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On The Mathematical Model of Two- Prey and Two-Predator Species

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Abstract

In this work, we study two species of predator with two species of prey model, where the two species of prey live in two diverse habitats and have the ability to group-defense. Only one of the two predators tends to switch between the habitats. The mathematical model has at most 13 possible equilibrium points, one of which is the point of origin, two are axial, tow are interior points and the others are boundary points. The model with $n = 2$, where n is the switching index, is discussed regarding the boundedness of its solutions and the local stability of its equilibrium points. In addition, a basin of attraction was created for the interior point. Finally, three numerical examples were given to support the theoretical results.

Keywords: Prey-Predator; Switching; Group defense; Equilibrium point; Local stability; Basin of attraction.

حول النموذج الرياضي لنوعين من الفرائس ونوعين من المفترسات

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الخلاصه

في هذا العمل ، ندرس نموذج لنوعين من الفرائس مع نوعين من المفترسات ، حيث يعيش النوعان من الفرائس في موئلين مختلفين ولديهما القدرة على الدفاع الجماعي. نوع واحد فقط من الحيوانات المفترسة يميل إلى التبديل بين الموئلين. للنموذج الرياضي ثلاثة عشر نقطة توازن ممكنة على الأكثر ، إحداها هي نقطة الأصل ، اثنتين محوريتين ، نقطتين داخليتين والأخريات نقاط حدودية. تمت مناقشة النموذج من حيث محدودية حلوله والاستقرار المحلي لنقاط التوازن في حالة $n = 2$ ، حيث أن n هو مؤشر التبديل. بالإضافة إلى ذلك ، تم إنشاء حوض جذب للنقطة الداخلية. وأخيرًا ، أعطيت ثلاثة أمثلة عددية لدعم النتائج النظرية.

1. INTRODUCTION

Predators feed themselves for some time in a habitat that contains adequate food, that is, there exists a large number (large size) of prey or the prey has no ability to defend itself. When the food is reduced, in other words, there is a small number (small size) of prey, the predators move to another habitat, which is the so-called the switching phenomenon [1, 2]. While, the Group-defense means that a prey can defend itself by attacking the predator collectively instead of waiting and giving the

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predator a chance to kill it. In 1920,Volera proposed a mathematical model for one predator and its prey. Later, other mathematical models have been proposed by many researchers, which deal with the relationship between prey and predators. Some of these models provided two prey with one predator [3, 4, 5], one prey with two predators [6], two prey with two predators [4, 7, 8, 9, 10], and the insect predation [11].

In one study [4], Bhattacharyya and Mukhopadhyay proposed and studied two models of prey–predator . The first model was considered with a prey group defense, while the second was considered without a prey group defense. The two prey species are supposed to live in two diverse habitats, while the two predator species tend to switch between habitats. In another investigation [12], the first model (with a prey group defense) was expanded by the following mathematical model:

$$\begin{aligned}\dot{x}_1 &= x_1\left[g_1\left(1-\frac{x_1}{k_1}\right)-\frac{\alpha_1 x_2^n y_1}{x_1^n+x_2^n}-\beta y_2\right],\\ \dot{x}_2 &= x_2\left[g_2\left(1-\frac{x_2}{k_2}\right)-\frac{\alpha_2 x_1^n y_1}{x_1^n+x_2^n}\right],\\ \dot{y}_1 &= y_1\left[-\mu_1+\frac{\delta_1 x_1 x_2^n}{x_1^n+x_2^n}+\frac{\delta_2 x_1^n x_2}{x_1^n+x_2^n}\right],\\ \dot{y}_2 &= y_2[-\mu_2+\gamma x_1],\end{aligned} \tag{1.1}$$

where $x_i$represents the density of the prey in their two divers' habitats; y_i represents the density of the predators. The two species of prey are supposed to grow logistically with a certain growth rate g_i and carrying environmental capacity $k_i$to x_i; α_i represents the rate of predation by the predator y_1, on prey x_i; β represents the rate of predation by the predator y_2 on x_1; μ_i represents the mortality rate of predators y_i such that $i = 1,2$, and δ_1, δ_2 and γ are the corresponding conversion rates. The two functions $\alpha_1 x_2^n y_1(x_1^n+x_2^n)^{-1}$and $\alpha_1 x_1^n y_1(x_1^n+x_2^n)^{-1}$explain the behavior of predator switching y_1. This Model includes two prey and two predators. The phenomenon of switching occurs only with one of the two predators, while the two prey species live in two diverse habitats and have the ability of group-defense against one of the two predators.

The model (1.1) was discussed for $n = 1$, as the switching index in the last mentioned work [12]. In this work, we discuss this model (1.1) for $n = 2$. At most, thirteen possible equilibrium points are found for the system (1.1). One of these equilibrium points is the origin point, which means the absence of all species, whereas two equilibrium points are axial (the absence of three species), two are interior points (coexistent state or normal steady state), and the last equilibrium points are boundary. Therefore, there are three states; one is the absence of two predators, another is the absence of one prey and one predator, while the others are the absence of one predator for each state.

2. THE MATHEMATICAL MODEL

Consider the model (1.1), that was proposed in the aforementioned study [12], when $n = 2$, the model (2.1) takes the form:

$$\begin{cases}\dot{x}_1 = x_1\left[g_1\left(1-\frac{x_1}{k_1}\right)-\frac{\alpha_1 x_2^2 y_1}{x_1^2+x_2^2}-\beta y_2\right],\\ \dot{x}_2 = x_2\left[g_2\left(1-\frac{x_2}{k_2}\right)-\frac{\alpha_2 x_1^2 y_1}{x_1^2+x_2^2}\right],\\ \dot{y}_1 = y_1\left[-\mu_1+\frac{\delta_1 x_1 x_2^2}{x_1^2+x_2^2}+\frac{\delta_2 x_1^2 x_2}{x_1^2+x_2^2}\right],\\ \dot{y}_2 = y_2[-\mu_2+\gamma x_1].\end{cases} \tag{2.1}$$

3. BOUNDEDNESS OF THE SOLUTIONS

In this section, we prove that all the positive solutions of system (2.1) that start with $\mathbb{D}$ are bounded Let

$$\mathbb{D} \coloneqq \{(x_1, x_2, y_1, y_2) \in \mathcal{R}^4, 0 < x_i < k_i, y_i > 0, i = 1,2\}. \tag{3.1}$$

Lemma1: If $\delta_i \leq \alpha_i, i = 1,2,\ \gamma \leq \beta$, then all the positive solutions of the system (2.1) that start in $\mathbb{D}$ are bounded.
Proof: Let the real function:
$u(t) = x_1(t) + x_2(t) + y_1(t) + y_2(t)$, be a positive definitive on $\mathbb{D}$.
From (2.1), we have:

$$\dot{u} = x_1 g_1\left(1 - \frac{x_1}{k_1}\right) + x_2 g_2\left(1 - \frac{x_2}{k_2}\right) - (y_1\mu_1 + y_2\mu_2) + (\gamma - \beta)x_1 y_2 + \frac{\left((\delta_1 - \alpha_1)x_1 x_2^2 + (\delta_2 - \alpha_2)x_1^2 x_2\right)y_1}{x_1^2 + x_2^2}.$$

If $0 < \rho \leq max\{\mu_1, \mu_2\}$, then we obtain:

$$\dot{u} + \rho u = x_1 g_1\left(1 - \frac{x_1}{k_1} + \frac{\rho}{g_1}\right) + x_2 g_2\left(1 - \frac{x_2}{k_2} + \frac{\rho}{g_2}\right) + (y_1(\rho - \mu_1) + y_2(\rho - \mu_2)) + \frac{\left((\delta_1 - \alpha_1)x_1 x_2^2 + (\delta_2 - \alpha_2)x_1^2 x_2\right)y_1}{x_1^2 + x_2^2}.$$

It is clear that

$$\dot{u} + \rho u \leq \frac{x_1}{k_1}(k_1 g_1 - g_1 x_1 + k_1\rho) + \frac{x_2}{k_2}(k_2 g_2 - g_2 x_2 + k_2\rho) <$$
$$< \frac{x_1}{k_1}(k_1 g_1 + k_1\rho) + \frac{x_2}{k_2}(k_2 g_2 + k_2\rho) <$$
$$< k_1(g_1 + \rho) + k_2(g_2 + \rho)$$

This leads to $0 \leq u(t) \leq \frac{\alpha}{\rho} + u(0)e^{-\rho t}$, and for t→∞,

$$0 \leq u(t) \leq \frac{\alpha}{\rho},$$

where $\alpha = k_1(g_1 + \rho) + k_2(g_2 + \rho)$.
So that, we obtain that the positive solutions of the system(2.1), with initial conditions that belong to $\mathbb{D}$, and satisfy $\delta \leq \alpha_1 + \alpha_2, \gamma \leq \beta$, are bounded.

4. LOCAL STABILITY OF EQUILIBRIUM POINTS

It is well known that the model (2.1) has at most thirteen equilibrium points, four of them always exist, but the existence of the other depends on the values of the parameters that defined the model and that will be shown throughout this section. In this section, we will discuss the local stability of the so-called equilibrium points.

I. The equilibrium point $P_1 = (0,0,0,0)$ of the model (2.1) always exists. The following lemma shows that the $P_1 = (0,0,0,0)$ is not stable:
Lemma 2: If $x_1(0) > 0$ or $x_2(0) > 0$, then there is no trajectory of the system (2,1)can converge to (0,0,0,0), which means that $P_1 = (0,0,0,0)$ is unstable.
Proof: Let $x_1(0) > 0, (x_1, x_2, y_1, y_2) \to (0,0,0,0)$, as t → ∞,then

$$\frac{d}{dt}(\ln x_1) \to g_1.$$

It is clear that
$\frac{d}{dt}(\ln x_1) \geq \frac{g_1}{2}$. If$(x_1, x_2, y_1, y_2) \to (0,0,0,0)$ as $t \to \infty$,
then there exists $t_0 > 0$, such that;
$x_1(t_0) > 0 x_1(t) \geq x_1(t_0) exp\left(\frac{g_1(t - t_0)}{2}\right) \to \infty$, as $t \to \infty$. So $x_1 \to \infty$.
In the same way , if $x_1(0) > 0, x_1 \to \infty$, then there is no a trajectory to the system (2.1) that converges to (0,0,0,0)
Hence, the equilibrium point $P_1 = (0,0,0,0)$ is unstable.
II. The equilibrium point**s** $P_2 = (k_1, 0,0,0)$ and $P_3 = (0, k_2, 0,0)$ of the system (2.1) always exist. The Jacobian matrix of the system (2.1) at the equilibrium points P_2 is

$$J(P_2) = \begin{bmatrix} -g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & -\mu_1 & 0 \\ 0 & 0 & 0 & \gamma k_1 - \mu_2 \end{bmatrix},$$

And the Jacobian matrix of the system (2.1)at$P_3$is

$$J(P_3)=\begin{bmatrix} g_1 & 0 & 0 & 0\\ 0 & -g_2 & 0 & 0\\ 0 & 0 & -\mu_1 & 0\\ 0 & 0 & 0 & -\mu_2 \end{bmatrix}.$$

It is clear that the sets of eigenvalues of the two matrices $J(P_2)$ and $J(P_3)$ are:
$S\big(J(P_2)\big)=\{-g_1, g_2, -\mu_1, \gamma k_1-\mu_2\}$ and
$S\big(J(P_3)\big)=\{g_1, -g_2, -\mu_1, -\mu_2\}$, respectively.
Note that $S\big(J(P_i)\big), i=2,3,$ has at least one positive eigenvalue and at least two negative eigenvalues, which means that the two equilibrium points are saddle points (unstable).

III. The equilibrium point $P_4=(k_1,k_2,0,0)$ always exists. The Jacobian matrix of the system (2.1) at P_4 is:

$$J(P_4)=\begin{bmatrix} -g_1 & 0 & \alpha_1 k_1 k_2^2(k_1^2+k_2^2)^{-1} & -\beta k_1\\ 0 & -g_2 & \alpha_2 k_1^2 k_2 (k_1^2+k_2^2)^{-1} & 0\\ 0 & 0 & (\delta_1 k_1 k_2^2+\delta_2 k_1^2 k_2)(k_1^2+k_2^2)^{-1}-\mu_1 & 0\\ 0 & 0 & 0 & \gamma k_1-\mu_2 \end{bmatrix}$$

And the set of the eigenvalues of the characteristic equations of the matrix $J(P_4)$ is:

$$S(P_4)=\left\{-g_1, -g_2, \frac{(\delta_1 k_1 k_2^2+\delta_2 k_1^2 k_2)}{k_1^2+k_2^2}-\mu_1, \gamma k_1-\mu_2\right\}.$$

Due to the stability criteria given in the above mentined study [12] by Routh-Hurwitz, the equilibrium point $P_4=(k_1,k_2,0,0)$ should be locally asymptotically stable, if it satisfies the following two conditions:

$$\begin{cases}\delta_1 \mathrm{k}_1 k_2^2+\delta_2 k_1^2 k_2<(\mathrm{k}_1^2+\mathrm{k}_2^2)\mu_1\\ k_1\gamma<\mu_2\end{cases} \quad (4.1)$$

IV.The fifth equilibrium point is $P_5=(\breve{x}_1,0,0,\breve{y}_2)$, **which** exists under the condition $\mu_2<\gamma k_1$.
where:

$$\begin{cases}\breve{x}_1=\dfrac{\mu_2}{\gamma},\\ \breve{y}_2=\dfrac{g_1}{\beta}\left(1-\dfrac{\mu_2}{\gamma k_1}\right),\end{cases} \quad (4.2)$$

The characteristic equation of the Jacobian matrix of (2.1) near the point $P_5=(\breve{x}_1,0,0,\breve{y}_2)$ is:
$(\mu_1+\lambda)(g_2-\lambda)[k_1\lambda^2+(g_1\breve{x}_1+k_1\mu_2)\lambda+g_1\breve{x}_1\mu_2+k_1\beta\gamma\breve{x}_1\breve{y}_2]=0.$
So that, the eigenvalues are as follows:
$\lambda_1=-\mu_1<0, \lambda_2=g_2>0,$ and λ_3,λ_4 are the solutions of the second degree equation:
$k_1\lambda^2+(g_1\breve{x}_1+k_1\mu_2)\lambda+g_1\breve{x}_1\mu_2+k_1\beta\gamma\breve{x}_1\breve{y}_2=0.$
Now, we have that $\lambda_1<0$ and $\lambda_2>0$, which means that the equilibrium $P_5=(\breve{x}_1,0,0,\breve{y}_2)$ is a saddle point.

V.The equilibrium point $P_6=(\breve{x}_1,\breve{x}_2,0,\breve{y}_2)$, exists if the condition $\mu_2<\gamma k_1$ is satisfied.
where:

$$\begin{cases}\breve{x}_1=\dfrac{\mu_2}{\gamma}, \qquad \breve{x}_2=k_2,\\ \breve{y}_2=\dfrac{g_1}{\beta}\left(1-\dfrac{\mu_2}{\gamma k_1}\right),\end{cases} \quad (4.3)$$

The characteristic equation of the Jacobian matrix of (2.1) near the point $P_6=(\breve{x}_1,\breve{x}_2,0,\breve{y}_2)$ is:
$(A_5-\lambda)(g_2+\lambda)[\lambda^2-A_1\lambda-A_3A_6]=0,$
where:
$A_1=-\dfrac{g_1\breve{x}_1}{k_1}, A_2=-\alpha_1 D\breve{x}_2, A_3=-\beta\breve{x}_1, A_4=-\alpha_2 D\breve{x}_2,$
$A_5=(\delta_1\breve{x}_2+\delta_2\breve{x}_1)\breve{D}-\mu_1, A_6=\gamma\breve{y}_2.$
$\breve{D}=\dfrac{\breve{x}_1\breve{x}_2}{\breve{x}_1^2+\breve{x}_2^2}.$
So that, the eigenvalues are:
$\lambda_1=A_5=(\delta_1\breve{x}_2+\delta_2\breve{x}_1)\breve{D}-\mu_1, \lambda_2=-g_2<0,$

$$\lambda_3 = \frac{-g_1\breve{x}_1 - \sqrt[2]{(g_1\breve{x}_1)^2 - k_1^2\gamma\breve{y}_2\beta\breve{x}_1}}{2k_1},$$
$$\lambda_4 = \frac{-g_1\breve{x}_1 + \sqrt[2]{(g_1\breve{x}_1)^2 - k_1^2\gamma\breve{y}_2\beta\breve{x}_1}}{2k_1},$$
It is clear that λ_2, λ_3 and λ_4 are negatives.
Now, we have that $\lambda_2 < 0$, λ_3 and λ_4 are negatives.
If $(\delta_1\breve{x}_2 + \delta_2\breve{x}_1)\frac{\breve{x}_1\breve{x}_2}{\breve{x}_1^2+\breve{x}_2^2} < \mu_1$, then $\lambda_1 < 0$.
Based on the criteria for the stability of Routh-Hurwitz [13], the equilibrium point $P_6 = (\breve{x}_1, \breve{x}_2, 0, \breve{y}_2)$should be locally asymptotically stable, if it satisfies the following condition:
$$(\delta_1\breve{x}_2 + \delta_2\breve{x}_1)\frac{\breve{x}_1\breve{x}_2}{\breve{x}_1^2 + \breve{x}_2^2} < \mu_1.$$
VI.The existence of the equilibrium point $P_7 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, 0)$,is guaranteed under the condition:
$\mu_1(1 + \tilde{x}^2) < k_1(\delta_1 + \delta_2\tilde{x})$,
where
$$\begin{cases} \tilde{x}_1 = \dfrac{\mu_1(1 + \tilde{x}^2)}{\delta_1 + \delta_2\tilde{x}}, \qquad \tilde{x}_2 = \dfrac{\tilde{x}_1}{\tilde{x}}, \\ \tilde{y}_1 = \dfrac{g_1}{\alpha_1}\left(1 - \dfrac{\tilde{x}_1}{k_1}\right)(1 + \tilde{x}^2) = \dfrac{g_2}{\alpha_2}\left(1 - \dfrac{\tilde{x}_2}{k_2}\right)\dfrac{(1 + \tilde{x}^2)}{\tilde{x}^2}, \end{cases} \quad (4.4a)$$

and $\tilde{x}$ is a real positive root of thee equations:
$\pi_0\tilde{x}^5 + \pi_1\tilde{x}^4 + \pi_2\tilde{x}^3 + \pi_3\tilde{x}^2 + \pi_4\tilde{x} + \pi_5 = 0, (4.4b)$
such that:
$\pi_0 = -g_1k_2\alpha_2\mu_1, \quad \pi_1 = g_1k_1k_2\alpha_2\delta_2,$
$\pi_2 = g_1k_2\alpha_2(k_1\delta_1 - \mu_1), \quad \pi_3 = -g_2k_1\alpha_1(k_2\delta_2 - \mu_1),$
$\pi_4 = -g_2k_1k_2\alpha_1\delta_1, \quad \pi_5 = g_2k_1\alpha_1\mu_1.$

It is clear that $\pi_0 < 0$ and $\pi_5 > 0$. According to the criteria of Routh-Hurwitz [2], not all the roots of equation $(4.4b)$ are negatives. So that it is possible to obtain some positive real roots. However, the numerical study for three examples presented in section 6 shows that, in the first example, equation (4.4b) has only one real positive root, while in the second example, equation (4.4b) does not have any real positive root, and, in the third example, equation (4.4b) has three real positive roots.
Therefore, $P_7 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, 0)$ exists under the two conditions:
$$\begin{cases} \tilde{x} \text{ is positive}, \\ \mu_1(1 + \tilde{x}^2) \le k_1(\delta_1 + \delta_2\tilde{x}), \end{cases}$$
So that if m where $0 \le m \le 5$, is the number of the real positive roots $\tilde{x}$ of the equation $(4.4b)$, that satisfies $\mu_1(1 + \tilde{x}^2) \le k_1(\delta_1 + \delta_2\tilde{x})$, then we have m equilibrium points of the type P_7.
The characteristic equation of the Jacobian matrix of (2.1) at the point $P_7 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, 0)$ is:
$(\gamma\tilde{x}_1 - \mu_2 - \lambda)(\lambda^3 + \tilde{F}_1\lambda^2 + \tilde{F}_2\lambda + \tilde{F}_3) = 0,$
such that:
$$\tilde{F}_1 = \sum_{i=1}^{2}\left(\frac{g_i\tilde{x}_i}{k_i} - R\alpha_i\right),$$
$$\tilde{F}_2 = \frac{g_1g_2\tilde{x}_1\tilde{x}_2}{k_1k_2} + \sum_{i=1}^{2}\left(\frac{\tilde{y}_1M_i\alpha_i\tilde{x}_i}{(1 + \tilde{x}^2)} - R\frac{g_i\tilde{x}_i\alpha_{(3-i)}}{k_i}\right)$$
$$\tilde{F}_3 = \sum_{i=1}^{2}\left(\frac{\tilde{x}_1\tilde{x}_2\tilde{y}_1g_i\alpha_{(3-i)}}{k_i(1 + \tilde{x}^2)} - 2\frac{\alpha_1\alpha_2R\tilde{y}_1\tilde{x}_i}{k_i(1 + \tilde{x}^2)}\right)M_i$$

$$R = \frac{2\tilde{x}^2\tilde{y}_1}{(1 + \tilde{x}^2)^2},$$

$M_1 = \frac{\delta_1[1-\tilde{x}^2]+2\delta_2\tilde{x}}{(1+\tilde{x}^2)^2}\tilde{y}_1,$
$M_1 = \frac{2\delta_1\tilde{x}^3+\delta_2[\tilde{x}^4-\tilde{x}^2]}{(1+\tilde{x}^2)^2}\tilde{y}_1$

According to the criteria of Routh-Hurwitz [2] of the system (2.1), if $P_7 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, 0)$ exists, then the necessary and sufficient conditions for $P_7 = (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, 0)$ to be locally asymptotically stable are:

$$\begin{cases} \mu_2 - \gamma\tilde{x}_1 > 0, \\ \tilde{F}_1 > 0, \\ \tilde{F}_3 > 0, \\ \tilde{F}_1\tilde{F}_2 > \tilde{F}_3. \end{cases}$$

VII.THE INTERIOR EQUILIBRIUM POINT

Consider the interior equilibrium point $P_8 = (\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$, such that:

$$\begin{cases} \bar{x}_1 = \frac{\mu_2}{\gamma}, \quad \bar{x}_2 = \sigma, \\ \bar{y}_1 = \frac{g_2(k_2-\sigma)(\mu_2^2+\gamma^2\sigma^2)}{\alpha_2 k_2 \mu_2^2} \\ \bar{y}_2 = \frac{g_1(\gamma k_1 - \mu_2)}{\gamma\beta k_1} - \frac{\gamma^2\sigma^2 g_2(k_2-\sigma)}{\beta k_2 \mu_2^2}, \end{cases} \quad (4.5a)$$

where σ is a positive root of the equation:

$$(\gamma\mu_1 - \delta_1\mu_2)\gamma\sigma^2 - \delta_2\mu_2^2\sigma + \mu_1\mu_2^2 = 0. \quad (4.5b)$$

Now, if $\gamma\mu_1 = \delta_1\mu_2$, then $\sigma = \frac{\mu_1}{\delta_2}$ and if $\gamma\mu_1 < \delta_1\mu_2$, then equation (4.5b) has only one positive root. So that equation(4.5b) has only one positive root, if $\gamma\mu_1 \le \delta_1\mu_2$.

Otherwise, with the condition $\delta_2^2\mu_2^2 > 4\gamma\mu_1(\gamma\mu_1 - \delta_1\mu_2)$, equation$(4.5b)$ has exactly two positive roots.

So that, only one interior equilibrium point of the type $P_8 = (\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$ is obtained if:

$$\begin{cases} \gamma\mu_1 \le \delta_1\mu_2, \\ 0 < k_1 g_2 \gamma^3\sigma^2(k_2-\sigma) < k_2 g_1 \mu_2^2(\gamma k_1 - \mu_2). \end{cases}$$

While there are exactly two interior equilibrium points of the type $P_8 = (\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$ if:

$$\begin{cases} \gamma\mu_1 \le \delta_1\mu_2, \\ \delta_2^2\mu_2^2 > 4\gamma\mu_1(\gamma\mu_1 - \delta_1\mu_2), \\ 0 < k_1 g_2 \gamma^3\sigma^2(k_2-\sigma) < k_2 g_1 \mu_2^2(\gamma k_1 - \mu_2). \end{cases}$$

It is not too difficult to show that the Jacobian matrix of the system (2.1) at the point $P_8 = (\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$, can be written as:

$$J(P_8) = \begin{bmatrix} \pi\alpha_1 - \frac{g_1\bar{x}_1}{k_1} & -\pi\alpha_1\bar{x}_1\bar{x}_2^{-1} & -\varphi\alpha_1\bar{x}_2 - \beta\bar{x}_1 \\ -\pi\alpha_2\bar{x}_1^{-1}\bar{x}_2 & \pi\alpha_2 - \frac{g_2\bar{x}_2}{k_2} & -\varphi\alpha_2\bar{x}_1 & 0 \\ N_1 & N_2 & 0 & 0 \\ \gamma\bar{y}_2 & 0 & 0 & 0 \end{bmatrix},$$

Such that

$\pi = \frac{2\bar{x}_1^2\bar{x}_2^2\bar{y}_1}{(\bar{x}_1^2+\bar{x}_2^2)^2}, \varphi = \frac{-\bar{x}_1\bar{x}_2}{(\bar{x}_1^2+\bar{x}_2^2)}$

$N_1 = \frac{\delta_1[\bar{x}_2^4 - \bar{x}_1^2\bar{x}_2^2] + n\delta_2\bar{x}_1\bar{x}_2^3}{(\bar{x}_1^2+\bar{x}_2^2)^2}\tilde{y}_1,$

$N_1 = \frac{2\delta_1\bar{x}_1^3\bar{x}_2 + \delta_2[\bar{x}_1^4 - \bar{x}_1^2\bar{x}_2^2]}{(\bar{x}_1^2+\bar{x}_2^2)^2}\tilde{y}_1$

The characteristic equation of the Jacobian matrix of (2.1) at the point $P_8 = (\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$is:

$\lambda^4 + \bar{e}_1\lambda^3 + \bar{e}_2\lambda^2 + \bar{e}_3\lambda + \bar{e}_4 = 0,$

where

$$\bar{e}_1 = \sum_{i=1}^{2}\left[\frac{g_i\bar{x}_i}{k_i} - \pi\alpha_1\right],$$

$$\bar{e}_2 = \frac{g_1 g_2 \bar{x}_1\bar{x}_2}{k_1 k_2} + \beta\gamma\bar{x}_1\bar{y}_2 - \sum_{i=1}^{2}\left[\pi\alpha_{3-i}\frac{g_i\bar{x}_i}{k_i} - \varphi\bar{x}_{3-i}N_i\alpha_i\right]$$

$$\bar{e}_3 = \beta\gamma\bar{y}_2\bar{x}_1\left(\frac{g_2\bar{x}_2}{k_2} - \pi\alpha_2\right) + \varphi\sum_{i=1}^{2}\left[\alpha_{3-i}\frac{g_i\bar{x}_i^2}{k_i} - \pi\alpha_1\alpha_2 N_i\left(\bar{x}_{3-i} + \bar{x}_i^2\bar{x}_{3-i}^{-1}\right)\right]$$

$$\bar{e}_4 = -\varphi\alpha_1 N_2\beta\gamma\bar{x}_1^2\bar{y}_2$$

The necessary and sufficient conditions for $P_8 = (\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$, to be locally asymptotically for the system (2.1) are:

$$\begin{cases} \bar{e}_i > 0, \quad i = 1,2,3,4 \\ \bar{e}_1\bar{e}_2 - \bar{e}_3 > 0, \\ \bar{e}_3(\bar{e}_1\bar{e}_2 - \bar{e}_3) - \bar{e}_4\bar{e}_1^2 > 0. \end{cases}$$

Next, we will define an appropriate Lyapunov function to create a basin of attractions for $P_8 = (\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$.

THEOREM 1: Assume that the equilibrium point P_8 of (2.1) is locally asymptotically stable, and $\bar{x}_i \geq k_i$, $\delta_0\bar{x}_0 - \mu_1 \geq 0$, where ,

$\delta_0 = \min\{\delta_1, \delta_2\}$, $\bar{x}_0 = \min\{\bar{x}_1, \bar{x}_2\}$,

Then, a basin of attraction of P_8 can be created by the set:

$\bar{\mathbb{B}} = \{(x_1, x_2, y_1, y_2): x_i \geq \bar{x}_i, y_i \leq \bar{y}_2, i = 1,2\}$.

Proof: The function

$$V(x_1, x_2, y_1, y_2) = \sum_{i=1}^{2}\left(x_i - \bar{x}_i - \bar{x}_i \ln\frac{x_i}{\bar{x}_i} + y_i - \bar{y}_i - \bar{y}_i \ln\frac{y_i}{\bar{y}_i}\right)$$

is positive definite .

$$\dot{V}(x_1, x_2, y_1, y_2) = \sum_{i=1}^{2}\left(\dot{x}_i\left(1 - \frac{k_1}{x_i}\right) + \dot{y}_i\left(1 - \frac{y_i}{\bar{y}_i}\right)\right) =$$

$$= \sum_{i=1}^{2}[(x_i - \bar{x}_i)\mathbb{G}_i(x_1, x_2) + (y_i - \bar{y}_i)\mathbb{G}_{i+2}(x_1, x_2)],$$

Such that

$$\mathbb{G}_1(x_1, x_2) = \left[g_1\left(1 - \frac{x_1}{k_1}\right) - \frac{\alpha_1 x_2^2 y_1}{x_1^2 + x_2^2} - \beta y_2\right], \mathbb{G}_2(x_1, x_2) = \left[g_2\left(1 - \frac{x_2}{k_2}\right) - \frac{\alpha_2 x_1^2 y_1}{x_1^2 + x_2^2}\right],$$

$$\mathbb{G}_3(x_1, x_2) = \left[-\mu_1 + \frac{\delta_1 x_1 x_2^2}{x_1^2 + x_2^2} + \frac{\delta_2 x_1^2 x_2}{x_1^2 + x_2^2}\right],$$

and $\mathbb{G}_4(x_1, x_2) = [-\mu_2 + \gamma x_1]$.

It is clear that $\mathbb{G}_i(x_1, x_2) < 0, (x_i - \bar{x}_i) > 0, i = 1,2$, so that

$$\sum_{i=1}^{2}(x_i - \bar{x}_i)\mathbb{G}_i(x_1, x_2) < 0.$$

and

$0 \leq \delta_0\bar{x}_0 - \mu_1 \leq \mathbb{G}_3(x_1, x_2), \quad \forall x_i > \bar{x}_i, i = 1,2,$

$0 = \mathbb{G}_4(\bar{x}_1, \bar{x}_2) < \mathbb{G}_4(x_1, x_2), \quad \forall x_i > \bar{x}_i, i = 1,2.$

And since

$(y_1 - \bar{y}_1) < 0, (y_2 - \bar{y}_2) < 0, \forall <, y_i < \bar{y}_i, i = 1,2,$

then

$$\sum_{i=1}^{2}(y_i - \bar{y}_i)\mathbb{G}_{i+2}(x_1, x_2) < 0, \quad \forall x_i > \bar{x}_i, y_i < \bar{y}_i, i = 1,2$$

So that

$$\sum_{i=1}^{2}[(x_i-\bar{x}_i)\mathbb{G}_i(x_1,x_2)+(y_i-\bar{y}_i)\mathbb{G}_{i+2}(x_1,x_2)]<0,$$
$\dot{V}(x_1,x_2,y_1,y_2)<0, \forall(x_1,x_2,y_1,y_2)\in\overline{\mathbb{B}}-\{(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2)\}$,
And $\dot{V}(x_1,x_2,y_1,y_2)=0$.

Therefore, any trajectory with the initial condition $(x_{01},x_{02},y_{01},y_{02})\in\overline{\mathbb{B}}$ converges asymptotically to $(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2)$, which means that $\overline{\mathbb{B}}$ is a basin of attraction for $P_8=(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2)$. Thus, the proof is complete.

5. NUMERICAL SIMULATIONS

In this section, we will analyze three sets of parameter values for the system (2.1), which are provided in Tables-(1, 2, and 3), repectively. The system (2.1), that is defined by the parameters of table 1, has eight equilibrium points. While the system (2.1), that is defined by the parameters of Table- 2, has only four equilibrium points, and the system that is determined by the parameters of Table-3 has ten equilibrium points. These results support our argument that the existence of some equilibrium points depends on the values of the parameters.

The main results can be summarized as follows:

1- The equilibrium points assigned with the parameters of table 1 are:
$P_1=(0,0,0,0)$; $P_2=(2.8,0,0,0)$; $P_3=(0,2.4,0,0)$; $P_4=(2.8,2.4,0,0)$;
$P_5=(2.5,0,0,1.6071)$; $P_6=(2.5,2.4,0,1.6071)$; $P_7=(2.6642,1.0856,2.5545,0)$
and$P_8=(2.5,,1.0801,2.6105,0.7859)$.The first seven points are unstable while the eighth point (the interior point) is asymptotically locally stable, as shown in Figures-(1-3).

Table 1-

i	k_i	g_i	μ_i	δ_i	α_i	β	γ
1	2.8	1.5	0.6	0.6	0.2	0.1	0.1
2	2.4	2	0.25	0.4	0.5		

2- The equilibrium points assigned with the parameters of table 2 are four points, as follows:
$P_1=(0,0,0,0)$; $P_2=(1,0,0,0)$ and $P_3=(0,1.4,0,0)$ are unstable, while $P_4=(1,1.4,0,0)$ is asymptotically locally stable, is shown in Figure- 4.

Table 2-

i	k_i	g_i	μ_i	δ_i	α_i	β	γ
1	1	0.5	1	0.3	0.2	0.1	0.5
2	1.4	2	0.6	0.4	0.8		

3- The system (2.1) that is defined by the parameters of table 3 has ten equilibrium points, as follow:
$P_1=(0,0,0,0)$; $P_2=(2.5,0,0,0)$, $P_3=(0,2.5,0,0)$, while $P_4=(2.5,2.5,0,0)$ and
$P_5=(0.5,0,0,1.4063)$, $P_6=(0.625,2.5,0,1.4063)$ and
$P_8=(0.625,0.4575,3.7644,1.0779)$.
Equation (4.4b) has 5 roots, three of them are positive, that is $\tilde{x}=0.1636,\ 0.5884$ or 4.9432. Therefore, from each root, we obtain one equilibrium point, as follows:
$\tilde{x}=0.1636\Rightarrow P_7^{*}=(0.3858,2.3586,6.5124,0)$
$\tilde{x}=0.5884\ \Rightarrow P_7^{**}=(0.4029,0.6847,8.4697,0)$
$\tilde{x}=4.9432\ \Rightarrow P_7^{***}=(2.4672,0.4991,2.4993,0)$

All the equilibrium points of the system (2.1) that are defined by the parameters of table 3 are unstable, except P_7^{*}, that is P_7^{*} is locally asymptotically stable. see Figures- (5-9).

Table 3-

i	k_i	g_i	μ_i	δ_i	α_i	β	γ
1	2.5	1.5	0.25	0.6	0.2	0.8	0.4
2	2.5	1.5	0.25	0.4	0.5		

The next Figures-(1-8) show the numerical simulation to the trajectories of the system (2.1) with the different parameterS:

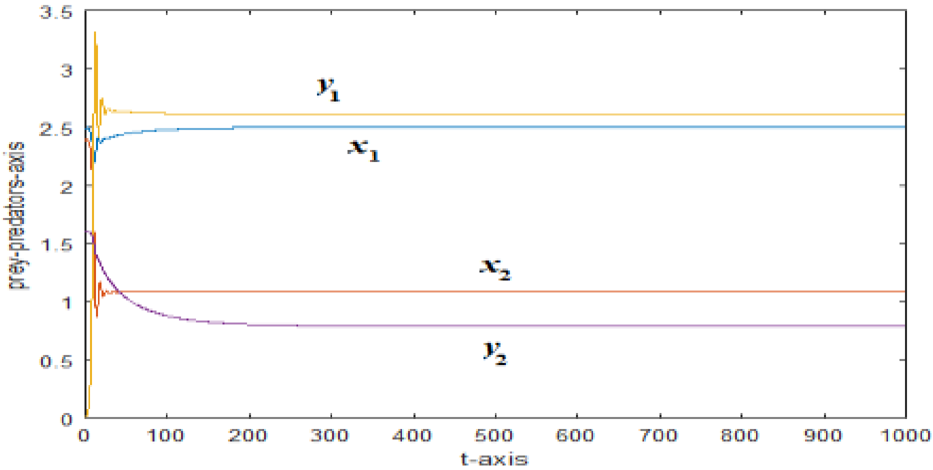

Figure 1-The trajectory of the system (2.1) with the parameters given in Table-1.We see that the initial point .(2.49,,2.39,0.01,1.60) is located close to P_6 and it is moving away from P_6 and approaching P_8.

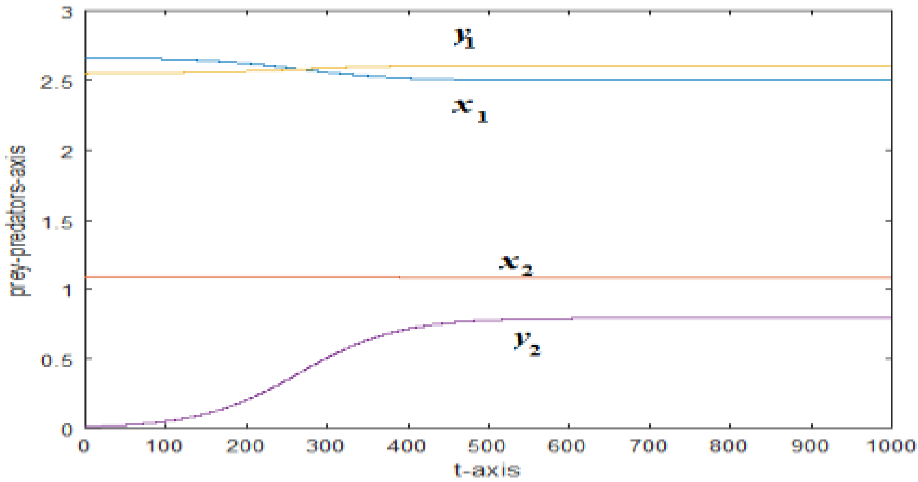

Figure 2-The trajectory of the system (2.1) with the parameters given in Table (1).We see that the initial point (2.66,,1.08,2.55,0.01) is located close to P_7 and it is moving away from P_7 and approaching P_8.

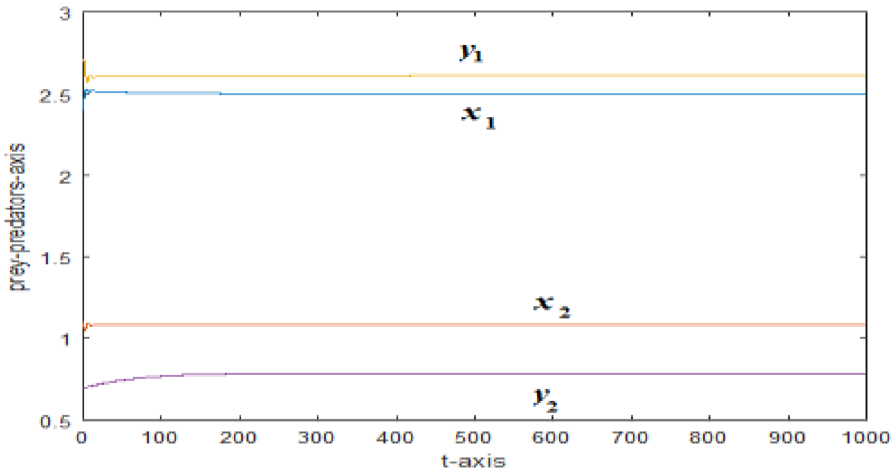

Figure 3-The trajectory of the system (2.1) with the parameters given in Table-1. We see that the initial point (2.4,,1.1,2.7,0.7)is located close to$P_8$and it is approaching P_8.

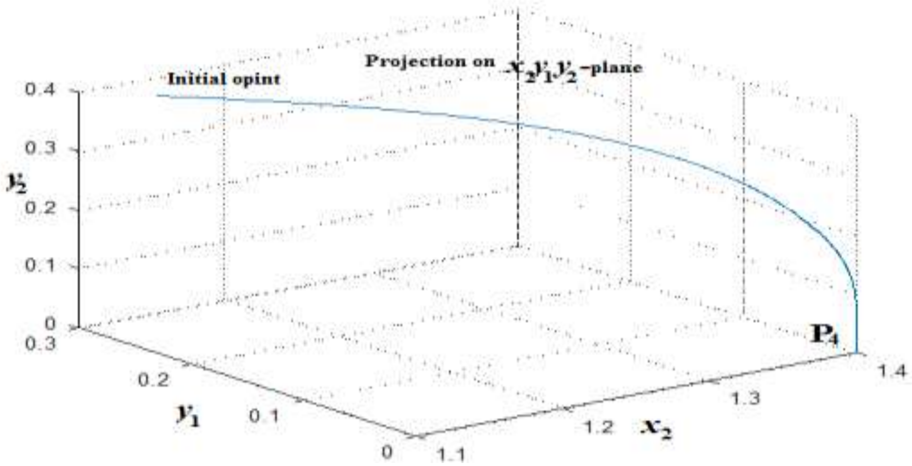

Figure 4-The trajectory of the system (2.1) with the parameters given in Table-2. We see that the initial point (0.7,,1.1,0.3,0.4) is approaching $\boldsymbol{P_4}$.

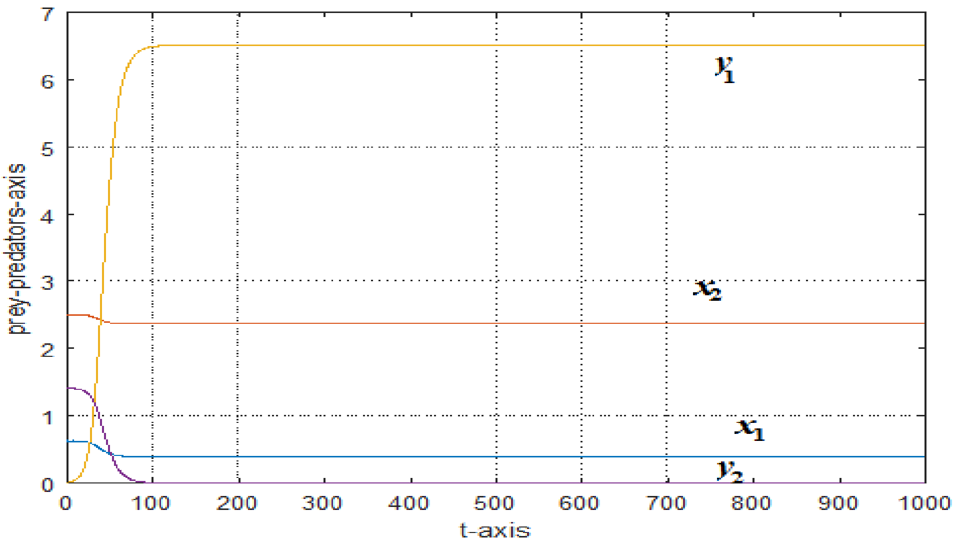

Figure 5-The trajectory of the system (2.1) with the parameters given in Table-3.We see that the initial point (0.624,2.49 ,0.01,1.4064), is located close to $\boldsymbol{P_6}$and it goes away from $\boldsymbol{P_6}$ and is approachingP_7*

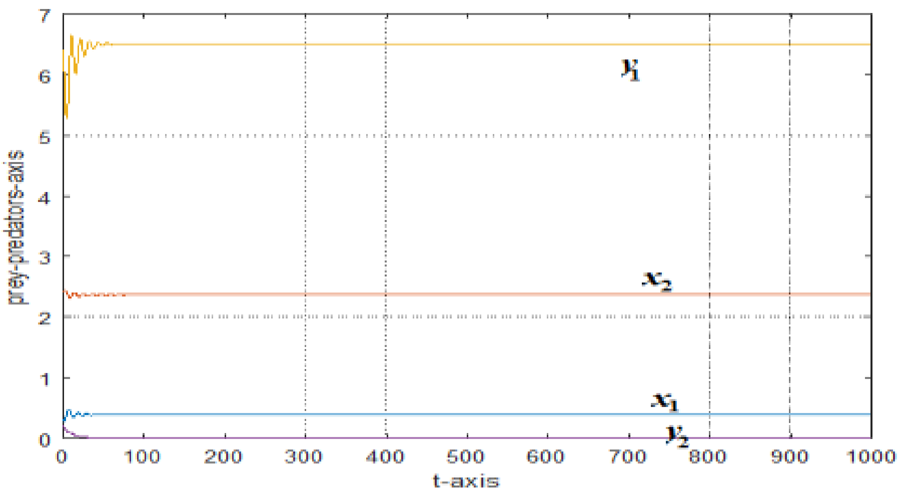

Figure 6-The trajectory of the system (2.1) with the parameters given in Table-4.We see that the initial point (0.3,2.4 ,6.4,0.2), is located close to P_7^* and is approaching P_7^*.

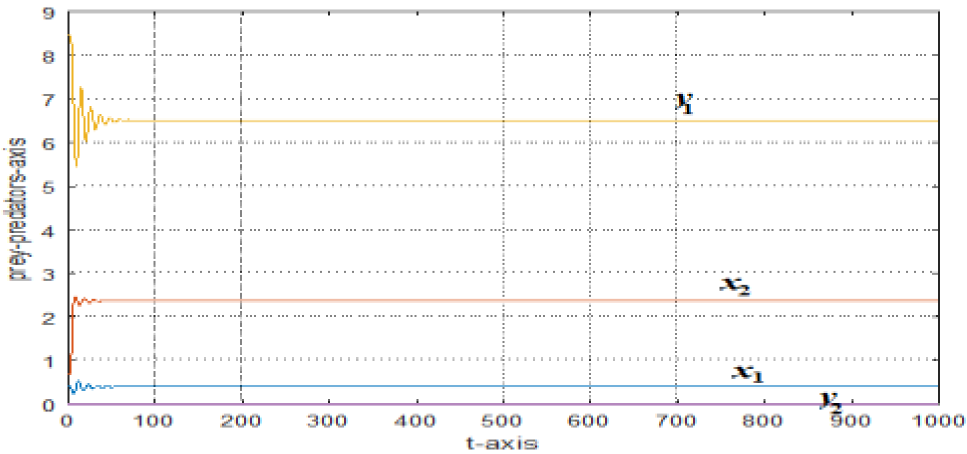

Figure 7-The trajectory of the system (2.1) with the parameters given in Table-4. We see that the initial point (0.402,0.684 ,8.469,0.004) , located close P_7^{**} and it goes away from P_7^{**} and is approaching P_7^{*}.

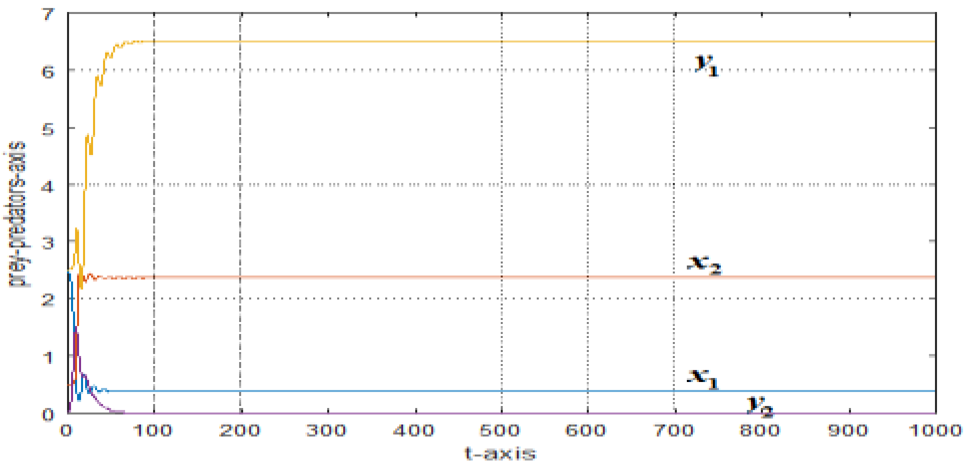

Figure 8-The trajectory of the system (2.1) with the parameters given in Table-4.We see that the initial point (2.47,0.5 ,2.5,0.02), is located close $P_7^{***}$and it goes away from P_7^{***} and is approaching P_7^{*}.

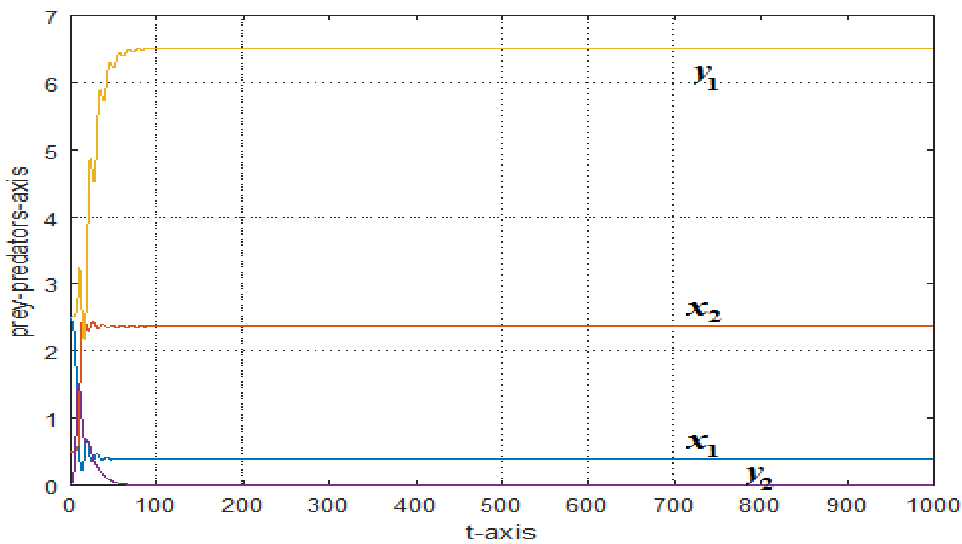

Figure 9-The trajectory of the system (2.1) with the parameters given in Table-4. We see that the initial point (0.264,0.458 ,3.76,1.08) , is located close to P_8 and it goes away from P_8 and is approaching P_7^{*}.

6. CONCLOSIONS

In this paper, we study a model of two species of predators and two species of prey, where the two species of prey live in two different habitats and have the ability to group-defense. Only one of the two predators tends to switch between the habitats. We have found that the system has, at most, thirteen possible equilibrium points, one of them is the origin, two are axial, two are interior points, and the rest of the equilibrium points are boundaries. Four of equilibrium points always exist, regardless of the values of the parameters, whereas the existence of the remaining equilibrium points depends on the values of the parameters, that implies that some sufficient conditions were given to ensure their existence. Moreover, we have studied the boundedness of the solutions and the local stability of the equilibrium points, and we have found that four points are unstable and the rest are locally asymptotically stable under certain conditions. Furthermore, We explained the general stability of the interior equilibrium point by means of the Lyapunov function to create a basin of attraction for the studied point.

Finally, we have given three examples; each of them is a set of parameter values. One of them shows that the system has eight points, seven of them are unstable and one (the interior point) is locally asymptotically stable. In the second example, the system has only four equilibrium points, three are unstable and one is locally asymptotically stable. The third example shows that the system has ten equilibrium points, nine of them are unstable and one is locally asymptotically stable. Our next work will be on studying the same model that we have discussed in this paper with $n \geq 1$ as the switching index.

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