REMARKS OF THE INTERSECTION OF YOUNG'S DIAGRAMS CORE

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Abstract
In this research, many simple new techniques will be used supported by numerical and theoretical proofs of the methods of the intersection of $\beta - numbers$ for any partition μ of r; which represented by Mahmood in 2010, in which he could appoint the location and the number of beads using "Guide value" and "The main diagram" methods

ملاحظات حول تقاطع قلب مخططات يونك

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ملخص البحث:
في هذا البحث يتم استخدام تقنيات جديدة وسهلة مدعومة ببراهين نظرية وحسابية لمفهوم تقاطع أعداد β لتجزئة معينة μ من r والتي قدمها محمود في 2010 حيث تمكن من تحديد موقع وعدد الكرات أو (العقد) باستخدام مفهومي "القيمة الدليل" و "المخطط الرئيسي".

I. Introduction:

Let r be a non-negative integer. A *composition* μ of r is a sequence $(\mu_1, \mu_2, ..., \mu_n)$ of non-negative integers such that $|\mu| = \sum_{j=1}^{n} \mu_j = r$.

A composition μ is a *partition* if $\mu_j \geq \mu_{j+1}$, for all $j \geq 1$.

β-numbers is defined by; see James [1]:

" Fix μ is a partition of r. Choose an integer b greater than or equal to the number of parts of μ and define $\beta_i = \mu_i + b - i, \quad 1 \leq i \leq b.$ The set $\{\beta_1, \beta_2, ..., \beta_b\}$ is said to be a set of β-numbers for μ."

For example, if $\mu = (5,4,4,2,1)$, then $n = 5$ and if we take $b = 7$, then β-numbers are $\{11,9,8,5,3,1,0\}$.

We can represent β-numbers by the diagram (A):

run.1	**run.2**		**run.e**
0	**1**	**…**	**e-1**
e	**e+1**	**…**	**2e-1**
2e	**2e+1**	**…**	**3e-1**
.	.		.
.	.		.
.	.		.

… **(A)**

where every β will be represented by a bead which takes its location in (diagram A). Returning to the above example we will see:

e = 2		b=7	
0	1	•	•
2	3	-	•
4	5	-	•
6	7	-	-
8	9	•	•
10	11	-	•

And

e = 3			b= 7		
0	1	2	•	•	-
3	4	5	•	-	•
6	7	8	-	-	•
9	10	12	•	-	•

Many researchers had been study this subject as it has a connection with representation theory of Iwahori-Hecke algebras, such as Fayers in [2] and [3].

II. The intersection of β-numbers :

In this part, we will deal with the main diagram and guide value methods which represented by Mahmood in [4]. We will try to give this subject by new technique which support the previous results.

Since the value of $b \geq n$, then we deal with an infinite numbers of values of b. Here we want to mention that these values have a special diagram (A) for it, but there is a repeated part of this diagram with other values where a "Down-shifted" or "Up-shifted", occurs when we take the following: b_1 *if* $b = n$, b_2 *if* $b = n+1$, ... , . b_e *if* $b = n+(e-1)$

<u>Definition (2.1)</u>:[4] For any $\beta - numbers$ in (diag.A), the values of $b_1, b_2, ..., b_e$ are called the *guides*.

By the above example, the guides values are $b_1 = 5$ and $b_2 = 6$ when $\mu = (5,4,4,2,1)$:

e = 2		b_1=5		b_1+1(e)	
0	1	-	•	•	•
2	3	-	•	-	•
4	5	-	-	-	•
6	7	•	•	-	-
8	9	-	•	•	•
10	11	-	-	-	•
12	13	-	-	-	-

And

e = 2		b_2 =6		b_2+1(e)	
0	1	•	-	•	•
2	3	•	-	•	-
4	5	•	-	•	-
6	7	-	•	•	-
8	9	•	-	-	•
10	11	•	-	•	-
12	13	-	-	•	-

we will define any (diagram A) that corresponds any b guides as a "*main diagram*" or "*guide diagram*".[4]

<u>Theorem (2.2)</u>: [4, 2.5] There is e of main diagrams for any partition μ of r.

Now, we return to the intersection method by:

1. let τ be the number of redundant part pf the partition μ of r, then we have $\mu = (\mu_1, \mu_2, ..., \mu_n) = (\mu_1^{\tau_1}, \mu_2^{\tau_2}, ..., \mu_m^{\tau_m})$ such that $r = \sum_{j=1}^{n} \mu_j = \sum_{l=1}^{m} \mu_l^{\tau_l}$.
2. We denote the intersection of main diagrams by $\bigcap_{s=1}^{e} m.d._{b_s}$.

We see in (2.3) that there is just one bead where there isn't any bead in (2.4). So the intersection result as a numerical value will be ϕ in the case of no existence for any bead, or, v in the case that v common beads exist in main diagrams (A).

m.d. $b_1=5$	m.d. $b_2=6$		$\bigcap_{S=1}^{2}$ m.d.$_{b_S}$
- •	• -		- -
- •	• -		- -
- -	- -	→	- -
• •	- •		- •
- •	• -		- -
- -	• -		- -

…..(2,3)

m.d. $b_1=5$	m.d. $b_2=6$	m.d. $b_3=7$		$\bigcap_{S=1}^{3}$ m.d.$_{b_S}$
- • -	• - •	• • -		- - -
• - -	- • -	• - •		- - -
• • -	- • •	- - •	→	- - -
• - -	- • -	• - •		- - -

..…(2.4)

Remark (2.5): For any e, all beads in all runners of main diagram $b = l$ will be "right-shifted" in the next main diagram $b = l + 1$ under consideration that the last runner will be right-shifted to the first runner adding 1 extra bead.

m.d. $b_1=5$	m.d. $b_2=6$	m.d. $b_3=7$		$\bigcap_{S=1}^{3}$ m.d.$_{b_S}$
- • -	• - •	• • - (+1)		- - -
• - -	- • -	• - •		- - -
• • -	- • •	- - •	→	- - -
• - -	- • -	• - •		- - -

This notice leads us to that, the main diagram for b_1 plays an important role in this problem, that we can neglect the remaining main diagrams to find the intersection. By this let us take the following proposition:

Proposition (2.6):

1. In the case of existence of repetitions equal to one, then this part will appear
2. only in the $(main\,diagram_{b_1})$ but not in other main diagrams where $b_1 < b_2 < ... < b_e$. So $\bigcap_{s=1}^{e} m.d._{b_s} = \phi.$
3. In the case that there exist repetitions equal to 2, this means that there exist 2 beads in $(m.d._{b_1})$. The least value of these 2 beads will be canceled, where the other will be exist.
4. If we choose $e > 2,$ this case in the same as in part one in this proposition, that we must cancel "*the two least values of two repetitions where* $e = 3,$ *the three least values of two repetitions where* $e = 4,$ …, *the least* $(m-1)$ *values of* $(e-1)$ *repetitions where* $e = m.$"

To explain this, let us take the example $\mu = (5,4^2,2,1)$ then we see:

β-numbers $b_1=5$	β-numbers $b_2=6$
5+5-1=9	5+6-1=10
4+5-2=7	4+6-2=8
4+5-3=6	4+6-3=7
2+5-4=3	2+6-4=4
1+5-5=1	1+6-5=2
	6-6=0

…..(2,7)

or if $\mu=(6,3^3,2,1^2)$ and $e=3$, then

β-numbers $b_1=7$	β-numbers $b_2=8$	β-numbers $b_3=9$
6+7-1=12	6+8-1=13	6+9-1=14
3+7-2=8	3+8-2=9	3+9-2=10
3+7-3=7	3+8-3=8	3+9-3=9
3+7-4=6	3+8-4=7	3+9-4=8
2+7-5=4	2+8-5=5	2+9-5=6
1+7-6=2	1+8-6=3	1+9-6=4
1+7-7=1	1+8-7=2	1+9-7=3
	8-8=0	9-8=1
		9-9=0

…..(2,8)

So instead of putting β-numbers in diagram (A), we will put the numbers as a numerical series in one row with the following steps:

	9	8	7	6	5	4	3	2	1	0
$\mu=(5,4^2,2,1)$	⊙	-	⊙	⊙	-	-	⊙	-	⊙	-
e=2	-	-	⊙	-	-	-	-	-	-	-
e=3	-	-	-	-	-	-	-	-	-	-

…(2.9)

	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
$\mu=(7^5,5^3,4^4,2^6,1^3)$		⊙	⊙	⊙	⊙	⊙ -	-		⊙ ⊙	⊙	-	⊙	⊙	⊙	⊙	-	-	⊙	⊙	⊙	⊙	⊙	⊙	-	⊙	⊙	⊙	-
e=2		⊙	⊙	⊙	⊙ -	-	-		⊙ ⊙	-	-	⊙	⊙	⊙	-	-	-	⊙	⊙	⊙	⊙	⊙	-	-	⊙	⊙	-	-
e=3		⊙	⊙	⊙ -	-	-	-		⊙ -	-	-	⊙	⊙	-	-	-	-	⊙	⊙	⊙	⊙	-	-	-	⊙	-	-	-
e=4		⊙	⊙ -	-	-	-	-	-	-	-	-	⊙	-	-	-	-	-	⊙	⊙	⊙	-	-	-	-	-	-	-	-
e=5		⊙ -	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	⊙	⊙	-	-	-	-	-	-	-	-	-
e=6	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	⊙	-	-	-	-	-	-	-	-	-	-
e=7	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
⋮	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

….(2.10)

We want to appoint that this simple technique which combine remark (2.5) and Proposition (2.6), will support the result of the intersection, by determining (2.7)-(2.9).

Many uses of technique in Proposition (2.6) and the studying of many cases as in (2.7)-(2.10) lead us numerically to the following relations such as knowing the new partition ρ_e directly from the main partition μ of r. First we must appoint that $\mu=(\mu_1^{\tau_1},\mu_2^{\tau_2},...,\mu_m^{\tau_m})$ which means that there are $m-sets$ which separated by a blank or set of blanks depending on the value of the difference between $(\mu_l-\mu_{l-1})$, for all $1\le l\le m$ where in every set of m there exists τ_l beads.

Theorem (2.11): Let $\mu=(\mu_1^{\tau_1},\mu_2^{\tau_2},...,\mu_m^{\tau_m})$ be a partition of r. The new partition $\rho_{e=2}$ after the intersection of the main diagrams is equal to:

$$\rho_2=((\mu_1+m)^{\tau_1-1},(\mu_2+(m-1))^{\tau_2-2},...,$$
$$(\mu_m+1)^{\tau_m-1}).$$

Proof: By Proposition (2.6) we will cancel the last bead in each set; (see (2.9)-(2.10)), i.e. we cancel m beads, that is, we add m blanks to μ_1, $(m-1)$ blanks to μ_2, … and one blank to

μ_m. Hence $\rho_2 = ((\mu_1 + m)^{\tau_1 - 1}, (\mu_2 + (m-1))^{\tau_2 - 2}, ..., (\mu_m + 1)^{\tau_m - 1})$.

Remark (2.12): Let some or all $\tau_l - 1 = 0$. Then this part with the new partition ρ_2 which related with it and we ordered ρ_2 without the existence of the part of order which equal zero, i.e. $\rho_2 = ((\mu_1 + m)^{\tau_1 - 1}, (\mu_2 + (m-1))^{\tau_2 - 2}, ...,. (\mu_m + 1)^{\tau_m - 1}) = (\sigma_1^{t_1}, \sigma_2^{t_2}, ..., \sigma_c^{t_c}),\ c < m$

Repeat the same steps of Theorem (2.11) and Remark (2.12) in order to find ρ_3 as the following:

Theorem (2.13):

1. $\rho_3 = \phi$ *if* $\rho_2 = \phi$, (*i.e.* $\tau_l = 1$) *for all* $1 \le l \le m$, *c. or* $t_x = 1$ *for all* $1 \le x \le c$
2. $\rho_3 = ((\mu_1 + 2m)^{\tau_1 - 2}, (\mu_2 + 2(m-1))^{\tau_2 - 2}, ..., (\mu_m + 2)^{\tau_m - 2})$ *if* $\tau_l > 2$ *for all* $1 \le l \le m$.

$3 - \rho_3 = ((\sigma_1 + c)^{t_1 - 1}, (\sigma_2 + (c-1))^{t_2 - 1}, ..., (\sigma_c + 1)^{t_c - 1})$.

if $t_1, t_2, ..., t_c \ge 1$

Remark (2.14): We can find any ρ_e after repeating the same methods of Theorems (2.11) and (2.13) and Remark (2.12).

Example: Let $\mu = (7^5, 5^3, 4^4, 2^6, 1^3)$, then the number of parts is 5. Then

$\rho_2 = (12^4, 9^2, 7^3, 4^5, 2^2),\quad n = 5,$

$\rho_3 = (17^3, 13, 10^2, 6^4, 3),\quad n = 5,$

$\rho_4 = (22^2, 17^0, 13, 8^3, 4^0) = (22^2, 13, 8^3),\quad n = 3,$

$\rho_5 = (25, 15^0, 9^2) = (25, 9^2),\quad n = 2,$

$\rho_6 = (27^0, 10) = (10),\quad n = 1,$

$\rho_7 = (11^0) = \phi,$

and $\rho_{e>7} = \phi$.

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