SEMI-ANALYTIC TECHNIQUE FOR SOLVING HIGH ORDER NONLINEAR ORDINARY INITIAL VALUE PROBLEM

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Abstract
The aim of this paper is to present method for solving high order nonlinear ordinary differential equations with initial conditions using semi-analytic technique with constructing polynomial solutions. The original problem is concerned using two-point osculatory interpolation with the fit equal numbers of derivatives at the end points of an interval [0, 1] and give example illustrate suggested method and accuracy, easily implemented. The accuracy of the method is confirmed by compared with conventional methods (RK4, RK-Butcher, DTM).
The existence, uniqueness and sensitivity of the solution is discussed.

الطريقة شبه التحليلية لحل مسائل القيم الابتدائية الاعتيادية غير الخطية ذات الرتب العالية

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المستخلص
الهدف من هذا البحث عرض طريقة لحل معادلات تفاضلية اعتيادية ذات رتب عالية لمسائل القيم الابتدائية باستخدام التقنية شبه التحليلية مع تكوين الحل كمتعددة حدود، أصل المسالة يتعلق باستخدام الاندراج التماسي ذو النقطتين والذي يتفق مع الدالة ومشتقاتها عند نقطتي نهاية الفترة [0, 1] المعرفة عليها كذلك وضحنا بمثال الطريقة المقترحة ، دقتها وسهولة الأداء حيث تم توضيح الدقة من خلال مقارنة الطريقة المقترحة مع طرق أخرى.
كما تضمن البحث مناقشة الوجود، الوحدانية والتحسس للمعادلات التفاضلية الاعتيادية.

1. Introduction

Many problems in engineering and science can be formulated in terms of differential equations. A differential equation is an equation involving a relation between an unknown function and one or more of its derivatives.

An ordinary differential equation (ODE) has only one independent variable, and all derivatives in it are taken with respect to that variable. Most often, this variable is time t. Gerald and Wheatley flip around a lot, using both t and x as the independent variable; pay careful attention to what the derivative is taken with respect to so you don't get confused .

The problems of solving an ODE are classified into initial value problems (IVP) and boundary value problems (BVP), depending on how the conditions at the endpoints of the domain are specified. All the conditions of IVP are specified at the initial point. On the other hand, the problem becomes a boundary value problem if the conditions are needed for both initial and final points. [1]

A general mth-order initial value problem:

$$y^{(m)}=f(x,y,y',y'',\dots,y^{(m-1)}),\ a\le x\le b \qquad (1)$$

With initial conditions:

$$y(a)=\alpha_1, y'(a)=\alpha_2,\dots,y^{(m-1)}(a)=\alpha_m \qquad (2)$$

The system of m first-order differential equations (meaning that only the first derivative of y appears in the equation and no higher derivatives) have the form :

$y'_1 = f_1 (x , y_1 , \ldots, y_m)$
$y'_2 = f_2 (x , y_1 , \ldots, y_m)$
.
.
$y'_m=f_m(x,y_1,y_2,\ldots,y_m)$ (3a)

with initial conditions :

$y_1(a)=\alpha_1,\ y_2(a) = \alpha_2, \ldots ,y_m(a) = \alpha_m$ (3b)

It is an easy to see that (3) can represent either an mth-order differential equation, a system of equations of mixed order but with total order of m, or system of m first -order equations.

The IVP is said to be homogeneous if both the differential equation and the initial conditions are homogeneous. Otherwise the problem is non homogeneous. [2]

Since there are relatively few differential equations arising from practical problems for which analytical solutions are unknown, one must resort to numerical methods. In this paper we study the solutions of high order ordinary differential equations with initial condition, where the problems define on the interval [0, 1] using semi – analytic technique that give solution with high accuracy and easy implemented from other numerical methods.

2. Existence And Uniqueness For Solution Of Higher-Order IVP

To discuss existence and uniqueness of solutions of high order ordinary differential equation with initial condition, we give the following theorem :

Theorem 1 [3], [4]

Consider the initial value problem :

$y^{(n)} + p_1(x)\, y^{(n-1)} + \ldots + p_{n-1}(x)\, y' + p_n(x)\, y = q(x)$, with :

$y(x_0) = y_0 ,\ y'(x_0) = y'_0 , \ldots ,$
$y^{(n-1)} (x_0) = y^{(n-1)}_0$

If the functions $\{p_i(x)\}$,i=1,2,…,n and q(x) are continuous on the open interval (a, b) , then there exists a unique solution to the problem .

3. Approximation Theory

The primary aim of a general approximation is to represent non-arithmetic quantities by arithmetic quantities so that the accuracy can be ascertained to a desired degree. Secondly, we are also concerned with the amount of computation required to achieve this accuracy. A complicated function f(x) usually is approximated by an easier function of the form $\varphi(x; a_0, \ldots , a_n)$ where $a_0, \ldots , a_n$ are parameters to be determined so as to characterize the best approximation of f .

In this paper, we shall consider only the interpolatory approximation. From Weierstrass Approximation Theorem, it follows that one can always find a polynomial that is arbitrarily close to a given function on some finite interval. This means that the approximation error is bounded and can be reduced by the choice of the adequate polynomial. Unfortunately Weierstrass Approximation Theorem is not a constructive one, i.e. it does not present a way how to obtain such a polynomial. i.e. the interpolation problem can also be formulated in another way, viz as the answer to the following question: How to find a good representative of a function that is not known explicitly, but only at some points of the domain .In this paper we use Osculatory Interpolation since has high order with the same given points in the domain .

3.1.Osculatory Interpolation [5]

Given the data $\{x_i\}$, i = 0,1, . . . ,n and values $f_i^{(0)}, \ldots , f_i^{(m_i)}$,where m_i are nonnegative integers and $f_i = f(x_i)$.We want to construct a polynomial P(x) such that :

$P^{(j)}(x_i) = f_i^{(j)}$ (4)

For each i = 0, 1. . . n and j = 0, . . . , m_i .

i.e. the osculating polynomial approximating a function $f \in C^m [a , b]$, where $m = \max \{m_0 , m_1 , \ldots , m_n \}$ and $x_i \in [a , b]$, for each i = 0, 1, . . . , n. Such a polynomial is said to be an osculatory interpolating polynomial of a function f.

Remark [5]

The degree of P(x) is at most $M=\sum_{i=0}^{n} m_i + n$ since the number of conditions to be satisfied is $\sum_{i=0}^{n} m_i + (n+1)$,and a polynomial of degree M has M + 1 coefficients that can be used to satisfy these conditions.

There exist various form for osculatory interpolation, but all of these differed only in formula, the following theorem illustrate this:

Theorem 2 [6], [7]

Given the nodes $\{x_i\}$, i = 0, . . . , n and values $\{ f_i^{(j)} \}$, j = 0, . . . , m_i ,
there exists a unique polynomial satisfying(4).

In this paper we use two-point osculatory interpolation [8]. The idea is to approximate a function y(x) by a polynomial P(x) in which values of y(x) and any number of its derivatives at given points are fitted by the corresponding function values and derivatives of P(x).

In this paper we are particularly concerned with fitting function values and derivatives at the two end points of a finite interval, say [0, 1],where a useful and succinct way of writing osculatory interpolant $P_{2n+1}(x)$ of degree 2n + 1 was given for example by Phillips [9] as :

$$P_{2n+1}(x)=\sum_{j=0}^{n} \{y^{(j)}(0)\ q_j(x) + (-1)^j\ y^{(j)}(1)\ q_j(1-x)\} \qquad (5)$$

$$q_j(x) = (x^j / j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s = Q_j(x)/j! \qquad (6)$$

so that (5) with (6) satisfies :

$y^{(j)}(0) = P_{2n+1}^{(j)}(0)$, $y^{(j)}(1) = P_{2n+1}^{(j)}(1)$, j = 0, 1, 2,…, n .

implying that $P_{2n+1}(x)$ agrees with the appropriately truncated Taylor series for y(x) about x = 0 and x = 1. The error on [0, 1] is given by [9]:

$$R_{2n+1} = y(x)-P_{2n+1}(x) = \frac{(-1)^{n+1} x^{(n+1)}(1-x)^{n+1} y^{(2n+2)}(\varepsilon)}{(2n+2)!} \text{ where}$$

$\varepsilon \in (0, 1)$ and $\mathbf{y}^{(2n+2)}$ is assumed to be continuous.

The osculatory interpolant for $P_{2n+1}(x)$ may converge to y(x) in [0 , 1] irrespective of whether the intervals of convergence of the constituent series intersect or are disjoint .The important consideration here is whether $R_{2n+1} \to 0$ as $n\to \infty$ for all x in [0 ,1]. We observe that (5) fits an equal number of derivatives at each end point but it is possible and indeed sometimes desirable to use polynomials which fit different numbers of derivatives at the end points of an interval .

Finally we observe that (5) can be rewritten directly in terms of the Taylor coefficients a_i and b_i about x = 0 and x = 1 respectively, as:

$$P_{2n+1}(x)=\sum_{i=0}^{n} \{a_i Q_i(x)+(-1)^i b_i Q_i(1-x)\} \qquad (7)$$

4. Solution Of Higher-Order Nonlinear IVP

Many important physical problems for example, electrical circuits and vibrating systems involve IVP whose equations have orders higher than one. This section contains an introduction to the semi - analytic solution of higher-order nonlinear differential equations subject to initial conditions (equations (1) and (2)). The techniques discussed are limited to those that transform a higher-order nonlinear equation into system of first-order nonlinear differential equations in the form equation (3). The object is to find m functions $y_1, \dots ,y_m$ that satisfy each of the differential equations together with all the initial conditions .

New techniques are not required for solving these problems; by relabeling the variables, can reduce a higher-order nonlinear differential equation into a system of first-order nonlinear differential equations and then apply semi-analytic technique.

A general mth-order initial value problem (1) with initial conditions (2) can be converted into a system of equations in form (3) by the following:

Let $u_1(x) = y(x), u_2(x) = y'(x),...,$ and $u_m(x) = y^{(m-1)}(x)$. This produces the first-order system :

$$\frac{du_1}{dx} = \frac{dy}{dx} = u_2$$

$$\frac{du_2}{dx} = \frac{dy'}{dx} = u_3$$

$$\vdots$$

$$\frac{du_{m-1}}{dx} = \frac{dy^{(m-2)}}{dx} = u_m$$

And

$$\frac{du_m}{dx} = \frac{dy^{(m-1)}}{dx} = y^{(m)} = f(x, y, y', \dots y^{(m-1)}) = f(x, u_1, u_2, \dots u_m),$$

With initial conditions:

$u_1(a) = y(a) = \alpha_1$, $u_2(a) = y'(a) = \alpha_2,...,$ $u_m(a) = y^{(m-1)}(a) = \alpha_m$

then solving by apply semi-analytic technique as the following :

first we discuss the method where m = 2 , i.e.:

$$y_1' = dy_1 / dx = f_1(x, y_1, y_2)$$
$$y_2' = dy_2 / dx = f_2(x, y_1, y_2) \qquad (8a)$$

For $0 \le x \le 1$, with the initial conditions:

$$y_1(0) = a_0 ,\ y_2(0) = b_0 , \qquad (8b)$$

where f_i , i = 1, 2 are in general nonlinear functions of their arguments .

The simple idea behind the use of two-point polynomials is to replace y(x) in problem (8a) – (8b), or an alternative formulation of it, by P_{2n+1} which enables any unknown derivatives of y(x) to be computed . The first step therefore is to construct the P_{2n+1}. To do this we need the Taylor coefficients of $y_1(x)$ and $y_2(x)$ respectively about x = 0:

$$y_1 = a_0 + a_1 x + \sum_{i=2}^{\infty} a_i x^i \qquad (9a)$$

$$y_2 = b_0 + b_1 x + \sum_{i=2}^{\infty} b_i x^i \qquad (9b)$$

where $y_1(0) = a_0$, $y_1'(0) = a_1$, … ,$y_1^{(i)}(0) / i! = a_i$, i = 2, 3, ……

and $y_2(0) = b_0$, $y_2'(0) = b_1$, … , $y_2^{(i)}(0) / i! = b_i$, i = 2, 3, ……

then insert the series forms (9a) and (9b) respectively into (8a) and equate coefficients of powers of x .

Also, we need Taylor coefficients of $y_1(x)$ and $y_2(x)$ about x = 1, respectively :

$$y_1 = c_0 + c_1 (x-1) + \sum_{i=2}^{\infty} c_i (x-1)^i \qquad (10a)$$

$$y_2 = d_0 + d_1 (x-1) + \sum_{i=2}^{\infty} d_i (x-1)^i \qquad (10b)$$

where $y_1(1) = c_0$, $y_1'(1) = c_1$, … ,$y_1^{(i)}(1) / i! = c_i$, i = 2, 3, ……

and $y_2(1) = d_0$, $y_2'(1) = d_1$, … ,$y_2^{(i)}(1) / i! = d_i$, i = 2, 3, ……

then insert the series forms (10a) and (10b) respectively into (8a) and equate coefficients of powers of (x – 1) .

The resulting system of equations can be solved using MATLAB version 7.9 to obtain a_i, b_i, c_i and d_i for all i ≥ 2, we see that c_i's and d_i's coefficients depend on indicated unknowns c_0 and d_0.

The algebraic manipulations needed for this process .We are now in a position to construct a $P_{2n+1}(x)$ and $\tilde{P}_{2n+1}$ (x) from (9) and (10) of the form (5) by the following:

$$P_{2n+1}(x) = \sum_{i=0}^{n} \{a_i Q_i(x) + (-1)^i c_i Q_i(1-x)\} \qquad (11a)$$

And

$$\tilde{P}_{2n+1}(x) = \sum_{i=0}^{n} \{b_i Q_i(x) + (-1)^i d_i Q_i(1-x)\} \qquad (11b)$$

Where $Q_i(x)$ defined in (6), we see that (11) have only two unknowns c_0 and d_0.

Now, integrate equation (8a) to obtain:

$$c_0 - a_0 = \int_0^1 f_1(x,y_1,y_2)dx \qquad (12a)$$

$$d_0 - b_0 = \int_0^1 f_2(x,y_1,y_2)dx \qquad (12b)$$

Use P_{2n+1} and $\tilde{P}_{2n+1}$ as a replacement of y_1 and y_2 respectively in (12).

Since we have only the two unknowns c_0 and d_0 to compute for any n we only need to generate two equations from this procedure as two equations are already supplied by (12) and initial condition (8b). Then solve this system of algebraic equations using MATLAB version 7.9 to obtain c_0 and d_0 ,so insert it into (11) thus (11) represents the solution of (8) .

Extensive computations have shown that this generally provides a more accurate polynomial representation for a given n.

Use the same manner to solve in general the system of more than two equations.

Now consider the following example illustrate suggested method where the results are presented in tables and figures for comparison solutions and errors between P_9 and exact, also, between P_9 , exact , RK4 , RK- Butcher, DTM (Differential Transformation method) to assign effectiveness and accuracy of the suggested method.

Example

Consider the following 4 rth order nonlinear initial value problem :

$y^{(4)} = e^{-x} y^2$, $0 \le x \le 1$, subject to the IC 's:

y(0) = y'(0) = y''(0) = y'''(0) = 1

The exact solution given in [10]: $y(x) = e^x$

Rewrite the 4 rth order IVP as a system of first order differential equations:

$y'_1 = y_2$ $\qquad y_1(0) = 1$

$y'_2 = y_3$ $\qquad y_2(0) = 1$

$y'_3 = y_4$ $\qquad y_3(0) = 1$

$y'_4 = e^{-x} y_1^2$ $\qquad y_4(0) = 1$

From equations (5) and (6) we have:

$P_9 = 0.00000460685x^9 + 0.00002072029x^8 + 0.0002027435x^7 + 0.0013866097x^6 + 0.0083338045\ x^5 + 0.0416666667\ x^4 + 0.1666666667\ x^3 + 0.5x^2 + x + 1.0$

$\tilde{P}_9 = 0.000004712x^9 + 0.00002025767x^8 + 0.0002034583\ x^7 + 0.0013862208x^6 + 0.00833379401\ x^5 + 0.0416666667\ x^4 + 0.1666666667\ x^3 + 0.5\ x^2 + x + 1.0$

$$T_9 = 0.0000046068\ x^9 + 0.00002167739x^8 + 0.0001989157\ x^7 + 0.0013919606x^6 + 0.0083311453\ x^5 + 0.04166666667\ x^4 + 0.1666666667\ x^3 + 0.5\ x^2 + x + 1.0$$

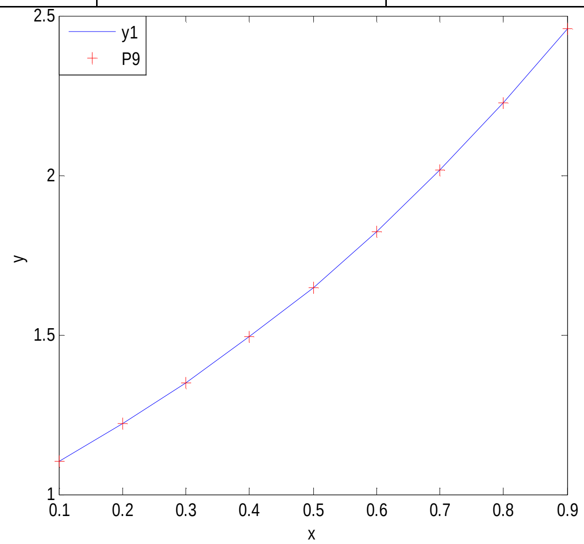
$$\tilde{T}_9 = -0.00002188043x^9 + 0.000139923x^8 - 0.00000162427x^7 + 0.0015455729x^6 + 0.0082861224\ x^5 + 0.04166666667\ x^4 + 0.16666666667\ x^3 + 0.5\ x^2 + x + 1.0$$

The result of the methods for n = 4 given in (Table 1).

This problem was studied by [10] but applying the DTM, RK4 and RK-Butcher, results are summarized in (Table 2) that represents the comparison between solution and errors for using the methods for solving the problem. (Figure 1) represents comparison between exact solution and semi- analytic method, (Figure 2) represents the comparison between different above methods .

Table 1: The result of the method for n = 4

		P_9	$\tilde{P}_9$	T_9	$\tilde{T}_9$
b_{10}		2.718281818221559456	2.718281818221559456	2.718281818221559456	2.718281818221559456
b_{20}		2.718281776141231591	2.718281776141231591	2.718281776141231591	2.718281776141231591
b_{30}		2.718281639144193335	2.718281639144193335	2.718281639144193335	2.718281639144193335
b_{40}		2.718281447154106455	2.718281447154106455	2.718281447154106455	2.718281447154106455
X	**y_1:exaxt**	P_9	$\tilde{P}_9$	T_9	$\tilde{T}_9$
0.1	1.105171	1.10517091807847	1.10517091807805	1.10517091805686	1.10517091774135
0.2	1.221403	1.22140275821099	1.22140275819076	1.22140275765595	1.22140275080204
0.3	1.349859	1.34985880777361	1.34985880759261	1.34985880443814	1.34985877039453
0.4	1.491825	1.49182469800689	1.49182469720188	1.49182468704941	1.49182459719638
0.5	1.648721	1.64872127103808	1.64872126861587	1.64872124537884	1.64872108205205
0.6	1.822119	1.82211880027067	1.82211879460860	1.82211875206183	1.82211852332721
0.7	2.013753	2.01375270629145	2.01375269521319	2.01375262882012	2.01375236660861
0.8	2.225541	2.22554092548648	2.22554090649895	2.22554081463729	2.22554055709926
0.9	2.459603	2.45960310527087	2.45960307587015	2.45960295986726	2.45960273177602



x	y_1:exact	RK4 solution	RK-Butcher solution	DTM solution	P_9 by using Osculatory
0.1	1.105171	1.105171	1.105167	1.10517	1.10517091807847
0.2	1.221403	1.221403	1.221333	1.221398	1.22140275821099
0.3	1.349859	1.349858	1.3495	1.34985	1.34985880777361
0.4	1.491825	1.491824	1.490667	1.491813	1.49182469800689
0.5	1.648721	1.648721	1.645833	1.648707	1.64872127103808
0.6	1.822119	1.822118	1.816	1.822104	1.82211880027067
0.7	2.013753	2.013751	2.002167	2.013741	2.01375270629145
0.8	2.225541	2.225539	2.205333	2.225537	2.22554092548648
0.9	2.459603	2.459601	2.4265	2.459612	2.45960310527087

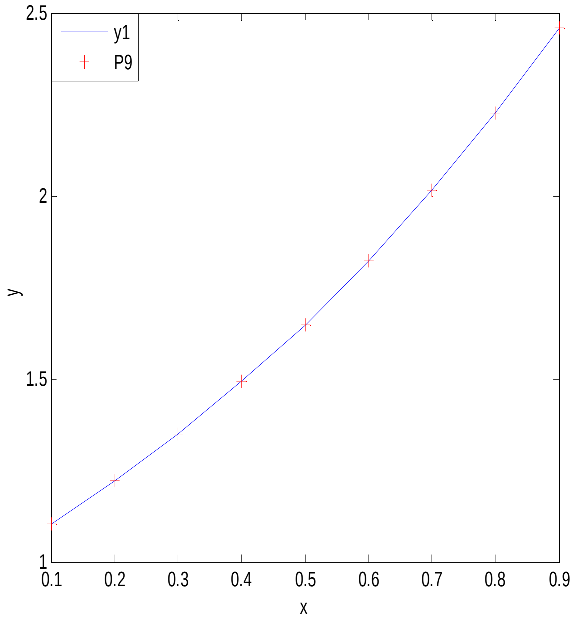


Figure 1: Comparison between the exact solution y_1 and semi-analytic solution P_9.

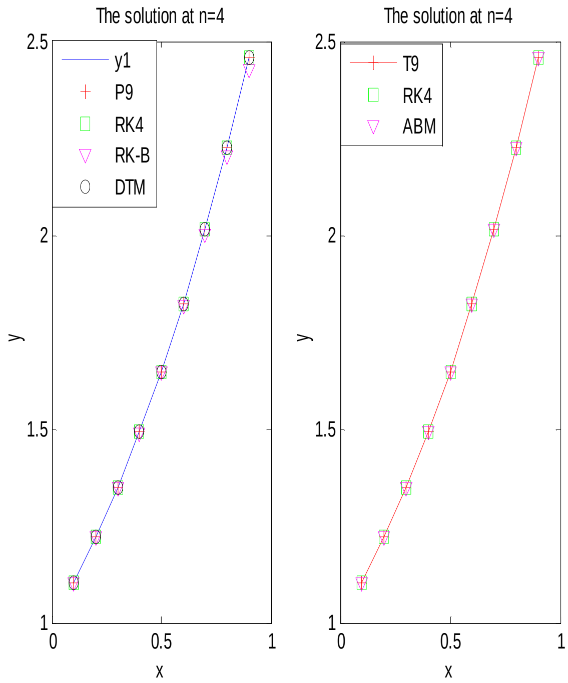


Figure 2: Comparison between different methods

Table 2: A comparison between semi-analytic method and other methods

X	y_1:exact	RK4 error	RK-Butcher error	DTM error	P_9 by using Osculatory error
0.1	1.1051	1.192093e-	5.364418	1.192093	8.19215300040809
0.2	1.2214	2.384186	7.05719	4.410744	2.41789009924886
0.3	1.3498	4.768372	3.601313	8.46386	1.92226389916783
0.4	1.4918	5.960464	1.15943	1.192093	3.01993109852460
0.5	1.6487	7.252557	2.889514	1.430511	2.71038079979746
0.6	1.8221	1.072884	6.120682	1.478195	1.99729329963815
0.7	2.0137	1.430511	1.1588 e-	1.144409	2.93708549925497
0.8	2.2255	2.145767	2.020979	4.053116	7.45135202429025
0.9	2.4596	2.622604	3.310585	8.583069	1.05270869799056
					S .S.E$_1$= 4.095762648509462e-013

5. Sensitivity Of Solution To The Data

In our study of IVP (1), we have been guided by the four questions: Does IVP (1) have any solution? How many? What are they? How do solutions respond to changes in the data? If the function f and $\partial f/\partial y$ are continuous in some region R in the xy-plane and (x_o, y_o) is a point of R, we gave satisfactory answers to the first, second, and third questions.

In this section we show that these same simple conditions on the data also lead to a satisfactory answer to the last question.
Loosely phrased, the question amounts to this: Is it always possible to find bounds on the determination of the data $\vec{f}$ (x, y) and $\vec{y}_0$ in IVP (1) which will guarantee that the corresponding solution will be within prescribed error bounds over a given x-interval?
If this question can be answered in the affirmative, one consequence is that any "small" enough change in the data of an IVP produces only a "small" change in the solution.

In addressing the last question, it would be extremely helpful to have a formula for the solution of IVP (1) in which the data appear explicitly.
But for general nonlinear differential equations, there rarely is a solution formula for IVP (1) in which the data appear explicitly .

To estimate the change in the solution to IVP(1)as the data $\vec{f}$ (x, y) and $\vec{y}_0$ are modified, we give the following theorem about perturbation estimate :

Theorem 3 [4]

Let the function f in IVP (1) be continuous with $\partial f/\partial y$ in a rectangle R described by the inequalities:

$x_0 \le x \le x_0 + a, |y - y_0| \le b$.

Suppose that $g(x,y)$ and $\partial g(x,y)/\partial y$ are also continuous functions on R and that on some common interval: $x_0 \le x \le x_0 + c$, which $c \le a$, the solution y(x) of IVP(1) and the solution $\tilde{y}$ (x) of the " perturbed " IVP :

$$y' = f(x,y) + g(x,y),\ y(x_0) = \tilde{y}_0 \qquad (13)$$

Both have solution curves which lie in R, then we have the estimate :

$$|y(x) - \tilde{y}(x)| \le |y_0 - \tilde{y}_0| e^{L(x-x_0)} + \frac{M}{L}\left(e^{L(x-x_0)} - 1\right),$$ Where

$$x_0 \le x \le x_0 + c \qquad (14)$$

L and M are any numbers such that

$M \le |g(x,y)|, L \le \left|\frac{\partial f}{\partial y}\right|, all\ (x,y)\ in\ R$ Now, in a position to answer that last of the basic questions , we give the following theorem about continuity in the data .

Theorem 4 [4]

let f ,$\partial f/\partial y$,g and $\partial g/\partial y$ be continuous functions of x and y on the rectangle R defined by: $x_0 \le x \le x_0 + a, |y - y_0| \le b$.

Let ε > 0 be a given error tolerance. Then there exist positive constants $H < b$ and c ≤ a such that the respective solution y(x) and $\tilde{y}$ (x) of the system of IVP :

$$(a)\ y' = f(x,y),\qquad y(x_o) = y_0.. \qquad (15)$$
$$(b)\ y' = f(x,y) + g(x,y)\ \ y(x_0) = \tilde{y}_o$$

Satisfy the inequality:

$$|y(x) - \tilde{y}(x)| \le \varepsilon\ ,\ x_0 \le x \le x_0 + c \qquad (16)$$

For any choice of $\tilde{y}_0$ for which $|y_0 - \tilde{y}_0| \le H$

6. Conclusions

A remarkable advantage of the semi-analytic technique for solving high order ordinary IVP is that it is easily implemented and gives a result with high accuracy. The high accuracy of the method is confirmed by example and the suggested method compared with conventional methods via example and is shown to be that seems to converge faster and more accurately than the conventional methods.
Another advantage of suggested method is that it gives the approximate solution on the continuous finite domain whereas other numerical techniques provide the solution on discrete only.

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