquad (10)$$

Using the Bayes' estimators $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4$; quasi-Bayes estimators of Weibull reliability function are as follows

$$\hat{R}_1(t) = e^{\left(\frac{t^p}{\hat{\lambda}_1}\right)}$$

$$\hat{R}_2(t) = e^{-\left(\frac{t^p}{\hat{\lambda}_2}\right)}$$

$$\hat{R}_3(t) = e^{-\left(\frac{t^p}{\hat{\lambda}_3}\right)}$$

$$\hat{R}_4(t) = e^{-\left(\frac{t^p}{\hat{\lambda}_4}\right)}$$

Simulation Results

After the Weibull reliability function, $\hat{R}_i(t)$ (i = 1,2,3,4) had been estimated; Mean Square Errors (MSE) and Mean Percentage Error (MPE) were calculated as an index for precision to compare the efficiency of the four estimators, where:

$$MSE[\hat{R}_i(t)] = \frac{\sum_{i=1}^{I}\left(\hat{R}_i(t) - R_i(t)\right)^2}{I}$$

and

$$MPE[\hat{R}_i(t)] = \frac{\sum_{i=1}^{I}\frac{|\hat{R}_i(t) - R_i(t)|}{R_i(t)}}{I}$$

where I is the number of replications. We generated I = 2000 samples of sizes n = 10, 30, 50 from the Weibull distribution with λ= 0.5. The mean square error and the mean percentage error were calculated. The results of the simulation study are reported in the following tables:

Table 1: MSE of estimated Weibull reliability function with λ= 0.5, p = 3, c = 1

n	r	$\hat{R}_1(t)$	$\hat{R}_2(t)$	$\hat{R}_3(t)$	$\hat{R}_4(t)$
10	2	0.011775	0.021651	0.017860	0.012507
	-2	0.011775	0.013933	0.011388	0.012952
30	2	0.004205	0.005140	0.004791	0.004273
	-2	0.004205	0.004430	0.004160	0.004325
50	2	0.002499	0.002838	0.002714	0.002513
	-2	0.002499	0.002566	0.002479	0.002544

Table 2: MSE of estimated Weibull reliability function with λ= 0.5, p = 10, c = 1

n	r	$\hat{R}_1(t)$	$\hat{R}_2(t)$	$\hat{R}_3(t)$	$\hat{R}_4(t)$
10	2	0.010743	0.019617	0.016250	0.011306
	-2	0.010743	0.012578	0.010360	0.011833
30	2	0.003795	0.004655	0.004338	0.003835
	-2	0.003795	0.003970	0.003747	0.003909
50	2	0.002252	0.002566	0.002453	0.002256
	-2	0.002252	0.002302	0.002231	0.002296

Table 3: MSE of estimated Weibull reliability function with λ= 0.5, p = 3, c = 3

n	r	$\hat{R}_1(t)$	$\hat{R}_2(t)$	$\hat{R}_3(t)$	$\hat{R}_4(t)$
10	2	0.178805	0.258187	0.231828	0.166910
	-2	0.178805	0.170623	0.191769	0.191769
30	2	0.095901	0.109257	0.104696	0.093933
	-2	0.095901	0.094807	0.094489	0.098068
50	2	0.072239	0.078804	0.076787	0.070804
	-2	0.072239	0.070977	0.071417	0.073442

Table 4: MSE of estimated Weibull reliability function with: λ= 0.5, p = 10, c = 3

n	r	$\hat{R}_1(t)$	$\hat{R}_2(t)$	$\hat{R}_3(t)$	$\hat{R}_4(t)$
10	2	0.012578	0.010743	0.010360	0.018737
	-2	0.012578	0.021543	0.014225	0.011306
30	2	0.003970	0.003795	0.003747	0.004704
	-2	0.003970	0.005058	0.004159	0.003835
50	2	0.002302	0.002252	0.002231	0.002561
	-2	0.002302	0.002689	0.002367	0.002256

Table 5: MPE of estimated Weibull reliability function with λ = 0.5, p = 3, c = 1

n	r	$\hat{R}_1(t)$	$\hat{R}_2(t)$	$\hat{R}_3(t)$	$\hat{R}_4(t)$
10	2	0.207091	0.303079	0.271482	0.190741
	-2	0.207091	0.194155	0.196965	0.223055
30	2	0.112190	0.128470	0.122965	0.109398
	-2	0.112190	0.110250	0.110386	0.114882
50	2	0.084510	0.092518	0.090073	0.082603
	-2	0.084510	0.082727	0.083472	0.085995

Table 6: MPE of estimated Weibull reliability function with λ= 0.5, p = 10, c = 1

n	r	$\hat{R}_1(t)$	$\hat{R}_2(t)$	$\hat{R}_3(t)$	$\hat{R}_4(t)$
10	2	0.013932	0.011775	0.011388	0.020672
	-2	0.013932	0.023694	0.015755	0.012506
30	2	0.004430	0.004205	0.004159	0.005260

	-2	0.004430	0.005656	0.004646	0.004273
50	2	0.002566	0.002499	0.002479	0.002862
	-2	0.002566	0.003006	0.002642	0.002513

Table 7: MPE of estimated Weibull reliability function with λ= *0.5, p = 3, c = 3*

n	r	$\hat{R}_1(t)$	$\hat{R}_2(t)$	$\hat{R}_3(t)$	$\hat{R}_4(t)$
10	2	0.194155	0.207091	0.196965	0.225703
	-2	0.194155	0.241425	0.201756	0.190741
30	2	0.110250	0.112190	0.110386	0.116881
	-2	0.110250	0.120376	0.111797	0.109398
50	2	0.082727	0.084510	0.083472	0.085332
	-2	0.082727	0.086932	0.083224	0.082603

Table 8: MPE of estimated Weibull reliability function with λ = 0.5, p = 10, c = 3

n	r	$\hat{R}_1(t)$	$\hat{R}_2(t)$	$\hat{R}_3(t)$	$\hat{R}_4(t)$
10	2	0.170623	0.178805	0.170807	0.200231
	-2	0.170623	0.214696	0.177950	0.166910
30	2	0.094807	0.095901	0.094489	0.100945
	-2	0.094807	0.104107	0.096277	0.093933
50	2	0.070970	0.072239	0.071417	0.073409
	-2	0.070970	0.074847	0.071468	0.070804

Discussion

It appears from tables 1, 2, 3 and 4 that in general comparison by MSE and MPE shows that when *c*, *p* are small and *r = 2*, the Weibull reliability function based upon squared error loss function was the best followed by the modified El-Sayyad's loss function. While comparison by MPE shows the reverse. As *c* and *p* become large *R(t)* based upon the modified El- Sayyad loss function was the best only when *r = -2* while El- Sayyad loss function was better for positive r.

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