STABILITY ANALYSIS OF AN ECOLOGICAL SYSTEM CONSISTING OF A PREDATOR AND STAGE STRUCTURED PREY

Raid Kamel Naji, Azhar Abbas Majeed
Department of Mathematics, College of Science, University of Baghdad. Baghdad-Iraq.

Abstract
In this paper, a mathematical model, consists from a predator interacting with stage structured prey, is proposed and analyzed. The existence, uniqueness and boundedness of the solution of the proposed model are discussed. The existence and the stability analyses of all possible equilibrium points are studied. The global stability of these equilibrium points are performed with suitable Lyapunov functions. Finally, the dynamical behavior of the model is investigated numerically.

تحليل الاستقرارية لنظام بيئي يتكون من مفترس وفريسة ذات مراحل عمرية مركبة

رائد كامل ناجي، ازهار عباس مجيد
قسم الرياضيات، كلية العلوم، جامعة بغداد. بغداد–العراق.

المستخلص
يتضمن هذا البحث اقـــتراح وتحليل نمـــوذج رياضي يتكون من مفترس يتفاعل مع فريسة ذات مراحل عمرية مركبة. تمت مناقشة وجود، وحدانيـــة وقيـــود الحل للنمـــوذج المقترح. كما قمنا بدراسة وجود واستقرارية النقاط الثابتـــة لهـــذا النموذج.كذلك درسنا الاستقرارية الشاملة للنقاط الثابة تحليليا باستخـــدام دوال ليابانوف. واخيرا السلوك الديناميكي الشامل بحث عدديا.

1. Introduction

Over the last decades there has been a considerable interest in the study of population dynamics with stage structure. Such studies are important, since the life cycle of the most of the animals and insects in nature have two stages: immature and mature. The species in the first stage can't interact or reproduce with the other species rather than that; it depends completely on its relative from mature species, see for example [1-4] and the references therein. Most of these studies were focused on prey-predator interactions involving a stage structured predators with or without time delay.

Later on Cui and Song [5] proposed and analyzed a prey-predator model with stage structure for prey. It is assumed that the predator consumed the immature prey according to Lotka-Volterra type of functional response. They obtained a set of sufficient and necessary conditions which guarantee the permanence of the system. However, Chen [6] studied the permanence of periodic predator–prey system with stage structure for prey. He obtained sufficient and necessary conditions which guarantee the predator and the prey species to be permanent. Recently, Chen and You [7] studied the permanence, extinction and periodic solution of the periodic predator–prey system with Beddington–DeAngelis functional response and stage structure for prey. They obtained a set of sufficient and necessary conditions which guarantee the permanent of the system.

In this paper however, we will propose and analyze Holling type-II prey-predator having stage structure for prey. The intraspecific competition for immature prey and predator is also included in the system.

2. Mathematical model

In this section, an ecological model consists of prey-predator system with stage structure for prey is proposed. In order to formulate the dynamic equations for such a model the following assumptions are made.

A1) The prey population is divided into two classes, immature prey population, whose population density at time T is denoted by $x_1(T)$, and mature prey population, whose population density at time T is denoted by $x_2(T)$.

A2) It is assumed that in the absence of predation only the mature prey population has the ability for reproduction logistically with carrying capacity k ($k>0$) and intrinsic growth rate α ($\alpha>0$). However, the immature prey population depends completely in his reproduction on the food supplied by mature prey. In addition to the above, the immature prey individuals still compete between each other for food and space with intraspecific rate constant η ($\eta>0$).

A3) The immature prey population transfer to mature prey population at a rate βx_1, where β ($\beta>0$) represents the conversion rate coefficient. Finally, both the immature and mature prey populations decreases due to the natural death rates r_1 ($r_1>0$) and r_2 ($r_2>0$) respectively. Thus, depending on the above assumptions the evolution equations for prey can be written as:

$$\frac{dx_1}{dT}=\alpha x_2\left(1-\frac{x_2}{k}\right)-r_1x_1-\beta x_1-\eta x_1^2 \qquad \text{……. (1a)}$$

$$\frac{dx_2}{dT}=\beta x_1-r_2x_2 \qquad (1b)$$

A4) In case of existence of predator, whose population density denoted by $x_3(T)$, it is assumed that the predator consumes the immature prey only (the immature prey is more vulnerable to predation than the mature prey) according to Holling type-II functional response $\frac{\beta_1 x_1}{\gamma_1+x_1}$, where β_1 ($\beta_1>0$) and γ_1 ($\gamma_1>0$) represent respectively, the maximum attack rate and half saturation constants. However, the predator's contribution from the prey species is assumed to be $\frac{c\beta_1 x_1}{\gamma_1+x_1}$ where c $(c>0)$ denotes to conversion rate constant.

A5) Finally, it is assumed that , the predator individuals still compete with each other for food and spaces with intraspecific rate constant η_1 ($\eta_1>0$), and decrease due to natural death rate r $(r>0)$.

Consequently, in the existence of predator species, the evolution equation (1) of a stage structure prey species becomes:

$$\frac{dx_1}{dT}=\alpha x_2\left(1-\frac{x_2}{k}\right)-r_1x_1-\beta x_1-\eta x_1^2-\frac{\beta_1x_1x_3}{\gamma_1+x_1} \qquad \text{……...(2a)}$$

$$\frac{dx_2}{dT}=\beta x_1-r_2x_2 \qquad \text{……….(2b)}$$

$$\frac{dx_3}{dT}=-rx_3+c\frac{\beta_1x_1x_3}{\gamma_1+x_1}-\eta_1x_3^2 \qquad \text{……….(2c)}$$

Now, for further simplifying the system (2), the following dimensionless variables are used

$y_1=\frac{c\beta_1}{\alpha\gamma_1}x_1$, $y_2=\frac{c\beta_1}{\alpha\gamma_1}x_2$, $y_3=\frac{\beta_1}{\alpha\gamma_1}x_3$, $t=\alpha T$.

Then system (2) can be turned into the following dimensionless form:

$$\frac{dy_1}{dt}=y_2(1-w_1y_2)-w_2y_1-w_3y_1-w_4y_1^2-\frac{y_1y_3}{1+w_5y_1} \qquad (3a)$$

$$\frac{dy_2}{dt}=w_3y_1-w_6y_2 \qquad (3b)$$

$$\frac{dy_3}{dt}=y_3\left(-w_7+\frac{y_1}{1+w_5y_1}-w_8y_3\right) \qquad (3c)$$

where $w_1=\frac{\alpha\gamma_1}{c\beta_1k}$; $w_2=\frac{r_1}{\alpha}$; $w_3=\frac{\beta}{\alpha}$; $w_4=\frac{\eta\gamma_1}{c\beta_1}$; $w_5=\frac{\alpha}{c\beta_1}$; $w_6=\frac{r_2}{\alpha}$; $w_7=\frac{r}{\alpha}$ and $w_8=\frac{\eta_1\gamma_1}{\beta}$ are the dimensionless parameters.

System (3) needs to analyzed with a specific initial condition, which may be taken as any point in the region $R_+^3=\{(y_1,y_2,y_3)\in R^3 : y_i\geq 0; i=1,2,3\}$.

Theorem 1:- All solutions of system (3), which are initiate in R_+^3 are uniformly bounded.

Proof:- Let $(y_1(t),y_2(t),y_3(t))$ be any solution of system (3) with non-negative initiate conditions.

Let $w(t)=y_1(t)+y_2(t)+y_3(t)$, then we get that

$$\frac{dw}{dt} = y_2 - w_1 y_2^2 - w_2 y_1 - w_4 y_1^2 - w_6 y_2 - w_7 y_3 - w_8 y_3^2$$
$$\leq y_2(1 - w_1 y_2) - w_2 y_1 - w_6 y_2 - w_7 y_3$$
$$\leq \frac{1}{4w_1} - N[y_1 + y_2 + y_3],$$

where $N = \min.\{w_2, w_6, w_7\}$. Thus we obtain:

$$\frac{dw}{dt} + Nw \leq \frac{1}{4w_1}$$

Now by comparing the above differential inequality with the associated linear differential equation, we obtain

$$w(t) \leq \frac{1}{4w_1 N}(1 - e^{-Nt}) + w(0)e^{-Nt}$$

Therefore $0 < w(t) \leq R_1$, as $t \to \infty$, where $R_1 = \max.\left\{ w(0), \frac{1}{4w_1 N} \right\}$.

Hence, all solutions of system (3) that initiate in R_+^3 are confined in the region $\left\{(y_1, y_2, y_3) \in R_+^3 : w(t) \leq R_1 + \varepsilon; \varepsilon > 0\right\}$. Thus all solutions are uniformly bounded, and then the proof is complete. ■

3. Existence and stability analysis of system (3):

The stage structured prey-predator model given by system (3) has at most three nonnegative equilibrium points, namely $E_0 = (0,0,0)$, $E_1 = (\bar{y}_1, \bar{y}_2, 0)$, and $E_2 = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$.

The equilibrium point E_0 always exists, however the equilibrium point E_1 exists in the $Int.R_+^2$ of $y_1 y_2$ −plane where

$$\bar{y}_1 = \frac{-(w_2 + w_3)w_6^2 + w_3 w_6}{(w_4 w_6^2 + w_1 w_3^2)} \quad (4a)$$

$$\bar{y}_2 = \frac{w_3}{w_6}\bar{y}_1 \quad (4b)$$

provided that:

$$w_3 > (w_2 + w_3)w_6 \quad (5)$$

Finally the positive equilibrium point $E_2 = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$ where

$$\hat{y}_2 = \frac{w_3}{w_6}\hat{y}_1 \quad (6a)$$

$$\hat{y}_3 = \frac{-w_7}{w_8} + \frac{\hat{y}_1}{w_8(1 + w_5 \hat{y}_1)} \quad (6b)$$

while $\hat{y}_1$ is a positive root of the following third order equation:

$$A_1 y_1^3 + A_2 y_1^2 + A_3 y_1 + A_4 = 0.$$

Here

$$A_1 = \frac{w_1 w_3^2 w_8 w_5^2}{w_6^2} + w_4 w_8 w_5^2 > 0,$$

$$A_2 = \frac{-w_3 w_8 w_5^2}{w_6} + \frac{2w_1 w_3^2 w_5 w_8}{w_6^2} + w_2 w_8 w_5^2 + w_3 w_8 w_5^2 + 2w_4 w_8 w_5,$$

$$A_3 = \frac{-2w_3 w_5 w_8}{w_6} + \frac{w_1 w_3^2 w_8}{w_6^2} + 2w_2 w_8 w_5 + 2w_3 w_8 w_5 + w_4 w_8 - w_5 w_7 + 1,$$

$$A_4 = \frac{-w_3 w_8}{w_6} + w_2 w_8 + w_3 w_8 - w_7,$$

Obviously E_2 exists uniquely in the $Int.R_+^2$ if and only if the following conditions hold

$$w_2 w_6 < w_3 < \frac{w_7}{w_8} \quad (8a)$$

$$-w_8 w_5^2 \left[\frac{w_3}{w_6} - w_2 \right] + R > 0; \text{ with} \quad (8b)$$

$$R = \frac{2w_1 w_3^2 w_5 w_8}{w_6^2} + w_3 w_8 w_5^2 + 2w_4 w_8 w_5, \text{ and}$$

$$\hat{y}_1 > w_7(1 + w_5 \hat{y}_1) \quad (8c)$$

In the following, the local dynamical behavior of system (3) around each of the above equilibrium points is discussed. First the jacobian matrix of system (3) at each point is determined and then the eigenvalues for the resulting matrix are computed. The jacobian matrix of system (3) at the equilibrium point $E_0 = (0,0,0)$ can be written by

$$J(E_0) = \begin{bmatrix} -(w_2 + w_3) & 1 & 0 \\ w_3 & -w_6 & 0 \\ 0 & 0 & -w_7 \end{bmatrix}.$$

Therefore, it is easy to verify that, the eigenvalues of $J(E_0)$, say $\lambda_{01}, \lambda_{02}$ and λ_{03} that describe the dynamics in the y_1, y_2 and y_3-direction respectively satisfy the following relations :

$$\lambda_{01} + \lambda_{02} = -(w_2 + w_3) - w_6 < 0 \quad (9a)$$

$\lambda_{01}.\lambda_{02} = (w_2 + w_3)w_6 - w_3$ (9b)

$\lambda_{03} = -w_7 < 0$ (9c)

Note that, according to Eq. (9b), the eigenvalues λ_{01} and λ_{02} have opposite sign provided that

$w_3 > (w_2 + w_3)w_6$. (10a)

Hence E_0 is a saddle point in the R_+^2 of $y_1 y_2$-plane and since the eigenvalue λ_{03} that describes the dynamics in y_3-direction is negative , hence E_0 is a saddle point in R_+^3 with locally stable manifold of dimension two and with locally unstable manifold of dimension one.

However, λ_{01} and λ_{02} are negative provided that

$w_3 < (w_2 + w_3)w_6$ (10b)

and then E_0 is a locally asymptotically stable in the R_+^3.

The Jacobian matrix of system (3) at the equilibrium point $E_1 = (\bar{y}_1, \bar{y}_2, 0)$ is given by: $J(E_1) = (b_{ij})_{3\times 3}$, where

$b_{11} = -w_2 - w_3 - 2w_4\bar{y}_1$, $b_{12} = 1 - 2w_1\bar{y}_2$,

$b_{13} = \frac{\bar{y}_1}{1+w_5\bar{y}_1}$, $b_{21} = w_3$, $b_{22} = -w_6$,

$b_{23} = b_{31} = b_{32} = 0$, $b_{33} = -w_7 + \frac{\bar{y}_1}{1+w_5\bar{y}_1}$.

Now, straightforward computation shows that , the eigenvalues of the jacobian matrix $J(E_1)$, say λ_{11}, λ_{12} and λ_{13} which describe the dynamics in the directions y_1, y_2 and y_3 respectively, satisfy the following relations:

$\lambda_{11} + \lambda_{12} = -(w_2 + w_3) - 2w_4\bar{y}_1 - w_6 < 0$, (11a)

$\lambda_{11}.\lambda_{12} = -M_1 + M_2\bar{y}_1$, (11b)

$$\lambda_{13} = -w_7 + \frac{\bar{y}_1}{1 + w_5\bar{y}_1} \quad (11c)$$

where $M_1 = w_3 - (w_2 + w_3)w_6$, which is positive under the existence condition of E_1, and $M_2 = 2(w_4 w_6 + \frac{w_1 w_3^2}{w_6})$.

Note that, according to the Eqs. (11a)-(11c) we have the following two cases :

Case 1:- If the following condition holds

$$\lambda_{13} < 0 \Leftrightarrow \frac{\bar{y}_1}{1 + w_5\bar{y}_1} < w_7 . \quad (12a)$$

Then E_1 is locally asymptotically stable in y_3-direction, and hence we have the following two subcases:

1. 1 E_1 is locally asymptotically stable in R_+^3 provided that the following condition holds
   $M_1 < M_2\bar{y}_1$ (12b)
2. E_1 is a saddle point in R_+^3 with locally stable manifold of dimension two and locally unstable manifold of dimension one provided that the following condition holds :
   $M_1 > M_2\bar{y}_1$ (12c)

Case 2:- If the following condition holds

$$\lambda_{13} > 0 \Leftrightarrow \frac{\bar{y}_1}{1 + w_5\bar{y}_1} > w_7 \quad (12d)$$

Then E_1 is a saddle point in R_+^3.

Finally, the jacobian matrix of the system (3) at the positive equilibrium point $E_2 = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$ can be written as:

$J(E_2) = (a_{ij})_{3\times 3}$ (13)

here $a_{11} = -(w_2 + w_3) - 2w_4\hat{y}_1 - \frac{\hat{y}_3}{N_0^2}$;

$a_{12} = 1 - 2w_1\hat{y}_2$; $a_{13} = \frac{\hat{y}_1}{N_0}$; $a_{21} = w_3$;

$a_{22} = -w_6$; $a_{23} = 0$; $a_{31} = \frac{\hat{y}_1}{N_0^2}$; $a_{32} = 0$;

$a_{33} = -w_7 + \frac{\hat{y}_1}{N_0} - 2w_8\hat{y}_3$; $N_0 = 1 + w_5\hat{y}_1$.

Accordingly the characteristic equation of $J(E_2)$ is given by

$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$ (14a)

where

$$A_1 = \frac{1}{N_0^2}\left[N_1 + N_0 N_2 + w_6 N_0^2\right] \quad (14b)$$

$$A_2 = \frac{w_6}{N_0^2}\left[N_1 + N_0 N_2\right] - \frac{1}{N_0^3}\left[\hat{y}_1^2 - N_1 N_2\right] - w_3 N_3 \quad (14c)$$

$$A_3 = \frac{-w_6}{N_0^3}\left[\hat{y}_1^2 - N_1 N_2\right] - \frac{w_3 N_2 N_3}{N_0} \quad (14d)$$

with

$N_1 = (w_2 + w_3 + 2w_4\hat{y}_1)N_0^2 + \hat{y}_3 > 0$,

$N_2 = (w_7 + 2w_8\hat{y}_3)N_0 - \hat{y}_1$, $N_3 = 1 - 2w_1\hat{y}_2$.

Note that, due to Routh-Harwitz criterion, the necessary and sufficient conditions for E_2 to be locally asymptotically stable in the $Int.R_+^3$, are $A_1 > 0$, $A_3 > 0$ and $\Delta = A_1 A_2 - A_3 > 0$.

Straightforward computation shows that, if the following condition holds

$N_2 > 0 \Leftrightarrow \hat{y}_1 < (w_7 + 2w_8\hat{y}_3)N_0$ (15a)

Then we obtain $A_1 > 0$. In addition to condition (15a), if the following conditions hold

$$N_3 < 0 \Leftrightarrow \hat{y}_2 > \frac{1}{2w_1} \text{, and} \qquad (15b)$$

$$\hat{y}_1^2 < N_1 N_2 \;, \qquad (15c)$$

Then we get that $A_3 > 0$.

Finally, substituting the values of A_i for $i = 1,2,3$ in $\Delta = A_1 A_2 - A_3$ and then simplifying the resulting term we get that

$$\Delta = \frac{(N_1 + N_0 N_2)}{N_0^2}\left[\frac{w_6}{N_0^2}(N_1 + N_0 N_2) - \frac{1}{N_0^3}(\hat{y}_1^2 - N_1 N_2) - w_3 N_3 + w_6^2\right] + w_3 N_3\left(\frac{N_2}{N_0} - w_6\right)$$

$$= N_4 + w_3 N_3 N_5 \;,$$

where

$$N_4 = \frac{(N_1 + N_0 N_2)}{N_0^2}\left[\frac{w_6}{N_0^2}(N_1 + N_0 N_2) - \frac{1}{N_0^3}(\hat{y}_1^2 - N_1 N_2) - w_3 N_3 + w_6^2\right] > 0$$

$$N_5 = \frac{N_2}{N_0} - w_6 \,,$$

obviously $\Delta > 0$ if and only if in addition to conditions (15a)-(15c) one of the following two conditions holds:

$$N_5 \le 0 \Leftrightarrow \frac{N_2}{N_0} \le w_6 \qquad (15d)$$

or

$$N_5 > 0 \text{ with } N_4 + w_3 N_3 N_5 > 0 \qquad (15e)$$

Consequently the following theorem for locally stability of E_2 can be proved easily.

Theorem 2:- Assume that the positive equilibrium point E_2 of system (3) exists. Then E_2 is locally asymptotically stable in the $Int.R_+^3$ if the conditions (15a)-(15c) with (15d) or (15e) are satisfied.

In the following theorems, the global dynamical behaviors of all equilibrium points of system (3) are studied analytically.

Theorem 3: Assume that the equilibrium point $E_0 = (0,0,0)$ is locally asymptotically stable in R_+^3. Then it is globally asymptotically stable in R_+^3 provided that $w_6 > 1$.

Proof:- Consider the following positive definite function

$$V_1 = \sum_{i=1}^{3} \alpha_i y_i \;,$$

where α_i; $(i = 1,2,3)$ are positive constants to be determined. Clearly $V_1 : R_+^3 \to R$ be a continuously different-tiable function so that $V_1(0,0,0) = 0$ and $V_1(y_1, y_2, y_3) > 0\,; \forall\,(y_1, y_2, y_3) \in Int.R_+^3$. Now since

$$\frac{dV_1}{dt} = (\alpha_1 - \alpha_2 w_6) y_2 + (\alpha_2 - \alpha_1) w_3 y_1 + (\alpha_3 - \alpha_1)\frac{y_1 y_3}{1 + w_5 y_1} - \alpha_1 w_1 y_2^2 - \alpha_1 w_2 y_1 - \alpha_1 w_4 y_1^2 - \alpha_3 w_7 y_3 - \alpha_3 w_8 y_3^2$$

Then by choosing $\alpha_1 = \alpha_2 = \alpha_3 > 0$ we obtain that:

$$\frac{dV_1}{dt} \le (1 - w_6)\alpha_1 y_2 - \alpha_1 w_2 y_1 - \alpha_3 w_7 y_3$$

Clearly, $\frac{dV_1}{dt} = 0$ if and only if $(y_1, y_2, y_3) = E_0$ and $\frac{dV_1}{dt} < 0$ otherwise.

Thus $\frac{dV_1}{dt}$ is negative definite and hence V_1 is a Lyapunov function. Therefore $E_0 = (0,0,0)$ is a globally asymptotically stable in R_+^3.

Theorem 4: Assume that the equilibrium point $E_1 = (\bar{y}_1, \bar{y}_2, 0)$ is locally asymptoti-cally stable in R_+^3. Then it is globally asymptotically stable in R_+^3 provided that the following conditions are satisfies.

$$\frac{w_1}{y_1}(y_2 + \bar{y}_2) + \frac{y_1 + y_2}{y_1 y_2} \le 2\sqrt{\frac{\bar{y}_2}{y_1 \bar{y}_1}}\sqrt{\frac{\bar{y}_1}{y_2 \bar{y}_2}} \qquad (16a)$$

$$y_1 > B_1 \qquad (16b)$$

where $B_1 = \text{Max}\left\{\frac{\bar{y}_1 - w_7}{w_4 w_7}, \frac{w_1 \bar{y}_2^{\,2}}{w_4 \bar{y}_1}\right\}$.

Proof:- Consider the following positive definite function

$$V_2 = \sum_{i=1}^{2} \alpha_i \left(y_i - \bar{y}_i - \bar{y}_i \ln\!\left(\frac{y_i}{\bar{y}_i}\right)\right) + \alpha_3 y_3 \;,$$

where α_i; $(i = 1,2,3)$ are positive constants to be determined.

Clearly $V_2 : R_+^3 \to R$ be a continuously differentiable function so that: $V_2(\bar{y}_1, \bar{y}_2, 0) = 0$ and $V_2(y_1, y_2, y_3) > 0$ otherwise. Now since

$$\frac{dV_2}{dt} = \sum_{i=1}^{2} \alpha_i \left[\frac{(y_i - \bar{y}_i)}{y_i}\right] \frac{dy_i}{dt} + \alpha_3 \frac{dy_3}{dt}$$

$$= \alpha_1 (y_1 - \bar{y}_1) \left[\frac{y_2 \bar{y}_1 - y_1 \bar{y}_2}{y_1 \bar{y}_1}\right] - \alpha_1 w_1 (y_1 - \bar{y}_1) \left[\frac{y_2^2 \bar{y}_1 - \bar{y}_2{}^2 y_1}{y_1 \bar{y}_1}\right] - \alpha_1 (y_1 - \bar{y}_1) \frac{y_3}{1 + w_5 y_1} - \alpha_1 w_4 (y_1 - \bar{y}_1)^2 + \alpha_2 w_3 (y_2 - \bar{y}_2) \left[\frac{y_1 \bar{y}_2 - \bar{y}_1 y_2}{y_2 \bar{y}_2}\right] - \alpha_3 w_7 y_3 + \alpha_3 \frac{y_1 y_3}{1 + w_5 y_1} - \alpha_3 w_8 y_3^2.$$

Then by choosing $\alpha_1 = w_3 \alpha_2 = \alpha_3 > 0$, and then simplifying the resulting terms according to conditions (16a-16b), we obtain

$$\frac{dV_2}{dt} \le -\alpha_1 \left[\sqrt{\frac{\bar{y}_2}{y_1 \bar{y}_1}} (y_1 - \bar{y}_1) - \sqrt{\frac{\bar{y}_1}{y_2 \bar{y}_2}} (y_2 - \bar{y}_2)\right]^2 - \alpha_1 (y_1 - \bar{y}_1)^2 \left[w_4 - \frac{w_1}{y_1 \bar{y}_1} \bar{y}_2{}^2\right] + \alpha_1 y_3 \left(\frac{\bar{y}_1}{1 + w_5 y_1} - w_7\right)$$

Obviously, $\frac{dV_2}{dt} = 0$ if and only if $(y_1, y_2, y_3) = E_1$ while $\frac{dV_2}{dt} < 0$ otherwise. Thus $\frac{dV_2}{dt}$ is negative definite and hence V_2 is a Lyapunov function. Therefore $E_1 = (\bar{y}_1, \bar{y}_2, 0)$ is a globally asymptoti-cally stable in R_+^3.

Theorem 5: Assume that the equilibrium point $E_2 = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$ is locally asymp-totically stable in $Int.R_+^3$ .Then it is globally asymptotically stable in $Int.R_+^3$ provided that the following conditions are satisfies.

$$\frac{w_1}{y_1}(y_2 + \hat{y}_2) + \frac{y_1 + y_2}{y_1 y_2} \le 2\sqrt{\frac{\hat{y}_2}{y_1 \hat{y}_1}} \sqrt{\frac{\hat{y}_1}{y_2 \hat{y}_2}} \quad (17a)$$

$$y_1 > B_2, \quad (17b)$$

where $B_2 = \text{Max}\left\{\hat{y}_1, \frac{\hat{y}_3}{\hat{y}_1 y_3}, \frac{w_1 \hat{y}_2^2}{w_4 \hat{y}_1}\right\}$.

Proof:- Consider the following positive definite function

$$V_3 = \sum_{i=1}^{3} \alpha_i \left(y_i - \hat{y}_i - \hat{y}_i \ln\left(\frac{y_i}{\hat{y}_i}\right)\right),$$

where α_i; $(i = 1,2,3)$ are positive constants to be determined.

Clearly $V_3 : R_+^3 \to R$ be a continuously differentiable function so that $V_3(\hat{y}_1, \hat{y}_2, \hat{y}_3) = 0$ and $V_3(y_1, y_2, y_3) > 0$ for all $(y_1, y_2, y_3) \in IntR_+^3$.

Now since

$$\frac{dV_3}{dt} = \sum_{i=1}^{3} \alpha_i \left(\frac{y_i - \hat{y}_i}{y_i}\right) \frac{dy_i}{dt}$$

$$= \alpha_1 (y_1 - \hat{y}_1) \left[\frac{y_2 \hat{y}_1 - y_1 \hat{y}_2}{y_1 \hat{y}_1}\right] - \alpha_1 w_1 (y_1 - \hat{y}_1) \left[\frac{y_2^2 \hat{y}_1 - \hat{y}_2^2 y_1}{y_1 \hat{y}_1}\right] - \alpha_1 w_4 (y_1 - \hat{y}_1)^2 - \frac{\alpha_1 (y_1 - \hat{y}_1)(y_3 - \hat{y}_3)}{(1 + w_5 y_1)(1 + w_5 \hat{y}_1)} - \frac{\alpha_1 w_5 (y_1 - \hat{y}_1)(\hat{y}_1 y_3 - \hat{y}_3 y_1)}{(1 + w_5 y_1)(1 + w_5 \hat{y}_1)} + \alpha_2 w_3 (y_2 - \hat{y}_2) \left[\frac{y_1 \hat{y}_2 - \hat{y}_1 y_2}{y_2 \hat{y}_2}\right] - \alpha_3 w_8 (y_3 - \hat{y}_3)^2 + \frac{\alpha_3 (y_1 - \hat{y}_1)(y_3 - \hat{y}_3)}{(1 + w_5 y_1)(1 + w_5 \hat{y}_1)}$$

.

Then by choosing $\alpha_1 = w_3 \alpha_2 = \alpha_3 > 0$, and using the given conditions (17a)-(17b), we obtain

$$\frac{dV_3}{dt} \le -\alpha_1 \left[\sqrt{\frac{\hat{y}_2}{y_1 \hat{y}_1}} (y_1 - \hat{y}_1) - \sqrt{\frac{\hat{y}_1}{y_2 \hat{y}_2}} (y_2 - \hat{y}_2)\right]^2 - \alpha_1 (y_1 - \hat{y}_1)^2 \left[w_4 - \frac{w_1}{y_1 \hat{y}_1} \hat{y}_2^2\right] - \frac{\alpha_1 w_5 (y_1 - \hat{y}_1)(\hat{y}_1 y_3 - \hat{y}_3 y_1)}{(1 + w_5 y_1)(1 + w_5 \hat{y}_1)} - \alpha_3 w_8 (y_3 - \hat{y}_3)^2$$

Hence $\frac{dV_3}{dt} = 0$ if and only if $(y_1, y_2, y_3) = E_2$ and $\frac{dV_3}{dt} < 0$ otherwise. Thus $\frac{dV_3}{dt}$ is negative definite and hence V_3 is a Lyapunov function. Therefore $E_2 = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$ is a globally asymp-totically stable in $Int.R_+^3$.

4-Numerical Simulation:-

In this section, the global dynamics of system (3) is further investigated by solving it numerically. The objective is to verify our previous analytical results and understand the effect of varying the parameters values.

For the following set of parameters value:

$$w_1 = 0.02, w_2 = 0.1, w_3 = 0.1, w_4 = 0.2, w_5 = 0.1, \quad w_6 = 0.2, w_7 = 0.1, w_8 = 0.1 \tag{18}$$

The trajectories of system (3) approach asymptotically to global stable point in the $Int.R_+^3$, as shown in Figure 1(a-c).Clearly, for this set of data, the numerical result confirms our analytical result. Moreover it is observed that, increasing the conversion rate from immature prey to mature prey further, i.e $w_3 > 0.1$, system (3) still have a globally stable point in $Int.R_+^3$.

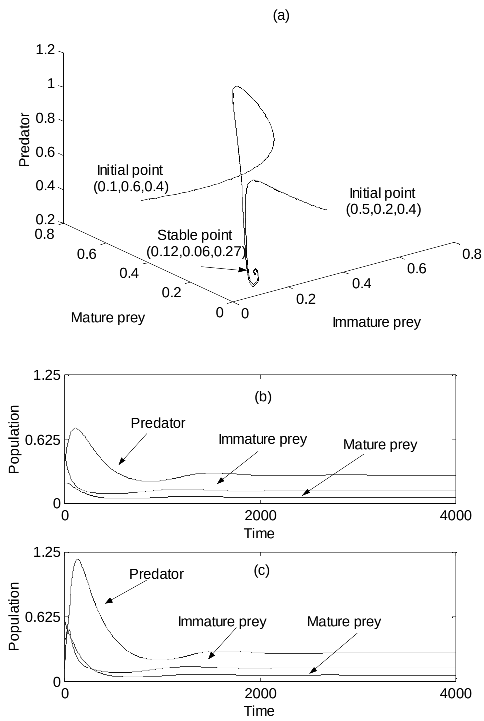

Figure1: (a) Globle stable point in $Int.R_+^3$.
(b) Time series of (a) starting from (0.5, 0.2, 0.4).
(c) Time series of (a) starting from (0.1, 0.6, 0.4).

However, for the parameters set (18) with $w_3 = 0.02$, condition (10b) holds and the trajectory of system (3) approaches asymptotically to the origin as shown in Figure. 2(a-b). Similar conclusion is drawn for $w_6 > 0.5$.

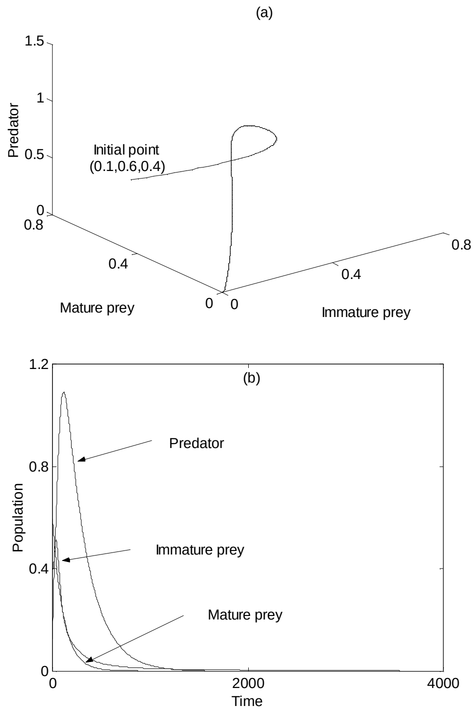

Figure 2: (a) (0, 0, 0) is global stable point.
(b) Time series of (a) starting from (0.1, 0.6, 0.4).

Finally, for the parameter set (18) with $w_7 > 1.3$ system (3) has a globally asymptotically stable point in $Int.R_+^2$ of $y_1 y_2$ -plane, see Figure 3(a-b).

Note that, it is easy to verify that, for the parameters set (18) with $w_7 > 1.276$, E_1 exists and the conditions (12)(a-b) are satisfied.

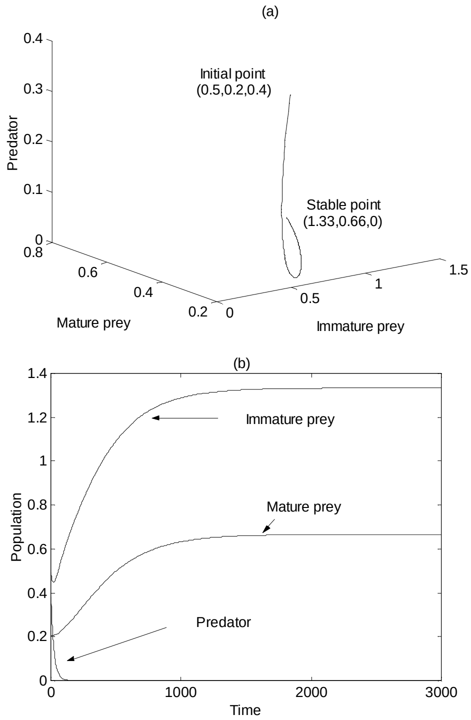

Figure 3: (a) Globle stable point in $Int.R_+^2$ of $y_1 y_2$ -plane. (b) Time series of (a) starting from (0.5, 0.2, 0.4).

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