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The $\Gamma -$ permutation BCK-algebras with their $\Gamma -$ permutation ideals

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Abstract

In this work, we propose the notion of $\Gamma -$ permutation $BCK -$algebras as an extension of $BCK -$algebras and investigate them. We also present $\Gamma -$ permutation $BCK -$subalgebra, $\Gamma -$ permutation $BCK -$ideal, $\Gamma -$ permutation $BCK -$closed ideal, $\Gamma -$ permutation normal $BCK -$subalgebra, $\Gamma -$ permutation normal $BCK -$ideal, and quotient $\Gamma -$ permutation $BCK -$algebra. We show that if $h: X \to Y$ is a normal homomorphism of $\Gamma -$permutation $BCK -$algebras X and Y, the quotient $\Gamma -$permutation $BCK -$algebra Y/V is isomorphic to $Im(h)$, where $V = ker(h)$.

Keywords: Symmetric group, $BCK -$algebra, Permutation, Quotient, Equivalences, Normal Ideal.
MSC: 03G25, 08A35.

جبر–بي سي كي التباديلي–كاما مع مثالياته التبادلية–كاما

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الخلاصة

في هذا البحث تم تقديم مفهوم, جبر–بي سي كي التباديلي–كاما كتوسيع لجبر–بي سي كي ,ايضاً تم تقديم المفاهيم مثل الجبر الجزئي–بي سي كي التباديلي–كاما و المثالية –بي سي كي التبادلية–كاما و المثالية المغلقة –بي سي كي التبادلية–كاما و الجبر الجزئي–بي سي كي العادي التباديلي–كاما و المثالية العادية –بي سي كي التبادلية –كاما و جبر–بي سي كي القسمة التباديلي–كاما , كذلك بينا اذا كان $h: X \to Y$ هو الهومومورفزم العادي حول جبور –بي سي كي التباديلية–كاما X و Y , فان Y/V جبر–بي سي كي القسمة التباديلي–كاما يتشاكل مع صورة التطبيق $Im(h)$ حيث $V = ker(h)$.

1. Introduction

The collection of all negative integers Z is not a semiring in terms of normal addition and multiplication, but it does form a $\Gamma -$semiring, where $\Gamma = Z$. The primary reason for the evolution of $\Gamma -$semiring consists of the development of outcomes of rings, $\Gamma -$rings, ternary semirings, semigroups, and semirings. In 1966, Y. Imai and K. Iseki [1] established two classes of abstract algebras, namely BCK and $BCI -$algebras, to broaden the concept of set$-$theoretic difference and non-classical propositional calculus. Every $BCI -$algebra M satisfies $0 \# r = 0$ for every $r \in X$ is a $BCK -$algebra. Every abelian group is a

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BCK −algebra, with # representing group subtraction and 0 representing group identity. As a result, various researchers have investigated generalizations of $BCK/BCI/RHO$ −algebras (see [2-10]). Moreover, there are other classes are studied by many researchers, [11-16]. The product of disjoint cycles $\beta = (b_1^1, b_2^1, ..., b_{\alpha_1}^1)\ (b_1^2, b_2^2, ..., b_{\alpha_2}^2)\ ...(b_1^{c(\beta)}, b_2^{c(\beta)}, ..., b_{\alpha_{c(\beta)}}^{c(\beta)})$ can be decomposed virtually uniquely for each permutation β in the symmetric group S_n, [17]. As a result, β can be written as $\lambda_1\lambda_2...\lambda_{c(\beta)}$. The division is referred to as the cycle type of β, [18]. The particular laws for union, intersection, and other laws of permutation sets (for short, β −sets) are described, [19]. Following that, this concept and its applications are investigated and discussed in a variety of domains such as topology and algebra, and others [20], [21].

The research structure in this paper as following, in the second section the basic notions are recalled which are needed in our work. After that, in the third section, the concept of $\Gamma -$ permutation BCK −algebras $[(\Gamma - (PBCK - A)]$ have been considered. Also, the notions of $\Gamma - (PBCK - SA)$, $\Gamma - (PNBCK - SA)$, $\Gamma - (PCBCK - A)$, and quotient $\Gamma - (PBCK - A)$ are given and we discussed their properties and equivalence classes. Next, in the fourth section some notions like $\Gamma -$ permutation BCK −ideal, $\Gamma -$ permutation BCK −closed ideal, and $\Gamma -$ permutation normal BCK −ideal, are given and we discussed their properties. Finally, in the last section, the conclusions for our results and future work are shown.

2. Basic fundamentals

In this section, we recall the following definitions which are necessary for completeness.

Definition 2.1: [19]

Let S_n be a symmetric group on the set $\Omega = \{1,2, ..., n\}$ with $B \in S_n$ and the cycle type of permutation B is $\propto (B) = (\propto_1, \propto_2, ..., \propto_{c(B)})$, then B composite of pairwise disjoint cycles $\{\mu_t\}_{t=1}^{c(B)}$ where $\mu_t = (\mu_1^t, \mu_2^t, .., \mu_{\alpha_t}^t)$, $1 \le t \le c(B)$. If $\mu = (\mu_1, \mu_2, .., \mu_h)$ is h −ycle in S_n. We say $\mu^B = \{\mu_1, \mu_2, .., \mu_h\}$ is a B −set of cycle μ, where $\{\mu_t\}_{t=1}^{c(B)} = \{\mu_t^B = \{\mu_1^t, \mu_2^t, .., \mu_{\alpha_t}^t\} | 1 \le t \le c(B)\}$.

Definition 2.2: [2]

A BCK −algebra is an algebra $(X,*,0)$ of type (2,0) satisfies the following axioms for every $x, y, z \in X$

1. $((x * y) * (x * z)) * (z * y) = 0;$
2. $(x * (x * y)) * y = 0;$
3. $x * x = 0;$
4. $0 * x = 0;$
5. $x * y = 0$ and $y * x = 0$ implies $x = y$.

3. On $\Gamma -$ permutation BCK −algebras

This section introduces and investigates the concept of $\Gamma -$ permutation BCK −algebras $[(\Gamma - (PBCK - A)]$. Also, the notions of $\Gamma - (PBCK - SA)$, $\Gamma - (PNBCK - SA)$, and $\Gamma - (PCBCK - A)$ are given and we discussed their properties and equivalence classes.

Definition 3.1: Let $V \subseteq X = \left\{\lambda_i^\beta\right\}_{i=1}^{c(\beta)}$, where $\beta \in S_n$, and let Γ be non-empty sets. Assume T is a constant element in V and $\mu: V \times \Gamma \times V \to V$ is a map, where $\mu(\lambda_i^\beta, \alpha, \lambda_j^\beta) = \lambda_i^\beta\, \alpha\, \lambda_j^\beta$, for all λ_i^β, $\lambda_j^\beta \in V$ and $\alpha \in \Gamma$, if μ have the following identities:

i) $\mu(\mu\left(\mu\left(\lambda_i^{\beta},\alpha,\lambda_j^{\beta}\right),\beta,\mu\left(\lambda_i^{\beta},\alpha,\lambda_m^{\beta}\right)\right),\beta,\mu(\lambda_m^{\beta},\alpha,\lambda_j^{\beta}))=T$,
ii) $\mu(\lambda_i^{\beta},\alpha,\lambda_j^{\beta})=\mu(\lambda_j^{\beta},\alpha,\lambda_i^{\beta})=T \Longrightarrow \lambda_i^{\beta}=\lambda_j^{\beta}$,
iii) $\mu(\lambda_i^{\beta},\alpha,\lambda_i^{\beta})=T$,
iv) $\mu(T,\alpha,\lambda_i^{\beta})=T$ for all $\alpha,\beta\in\Gamma$, λ_i^{β}, λ_j^{β}, $\lambda_m^{\beta}\in V$.
Then V is called a $\Gamma-$ permutation $BCK-$algebra [for short, $\Gamma-(PBCK-A)$].

Remark 3.2: If V is a $\Gamma-(PBCK-A)$ and $\alpha\in\Gamma$. Let $: V\times V\to V$ be a map, where $\tau(\lambda_i^{\beta},\lambda_j^{\beta})=\lambda_i^{\beta}\alpha\lambda_j^{\beta}, \forall\lambda_i^{\beta},\lambda_j^{\beta}\in V$. Then $(\Re,\tau,T)$ is symbolized by $\Re\alpha$, if it is a $(PBCK-A)$.

Example 3.3: Let $\beta=\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12\\4&6&2&1&9&3&10&8&5&7&12&11\end{pmatrix}$ be a permutation in S_{12}. So, $\beta=(1\ 4)(2\ 6\ 3)\ (5\ 9)(7\ 10)(11\ 12)(8)$. Therefore, we have $X=\left\{\lambda_i^{\beta}\right\}_{i=1}^{6}=\{\{1,4\},\{2,3,6\},\{5,9\},\{7,10\},\{8\},\{11,12\}\}$ and $T=\{1,4\}$. Let $\Gamma=\{\mu_1,\mu_2,\mu_3,\mu_4,\mu_5\}$ be a set of operations that are defined on X by Tables [(1)-(5)].

Table 1: $\mu_1: X\times X\to X$

μ_1	{1,4}	{2,3,6}	{5,9}	{7,10}	{8}	{11,12}
{1,4}	{1,4}	{1,4}	{1,4}	{1,4}	{1,4}	{1,4}
{2,3,6}	{2,3,6}	{1,4}	{2,3,6}	{2,3,6}	{2,3,6}	{2,3,6}
{5,9}	{5,9}	{5,9}	{1,4}	{5,9}	{5,9}	{5,9}
{7,10}	{7,10}	{7,10}	{7,10}	{1,4}	{7,10}	{7,10}
{8}	{8}	{8}	{8}	{8}	{1,4}	{8}
{11,12}	{11,12}	{11,12}	{11,12}	{11,12}	{11,12}	{1,4}

Table 2 : $\mu_2: X\times X\to X$

μ_2	{1,4}	{2,3,6}	{5,9}	{7,10}	{8}	{11,12}
{1,4}	{1,4}	{1,4}	{1,4}	{1,4}	{1,4}	{1,4}
{2,3,6}	{2,3,6}	{1,4}	{5,9}	{5,9}	{5,9}	{5,9}
{5,9}	{5,9}	{7,10}	{1,4}	{7,10}	{7,10}	{7,10}
{7,10}	{7,10}	{8}	{8}	{1,4}	{8}	{8}
{8}	{8}	{11,12}	{11,12}	{11,12}	{1,4}	{11,12}
{11,12}	{11,12}	{2,3,6}	{2,3,6}	{2,3,6}	{2,3,6}	{1,4}

Table 3 : $\mu_3: X\times X\to X$

μ_3	{1,4}	{2,3,6}	{5,9}	{7,10}	{8}	{11,12}
{1,4}	{1,4}	{1,4}	{1,4}	{1,4}	{1,4}	{1,4}
{2,3,6}	{2,3,6}	{1,4}	{7,10}	{7,10}	{7,10}	{7,10}
{5,9}	{5,9}	{8}	{1,4}	{8}	{8}	{8}
{7,10}	{7,10}	{11,12}	{11,12}	{1,4}	{11,12}	{11,12}
{8}	{8}	{2,3,6}	{2,3,6}	{2,3,6}	{1,4}	{2,3,6}
{11,12}	{11,12}	{5,9}	{5,9}	{5,9}	{5,9}	{1,4}

Table 4 : $\mu_4 : X \times X \to X$

μ_4	{1,4}	{2,3,6}	{5,9}	{7,10}	{8}	{11,12}
{1,4}	{1,4}	{1,4}	{1,4}	{1,4}	{1,4}	{1,4}
{2,3,6}	{2,3,6}	{1,4}	{8}	{8}	{8}	{8}
{5,9}	{5,9}	{11,12}	{1,4}	{11,12}	{11,12}	{11,12}
{7,10}	{7,10}	{2,3,6}	{2,3,6}	{1,4}	{2,3,6}	{2,3,6}
{8}	{8}	{5,9}	{5,9}	{5,9}	{1,4}	{5,9}
{11,12}	{11,12}	{7,10}	{7,10}	{7,10}	{7,10}	{1,4}

Table 5 : $\mu_5 : X \times X \to X$

μ_5	{1,4}	{2,3,6}	{5,9}	{7,10}	{8}	{11,12}
{1,4}	{1,4}	{1,4}	{1,4}	{1,4}	{1,4}	{1,4}
{2,3,6}	{2,3,6}	{1,4}	{11,12}	{11,12}	{11,12}	{11,12}
{5,9}	{5,9}	{2,3,6}	{1,4}	{2,3,6}	{2,3,6}	{2,3,6}
{7,10}	{7,10}	{5,9}	{5,9}	{1,4}	{5,9}	{5,9}
{8}	{8}	{7,10}	{7,10}	{7,10}	{1,4}	{7,10}
{11,12}	{11,12}	{8}	{8}	{8}	{8}	{1,4}

Hence, X is $\Gamma-(PBCK-A)$.
Example 3.4:

Let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & 1 & 9 & 8 & 2 & 4 & 5 & 7 \end{pmatrix}$ be a permutation in S_9. So, $\beta = (1\ 3)(2\ \ 6)\ (5\ \ 8)(7\ 4\ \ 9)$. Therefore, we have $Y = \left\{\lambda_i^{\beta}\right\}_{i=1}^{4} = \{\{1,3\},\{2,6\},\{5,8\},\{4,7,9\}\}$ and $T = \{1,3\}$. Let $\Gamma = \{\omega_1, \omega_2\}$. The operations ω_1, ω_2 are defined on Y by Tables (6) and (7), respectively.

Table 6 : $\omega_1 : Y \times Y \to Y$

ω_1	{1,3}	{2,6}	{5,8}	{4,7,9}
{1,3}	{1,3}	{1,3}	{1,3}	{1,3}
{2,6}	{2,6}	{1,3}	{5,8}	{5,8}
{5,8}	{5,8}	{4,7,9}	{1,3}	{4,7,9}
{4,7,9}	{4,7,9}	{2,6}	{2,6}	{1,3}

Table 7: $\omega_2 : Y \times Y \to Y$

ω_2	{1,3}	{2,6}	{5,8}	{4,7,9}
{1,3}	{1,3}	{1,3}	{1,3}	{1,3}
{2,6}	{2,6}	{1,3}	{2,6}	{2,6}
{5,8}	{5,8}	{5,8}	{1,3}	{5,8}
{4,7,9}	{4,7,9}	{4,7,9}	{4,7,9}	{1,3}

Hence, we consider Y as $\Gamma-(PBCK-A)$.

Definition 3.5: Let V be a $\Gamma-(PBCK-A)$ and $\emptyset \neq A \subseteq V$. We say A is $\Gamma-$ permutation BCK –subalgebra [for short, $\Gamma-(PBCK-SA)$], if $\lambda_i^{\beta}\,\alpha\,\lambda_j^{\beta} \in A, \forall\, \lambda_i^{\beta}, \lambda_j^{\beta} \in A, \alpha \in \Gamma$.

Definition 3.6: Let V be a $\Gamma-(PBCK-A)$ and $\emptyset \neq A \subseteq V$. We say A is a $\Gamma-$ permutation normal BCK –subalgebra [for short, $\Gamma-(PNBCK-SA)$], if
$\left(\lambda_i^\beta \alpha \lambda_m^\beta\right)\alpha\left(\lambda_j^\beta \alpha \lambda_s^\beta\right) \in A,\ \forall \lambda_i^\beta \alpha \lambda_j^\beta,\ \lambda_m^\beta \alpha \lambda_s^\beta \in A,\ \alpha \in \Gamma.$

Example 3.7: Let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 1 & 8 & 6 & 5 & 4 & 2 & 3 \end{pmatrix}$ be a permutation in S_8. So, $\beta = (1\ 7\ 2)(3\ 8)(4\ 6)(5)$. Therefore, we have $X = \left\{\lambda_i^\beta\right\}_{i=1}^{4} = \{\{1,2,7\},\{3,8\},\{4,6\},\{5\}\}$ and $T = \{5\}$. Let $= \{\mu\}$, where μ is defined on X by Table (8).

Table 8 : $\mu: X \times X \to X$

μ	{5}	{1,2,7}	{4,6}	{3,8}
{5}	{5}	{5}	{5}	{5}
{1,2,7}	{1,2,7}	{5}	{1,2,7}	{1,2,7}
{4,6}	{4,6}	{4,6}	{5}	{4,6}
{3,8}	{3,8}	{3,8}	{3,8}	{5}

Hence, $M = \{\{5\},\{1,2,7\}\}$ is $\Gamma-(PNBCK-SA)$.

Definition 3.8:
(i) A $\Gamma-(PBCK-A)$ V is called $\Gamma-$ permutation commutative BCK –algebra [for short, $\Gamma-(PCBCK-A)$], if $\lambda_j^\beta \alpha\left(\lambda_j^\beta \beta \lambda_i^\beta\right) = \lambda_i^\beta \alpha(\lambda_i^\beta \beta \lambda_j^\beta), \forall \lambda_i^\beta, \lambda_j^\beta \in V,$ and $\alpha, \beta \in \Gamma$.

(ii) A $\Gamma-(PBCK-A)$ V can be partially ordered by $\lambda_i^\beta \leq \lambda_j^\beta$ if and only if $\lambda_i^\beta \alpha \lambda_j^\beta = T$ for all $\alpha \in \Gamma$. This ordering is called a $\Gamma-BCK$ ordering.

Note: If V is a $\Gamma-(PBCK-A)$, then $\lambda_i^\beta \alpha T = \lambda_i^\beta$ is hold.

Example 3.9: Consider $X = \{\{1,2,7\},\{3,8\},\{4,6\},\{5\}\}$ and μ in Example 3.7. Let $\Gamma = \{\mu, \omega\}$, where ω is defined by Table (9).

Table 9 : $\omega: X \times X \to X$

ω	{5}	{1,2,7}	{4,6}	{3,8}
{5}	{5}	{5}	{5}	{5}
{1,2,7}	{1,2,7}	{5}	{1,2,7}	{3,8}
{4,6}	{4,6}	{1,2,7}	{5}	{1,2,7}
{3,8}	{3,8}	{4,6}	{1,2,7}	{5}

Hence, X is $\Gamma-(PCBCK-A)$.

Definition 3.10: Let X and Y be $\Gamma-(PBCK-As)$. A map $f: X \to Y$ is called a homomorphism if $f(\lambda_i^\beta \alpha \lambda_j^\beta) = f(\lambda_i^\beta)\alpha f(\lambda_j^\beta), \forall \lambda_i^\beta, \lambda_j^\beta \in X, \alpha \in \Gamma$.
Definition 3.11: Let X and Y be $\Gamma-(PBCK-As)$ with constant elements T_X and T_Y, respectively, and $f: X \to Y$ be a homomorphism. Then the set $\left\{\lambda_i^\beta \in X \mid f\left(\lambda_i^\beta\right) = T_Y\right\}$ is called the kernel of f and it is denoted by $ker(f)$ and the set $\{f(\lambda_i^\beta) \mid \lambda_i^\beta \in M\}$ is called the image of f and is denoted by $Im(f)$.

Theorem 3.12:
If V is $\Gamma-(PBCK-A)$, then $\lambda_i^{\beta}\alpha\left(\lambda_i^{\beta}\beta\,\lambda_j^{\beta}\right)\alpha\,\lambda_j^{\beta} = T$, $\forall\,\lambda_i^{\beta},\lambda_j^{\beta} \in V$, and $\alpha,\beta \in \Gamma$.
Proof.
From Definition 3.8 - (i) of $\Gamma-(PBCK-A)$,

$$\left(\left(\lambda_i^{\beta}\beta\,\lambda_j^{\beta}\right)\alpha\left(\lambda_i^{\beta}\beta\,\lambda_m^{\beta}\right)\right)\alpha\left(\lambda_m^{\beta}\beta\,\lambda_j^{\beta}\right) = T. \quad (1)$$

$\forall\,\lambda_i^{\beta},\lambda_j^{\beta},\lambda_m^{\beta} \in V$, $\alpha,\beta \in \Gamma$. Put λ_j^{β} = T, $\lambda_m^{\beta} = \lambda_j^{\beta}$, in (1), then $\left(\left(\lambda_i^{\beta}\beta T\right)\alpha\left(\lambda_i^{\beta}\beta\,\lambda_j^{\beta}\right)\right)\alpha\left(\lambda_j^{\beta}\beta T\right) = \lambda_i^{\beta}\alpha\left(\lambda_i^{\beta}\beta\,\lambda_j^{\beta}\right)\alpha\,\lambda_j^{\beta} = T$.

Theorem 3.13: Let M be a $\Gamma-(PBCK-A)$. Then the following are equivalent:
(i) M is commutative;
(ii) $\lambda_i^{\beta} \le \lambda_j^{\beta} \Rightarrow \lambda_i^{\beta} = \lambda_j^{\beta}\alpha\left(\lambda_j^{\beta}\beta\,\lambda_i^{\beta}\right)$ for all $\lambda_i^{\beta},\lambda_j^{\beta} \in M, \alpha,\beta \in \Gamma$.
Proof.
$(i) \Rightarrow (ii)$ Suppose that M is commutative. Then $\lambda_i^{\beta}\alpha\left(\lambda_i^{\beta}\beta\lambda_j^{\beta}\right) = \lambda_j^{\beta}\alpha\left(\lambda_j^{\beta}\beta\,\lambda_i^{\beta}\right)$. for all $\lambda_i^{\beta},\lambda_j^{\beta} \in M$ and $\alpha,\beta \in \Gamma$. So, if $\lambda_i^{\beta} \le \lambda_j^{\beta}$ from Definition 3.8 - (ii), hence $\lambda_i^{\beta}\alpha\,\lambda_j^{\beta} =$ T.
That implies $\lambda_i^{\beta} = \lambda_i^{\beta}\alpha T = \lambda_i^{\beta}\alpha\left(\lambda_i^{\beta}\beta\lambda_j^{\beta}\right) = \lambda_j^{\beta}\alpha\left(\lambda_j^{\beta}\beta\,\lambda_i^{\beta}\right)$, hence $\lambda_i^{\beta} = \lambda_j^{\beta}\alpha(\lambda_j^{\beta}\beta\,\lambda_i^{\beta})$.
(ii)$\Rightarrow$ (i) Suppose $\lambda_j^{\beta} \le \lambda_i^{\beta}$ then $\lambda_i^{\beta} = \lambda_j^{\beta}\alpha(\lambda_j^{\beta}\beta\,\lambda_i^{\beta})$. On the other hand, $\lambda_i^{\beta} \le \lambda_j^{\beta}$, then $\lambda_i^{\beta}\alpha\lambda_j^{\beta} = T$. Then $\lambda_i^{\beta} = \lambda_i^{\beta}\alpha T = \lambda_i^{\beta}\alpha(\lambda_i^{\beta}\beta\lambda_j^{\beta})$. Hence, $\lambda_i^{\beta}\alpha(\lambda_i^{\beta}\beta\lambda_j^{\beta}) = \lambda_j^{\beta}\alpha(\lambda_j^{\beta}\beta\,\lambda_i^{\beta})$.

Lemma 3.14: Let M be a $\Gamma-(PBCK-A)$. Then $T \le \lambda_i^{\beta}$ for all $\lambda_i^{\beta} \in M$.
Proof. We have $T\alpha\,\lambda_i^{\beta} = T$ for all $\alpha \in \Gamma$, then $T \le \lambda_i^{\beta}$. Hence, T is the least element of the $\Gamma-(PBCK-A)$ M.

Theorem 3.15: If $f: M \to N$ is a homomorphism of $\Gamma-(PBCK-A)$ M and N, then $Im(f)$ is a subalgebra of N.
Proof. Let $f: M \to N$ be a homomorphism of $\Gamma-(PBCK-A)$ M and N, $\lambda_i^{\beta},\lambda_j^{\beta} \in Im(f)$. Then there exist $u_i^{\beta}, v_j^{\beta} \in M$ such that $f\left(u_i^{\beta}\right) = \lambda_i^{\beta}$, $f\left(v_j^{\beta}\right) = \lambda_j^{\beta}$. That implies $f\left(u_i^{\beta}\right)\alpha f\left(v_j^{\beta}\right) = \lambda_i^{\beta}\alpha\lambda_j^{\beta}$ for all $\alpha \in \Gamma$. That implies $f\left(u_i^{\beta}\alpha\,v_j^{\beta}\right) = \lambda_i^{\beta}\alpha\lambda_j^{\beta}$. Then $\lambda_i^{\beta}\alpha\lambda_j^{\beta} \in Im(f)$. Hence, $Im(f)$ is a subalgebra of N.

Lemma 3.16: Let $f: M \to N$ be a homomorphism of $\Gamma-(PBCK-A)$ M and N. Then $ker(f)$ is a subalgebra of M.
Proof. Let $f: M \to N$ be a homomorphism of $\Gamma-(PBCK-A)$ M and N, $\lambda_i^{\beta},\lambda_j^{\beta} \in ker(f)$. Then $f\left(\lambda_i^{\beta}\right) = f\left(\lambda_j^{\beta}\right) = T$ and so $f\left(\lambda_i^{\beta}\alpha\lambda_j^{\beta}\right) = f\left(\lambda_i^{\beta}\right)\alpha f\left(\lambda_j^{\beta}\right) = T\alpha T = T$. Hence, $\lambda_i^{\beta}\alpha\lambda_j^{\beta} \in ker(f)$. Therefore, $ker(f)$ is a subalgebra of M.

Lemma 3.17: Let $f: M \to N$ be a homomorphism of $\Gamma-(PBCK-A)$, M and N. Then
$(i) f(T) = T$.
(ii) if $\lambda_i^{\beta}\alpha\lambda_j^{\beta} = T$, then $f(\lambda_i^{\beta})\alpha f(\lambda_j^{\beta}) = T$.

Proof. (i) Now, $f(T) = f(T\alpha T) = f(T)\alpha f(T) = T$. (ii) Suppose $\lambda_i^\beta \alpha \lambda_j^\beta = T$.Then $f\left(\lambda_i^\beta \alpha \lambda_j^\beta\right) = f(T)$ implies $f\left(\lambda_i^\beta\right)\alpha f\left(\lambda_j^\beta\right) = T$.

Theorem 3.18: Let $f : M \to N$ be a homomorphism of $\Gamma - (PBCK - A), M$ and N. Then f is injective if and only if $ker(f) = \{T\}$.
Proof. Suppose $ker(f) = \{T\}$ and $f\left(\lambda_i^\beta\right) = f\left(\lambda_j^\beta\right)$, for some $\lambda_i^\beta, \lambda_j^\beta \in M$. Then $f\left(\lambda_i^\beta \alpha \lambda_j^\beta\right) = f(\lambda_i^\beta)\alpha f(\lambda_j^\beta) = T$, that implies $\lambda_i^\beta \alpha \lambda_j^\beta \in ker(f) = \{T\}$, implies $\lambda_i^\beta \alpha \lambda_j^\beta = T, \forall \alpha \in \Gamma$. Similarly, $\lambda_j^\beta \alpha \lambda_i^\beta = T, \forall \alpha \in \Gamma$. Therefore, $\lambda_i^\beta = \lambda_j^\beta$. Conversely, suppose f is injective and $\lambda_i^\beta \in ker(f)$. Then $f(\lambda_i^\beta) = T = f(T)$, that implies $\lambda_i^\beta = T$, implies $ker(f) = T$.

Lemma 3.19: Let N be a $\Gamma - (PNBCK - SA)$ of a $\Gamma - (PBCK - A)$ M. If $\lambda_i^\beta \alpha \lambda_j^\beta \in N$, for all $\lambda_i^\beta, \lambda_j^\beta \in M$, then $\lambda_j^\beta \alpha \lambda_i^\beta \in N, \alpha \in \Gamma$.
Proof. Let $\lambda_i^\beta, \lambda_j^\beta \in M$ and $\lambda_i^\beta \alpha \lambda_j^\beta \in N$: We have $\lambda_j^\beta \alpha \lambda_j^\beta = T \in N$ for all $\alpha \in \Gamma$.Then $\lambda_j^\beta \alpha \lambda_i^\beta = \left(\lambda_j^\beta \alpha \lambda_i^\beta\right)\alpha(T) = \left(\lambda_j^\beta \alpha \lambda_i^\beta\right)\alpha\left(\lambda_j^\beta \alpha \lambda_j^\beta\right) \in N, \alpha \in \Gamma$. Since N is a $\Gamma - (PNBCK - SA)$ of M. Therefore, $\lambda_j^\beta \alpha \lambda_i^\beta \in N$.

Remark 3.20: Let N be a $\Gamma - (PNBCK - SA)$ of $a\, \Gamma - (PBCK - A)$ M. Define a relation " $\sim N$ " on M by $\lambda_i^\beta \sim N\, \lambda_j^\beta$ if and only if $\lambda_i^\beta \alpha \lambda_j^\beta \in N$ for any $\lambda_i^\beta, \lambda_j^\beta \in M, \alpha \in \Gamma$.

Theorem 3.21: Let N be a $\Gamma - (PNBCK - SA)$ of a $\Gamma - (PBCK - A)$ M. Then " $\sim N$ " is a congruence relation.
Proof. Let $\lambda_i^\beta \in M, \alpha \in \Gamma$. Then the relation $\sim_N$ is reflexive, since $\lambda_i^\beta \alpha \lambda_i^\beta = T \in N$. The relation $\sim_N$ is symmetric, follows from Lemma 3.19. Suppose $\lambda_i^\beta \sim_N \lambda_j^\beta, \lambda_j^\beta \sim_N \lambda_m^\beta \in N$. Then $\lambda_i^\beta \alpha \lambda_j^\beta \in N$ and $\lambda_j^\beta \alpha \lambda_m^\beta \in N$. By Lemma 3.19 $\lambda_m^\beta \alpha \lambda_j^\beta \in N$, thus $\left(\lambda_i^\beta \alpha \lambda_m^\beta\right)\alpha\left(\lambda_j^\beta \alpha \lambda_j^\beta\right) = (\lambda_i^\beta \alpha \lambda_m^\beta)\alpha T = (\lambda_i^\beta \alpha \lambda_m^\beta) \in N$, thus is $\Gamma - (PNBCK - SA)$. Hence $\lambda_i^\beta \sim_N \lambda_m^\beta$. Then " $\sim_N$ " is an equivalence relation. Let $\lambda_i^\beta \sim_N \lambda_j^\beta$ and $\lambda_s^\beta \sim_N \lambda_m^\beta$ for all $\lambda_i^\beta, \lambda_j^\beta, \lambda_s^\beta$ and $\lambda_m^\beta \in M$. Then $\lambda_i^\beta \alpha \lambda_j^\beta \in N, \lambda_s^\beta \alpha \lambda_m^\beta \in N$, we have $(\lambda_i^\beta \alpha \lambda_s^\beta)\alpha(\lambda_j^\beta \alpha \lambda_m^\beta) \in N$. Therefore, $\lambda_i^\beta \alpha \lambda_s^\beta \sim_N \lambda_j^\beta \alpha \lambda_m^\beta$, since N is a $\Gamma - (PNBCK - SA)$. Hence, $\sim_N$ is a congruence relation.

Definition 3.22: Let N be a congruence relation on a $\Gamma - (PBCK - A)$ M. Denote $M/N = \{\left[\lambda_i^\beta\right]_N / \lambda_i^\beta \in M\}$, where $\left[\lambda_i^\beta\right]_N = \{\lambda_j^\beta \in M / \lambda_i^\beta \sim_N \lambda_j^\beta\}$. Define $\left[\lambda_i^\beta\right]_N \alpha \left[\lambda_j^\beta\right]_N = \left[\lambda_i^\beta \alpha \lambda_j^\beta\right]_N, \alpha \in \Gamma$. The M/N is $a\, \Gamma - (PBCK - A)$. Then $\Gamma - (PBCK - A)$ M / N is called the quotient $\Gamma - (PBCK - A)$ and denoted by $\Gamma - (PQBCK - A)$.

Example 3.23:
Consider $Y = \{\{1,3\},\{2,6\},\{5,8\},\{4,7,9\}\}$ and $\Gamma = \{\omega_1, \omega_2\}$ in Example 3.4. Hence Y is $\Gamma - (PBCK - A)$. Let $V = \{\{1,3\},\{2,6\}\} \subseteq Y$. Then V is $\Gamma - (PNBCK - SA)$. Denote $\frac{Y}{V} = \{[\lambda_i^\beta]_v / \lambda_i^\beta \in Y\}$ and define $[\lambda_i^\beta]_v \omega [\lambda_j^\beta]_v = [\lambda_i^\beta \omega \lambda_j^\beta]_v$, where $\omega \in \Gamma$. Then Y/V is a quotient $\Gamma - (PBCK - A)$.

Theorem 3.24: Let N be a $\Gamma-(PNBCK-SA)$ of a $\Gamma-(PBCK-A)$ M. Then the mapping $f: M \to M/N$ defined by $f(\lambda_i^\beta) = \left[\lambda_i^\beta\right]_N$ is a surjective homomorphism and $ker(f) = N$.

Proof. Since $f(\lambda_i^\beta \alpha \lambda_j^\beta) = \left[\lambda_i^\beta \alpha \lambda_j^\beta\right]_N = \left[\lambda_i^\beta\right]_N \alpha \left[\lambda_j^\beta\right]_N = f(\lambda_i^\beta)\alpha f(\lambda_j^\beta)$, we have $f(M) = \{f(\lambda_i^\beta) / \lambda_i^\beta \in M\} = \{\left[\lambda_i^\beta\right]_N / \lambda_i^\beta \in M\} = M/N$. Therefore, f is surjective. The set $ker(f) = \{\lambda_i^\beta \in M / f(\lambda_i^\beta) = N\} = \{\lambda_i^\beta \in M / \left[\lambda_i^\beta\right]_N = N\} = \{\lambda_i^\beta \in M / \left[\lambda_i^\beta\right]_N = [[T]_N\} = \{\lambda_i^\beta \in M / \lambda_i^\beta \in N\} = N$.

4. $\Gamma-$ permutation $BCK-$ideals of the $\Gamma-(PBCK-A)$

Here, some notions like $\Gamma-$ permutation $BCK-$ideal, $\Gamma-$ permutation $BCK-$closed ideal, $\Gamma-$ permutation normal $BCK-$ideal, and quotient $\Gamma-$ permutation $BCK-$algebra are given and we discussed their properties.

Definition 4.1: Let M be a $\Gamma-(PBCK-A)$ and I be a non-empty subset of M. Then I is called an $\Gamma-$ permutation $BCK-$ ideal [for short, $\Gamma-(PBCK-I)$] of M if
i) $T \in I$,
ii) $\lambda_i^\beta \alpha \lambda_j^\beta \in I, \alpha \in \Gamma$, and $\lambda_j^\beta \in I$, then $\lambda_i^\beta \in I$.
Where, M and $\{T\}$ are trivial ideals. An ideal I is proper if $I \neq M$.

Example 4.2:
Consider $= \left\{\lambda_i^\beta\right\}_{i=1}^4 = \{\{1,2,7\},\{3,8\},\{4,6\}, \{5\}\}$ in Example 4.7. Let $\Gamma = \{\mu, \omega\}$, where μ and ω are defined by Tables (11) and (12), respectively.

Table 11: $\mu: X \times X \to X$

μ	{5}	{1,2,7}	{4,6}	{3,8}
{5}	{5}	{5}	{5}	{5}
{1,2,7}	{1,2,7}	{5}	{1,2,7}	{4,6}
{4,6}	{4,6}	{4,6}	{5}	{4,6}
{3,8}	{3,8}	{4,6}	{3,8}	{5}

Table 12: $\omega: X \times X \to X$

ω	{5}	{1,2,7}	{4,6}	{3,8}
{5}	{5}	{5}	{5}	{5}
{1,2,7}	{1,2,7}	{5}	{1,2,7}	{4,6}
{4,6}	{4,6}	{4,6}	{5}	{4,6}
{3,8}	{3,8}	{3,8}	{3,8}	{5}

Then $A = \{\{5\},\{1,2,7\},\{3,8\}\}$ and $B = \{\{5\},\{1,2,7\},\{4,6\}\}$ are $\Gamma-(PBCK-SAs)$. Also, A is $\Gamma-(PBCK-I)$, but B is not $\Gamma-(PBCK-I)$ since $\exists \mu \in \Gamma$ with $\{3,8\}\mu\{1,2,7\} = \{4,6\} \in B$ and $\{1,2,7\} \in B$, but $\{3,8\} \notin B$.

Theorem 4.3: Let I be $\Gamma-(PBCK-I)$ of a $\Gamma-(PBCK-A)$ M, if $\lambda_j^\beta \in M, \lambda_i^\beta \in I$ and $\lambda_j^\beta \leq \lambda_i^\beta$, then $\lambda_j^\beta \in I$.

Proof. Suppose $\lambda_j^\beta \leq \lambda_i^\beta, \lambda_i^\beta \in I, \lambda_j^\beta \in M$, then $\lambda_j^\beta \alpha \lambda_i^\beta = T$ for all $\alpha \in \Gamma$. So $\lambda_j^\beta \alpha \lambda_i^\beta \in I$, thus $\lambda_j^\beta \in I$, since $\lambda_i^\beta \in I$

Definition 4.4: A non-empty subset I of a $\Gamma-(PBCK-\text{A})$. Then I is called an implicative ideal of M if it satisfies
i) $T \in I$,
ii)If $\left(\lambda_i^\beta \alpha\left(\lambda_j^\beta \beta \lambda_i^\beta\right)\right)\gamma\lambda_m^\beta \in I$ and $\lambda_m^\beta \in I$ implies $\lambda_i^\beta \in I$ for all $\lambda_i^\beta, \lambda_j^\beta \in M$, and $\alpha, \beta, \gamma \in \Gamma$.

Theorem 4.5: Every implicative ideal of a $\Gamma-(BCK-A)\ M$ is $\Gamma-(PBCK-I)$ of M.

Proof. Suppose I is an implicative ideal of M. Let $\lambda_i^\beta\gamma\lambda_m^\beta \in I, \lambda_m^\beta \in I, \lambda_i^\beta \in M, \alpha, \gamma \in \Gamma$. Then $\lambda_i^\beta\gamma\lambda_m^\beta = \left(\lambda_i^\beta\alpha T\right)\gamma\lambda_m^\beta = (\lambda_i^\beta\alpha(\lambda_i^\beta\beta\lambda_i^\beta))\gamma\lambda_m^\beta \in I$. Therefore, $\lambda_i^\beta \in I$. Hence, I is $\Gamma-(PBCK-I)$.

Definition 4.6: Let I be a non-empty subset of a $\Gamma-(PBCK-A)\ M$. Then I is called a Γ −permutation BCK −closed ideal if I is both $\Gamma-(PBCK-I)$ and a $\Gamma-(PBCK-SA)$ of M and denoted by $\Gamma-(PBCK-CI)$.

Example 4.7:
Let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 7 & 2 & 3 & 1 & 5 \end{pmatrix}$ be a permutation in S_7. So, $\beta = (1\ \ 6)(2\ \ 4)(3\ \ 7\ \ 5)$. Therefore, we have $X = \left\{\lambda_i^\beta\right\}_{i=1}^3 = \{\{1,6\},\{2,4\},\{3,5,7\}\}$ and $T = \{2,4\}$. Let $= \{\mu, \omega\}$, where μ and ω are defined by Tables (12) and (13), respectively.

Table (12): $\mu: X \times X \to X$

μ	{2,4}	{1,6}	{3,5,7}
{2,4}	{2,4}	{2,4}	{2,4}
{1,6}	{1,6}	{2,4}	{1,6}
{3,5,7}	{3,5,7}	{3,5,7}	{2,4}

Table (13): $\omega: X \times X \to X$

ω	{2,4}	{1,6}	{3,5,7}
{2,4}	{2,4}	{2,4}	{2,4}
{1,6}	{1,6}	{2,4}	{3,5,7}
{3,5,7}	{3,5,7}	{1,6}	{2,4}

Hence, $A = \{\{1,6\},\{2,4\}\}$ is a $\Gamma-(PBCK-CI)$ of the $\Gamma-(PBCK-A)\ X$.

Lemma 4.8: Let N be a $\Gamma-(PNBCK-SA)$ of a $\Gamma-(PBCK-A)\ M$. Then $[T]_N$ is a $\Gamma-(PBCK-CI)$ of M.
Proof. We consider that $[T]_N = \{\lambda_i^\beta \in M/\lambda_i^\beta \sim_N T\} = \{\lambda_i^\beta \in M/\ \lambda_i^\beta\alpha T \in N\} = \left\{\lambda_i^\beta \in M/\lambda_i^\beta \in N\right\} = N$. Suppose that $\lambda_i^\beta\alpha\,\lambda_j^\beta,\ \lambda_j^\beta \in [T]_N$. Then $\lambda_i^\beta\alpha\,\lambda_j^\beta \sim_N T$ and $\lambda_j^\beta \sim_N T$. Then $\left(\lambda_i^\beta\alpha\,\lambda_j^\beta\right)\alpha T,\ \lambda_j^\beta\alpha T \in N$. That implies $\lambda_i^\beta\alpha\,\lambda_j^\beta,\ \lambda_j^\beta\alpha\,T \in N$. This means $\lambda_i^\beta\alpha\,\lambda_j^\beta, T\alpha\,\lambda_j^\beta \in N$. Which gives $\left(\lambda_i^\beta\alpha T\right)\alpha\left(\ \lambda_j^\beta\alpha\,\lambda_j^\beta\right) \in N$. That implies $\lambda_i^\beta\alpha T \in N$. Therefore, $\lambda_i^\beta \in [T]_N$. Hence, $[T]_N$ is $\Gamma-(PBCK-I)$ of M. Let $\lambda_i^\beta,\ \lambda_j^\beta \in [T]_N$. Then $\lambda_i^\beta \sim_N T$ and $\lambda_j^\beta \sim_N T$, thus $\lambda_i^\beta\alpha\,\lambda_j^\beta \sim_N T$, and hence $\lambda_i^\beta\alpha\,\lambda_j^\beta \in [T]_N$. Therefore, $[T]_N$ is a $\Gamma-(PBCK-CI)$ of M.

Theorem 4.9: Let $f : M \to N$ be a homomorphism of $\Gamma-(PBCK-A)\ M$ and N. Then $ker(f)$ is a $\Gamma-(PBCK-CI)$ of M.

Proof. Let $f : M \to N$ be a homomorphism and $T \in M$. Then $f(T\alpha T) = f(T)\alpha f(T) \Rightarrow$

$f(T) = T$, that implies $T \in ker(f)$. Let $\lambda_i^\beta \alpha \lambda_j^\beta \in ker(f), \alpha \in \Gamma, \lambda_i^\beta, \lambda_j^\beta \in M$ and $\lambda_j^\beta \in ker(f)$. Then $T = f(\lambda_i^\beta \alpha \lambda_j^\beta) = f(\lambda_i^\beta)\alpha f(\lambda_j^\beta) = f(\lambda_i^\beta)\alpha T = f(\lambda_i^\beta)$ so $\lambda_i^\beta \in ker(f)$. Hence, $ker(f)$ is $\Gamma-(PBCK-I)$ of M: Suppose $\lambda_i^\beta, \lambda_j^\beta \in ker(f)$ and $\alpha \in \Gamma$. Then $f\left(\lambda_i^\beta \alpha \lambda_j^\beta\right) = f\left(\lambda_i^\beta\right)\alpha f\left(\lambda_j^\beta\right)$, then $f\left(\lambda_i^\beta \alpha \lambda_j^\beta\right) = T\alpha T = T$, implies $\lambda_i^\beta \alpha \lambda_j^\beta \in ker(f)$. Hence, $ker(f)$ is a $\Gamma-(PBCK-CI)$ of M.

Definition 4.10: Let I be $\Gamma-(PBCK-I)$ of a $\Gamma-(PBCK-A)\,M$. Then I is called Γ −permutation normal BCK −ideal of M if it is a $\Gamma-(PNBCK-SA)$ and denoted by $\Gamma-(PNBCK-I)$.

Example 4.11: Consider $Y = \{\{1,3\},\{2,6\},\{5,8\},\{4,7,9\}\}$ and $\Gamma = \{\omega_1, \omega_2\}$ in Example 3.4. Hence Y is $\Gamma-(PBCK-A)$. Let $G = \{\{1,3\},\{2,6\}\} \subseteq Y$. Hence, $G = \{\{1,3\},\{2,6\}\}$ is a $\Gamma-(PNBCK-I)$ of the $\Gamma-(PBCK-A)\,X$.

Theorem 4.12: Let I be a $\Gamma-(PNBCK-I)$ of a $\Gamma-(PBCK-A)\,M$. Then I is a $\Gamma-(PBCK-SA)$ of M.

Proof. Let I be a $\Gamma-(PNBCK-I)$ of M and $\lambda_i^\beta, \lambda_j^\beta \in I, \alpha, \beta \in \Gamma$. Then $\lambda_i^\beta \alpha \lambda_i^\beta = T \in I$ and $\lambda_j^\beta \alpha T = \lambda_j^\beta \in I$. Therefore, $\left(\lambda_i^\beta \alpha \lambda_j^\beta\right)\beta\left(\lambda_i^\beta \alpha T\right) = \left(\lambda_i^\beta \alpha \lambda_j^\beta\right)\beta \lambda_i^\beta \in I$, since I is a $\Gamma-(PNBCK-I)$. Therefore, $\lambda_i^\beta \alpha \lambda_j^\beta \in I$. Hence, I is a $\Gamma-(PBCK-SA)$ of M.

Theorem 4.13: If I is a $\Gamma-(PNBCK-SA)$ of a $\Gamma-(PBCK-A)\,M$, then I is a $\Gamma-(PNBC-I)$ of M.

Proof. Let I be a $\Gamma-(PNBCK-SA)$ of M. Suppose $\lambda_i^\beta \alpha \lambda_j^\beta \in I$, $\lambda_j^\beta \in I$ and $\alpha, \beta \in \Gamma$, we have $T\alpha \lambda_j^\beta = T \in I$. Then $\left(\lambda_i^\beta \alpha T\right)\beta\left(\lambda_j^\beta \alpha \lambda_j^\beta\right) \in I$, that implies $\lambda_i^\beta \beta T \in I$, means $\lambda_i^\beta \in I$.
Hence, $\lambda_i^\beta \in I$.

Definition 4.14: A homomorphism $f : M \to N$, where M, N are $\Gamma-(PBCK-A)$, is said to be a normal homomorphism, if $ker\ f$ is a $\Gamma-(PNBCK-I)$ of M.

Example 4.15: Consider $= \left\{\lambda_i^\beta\right\}_{i=1}^4 = \{\{1,2,7\},\{3,8\},\{4,6\},\{5\}\}$ in Example 4.7. Let $\Gamma = \{\mu, \omega\}$, where μ and ω are defined by Tables (14) and (15), respectively.

Table (14): $\mu: X \times X \to X$

μ	{5}	{1,2,7}	{4,6}	{3,8}
{5}	{5}	{5}	{5}	{5}
{1,2,7}	{1,2,7}	{5}	{1,2,7}	{5}
{4,6}	{4,6}	{4,6}	{5}	{5}
{3,8}	{3,8}	{4,6}	{1,2,7}	{5}

Table (15): $\omega: X \times X \to X$

ω	{5}	{1,2,7}	{4,6}	{3,8}
{5}	{5}	{5}	{5}	{5}
{1,2,7}	{1,2,7}	{5}	{4,6}	{4,6}
{4,6}	{4,6}	{3,8}	{5}	{3,8}
{3,8}	{3,8}	{3,8}	{3,8}	{5}

Therefore, X is $\Gamma-(PBCK-A)$. Define $g: X \to X$ by $g(\{5\}) = g(\{1,2,7\}) = g(\{4,6\}) = \{5\}$ and $g(\{3,8\}) = \{3,8\}$. Hence, g is a normal homomorphism.

Theorem 4.16: Let $f : M \to L$ be a normal homomorphism of $\Gamma - (PBCK - A)$ M and N. Then $\Gamma - (PBCK - A)$ $M = N$ is isomorphic to $Im(f)$ where $N = \ker(f)$:

Proof. Let $f : M \to L$ be a normal homomorphism of $\Gamma - (PBCK - A)$ of M and N. Then by definition $\ker(f)$ is a $\Gamma - (PNBCK - I)$ of M. Let $N = ker(f)$. Therefore, $\ker(f)$ is a $\Gamma - (PNBCK - SA)$ of M. Define a mapping $\varphi : M/N \to Im(f)\ by\ \varphi\left(\left[\lambda_i^\beta\right]_N\right) = f\left(\lambda_i^\beta\right), \forall\, \lambda_i^\beta \in M$. Let $\left[\lambda_i^\beta\right]_N = \left[\lambda_j^\beta\right]_N$. Then $\lambda_i^\beta \sim_N \lambda_j^\beta$, [i. e., $\lambda_i^\beta \alpha\, \lambda_j^\beta \in N$ and $\lambda_j^\beta \alpha \lambda_i^\beta \in N$], that implies $\left(\lambda_i^\beta\right) \alpha f\left(\lambda_j^\beta\right) = [T]_L = f\left(\lambda_j^\beta\right) \alpha f\left(\lambda_i^\beta\right)$, implies $\left(\lambda_i^\beta\right) = f\left(\lambda_j^\beta\right)$, therefore $\varphi(\left[\lambda_i^\beta\right]_N) = \varphi(\left[\lambda_j^\beta\right]_N)$. Hence, φ is well defined. $\varphi\left(\left[\lambda_i^\beta\right]_N \alpha \left[\lambda_j^\beta\right]_N\right) = \varphi\left(\left[\lambda_i^\beta \alpha\, \lambda_j^\beta\right]_N\right) = f(\lambda_i^\beta \alpha\, \lambda_j^\beta) = f(\lambda_i^\beta) \alpha f(\lambda_j^\beta) = \varphi([x]_N) \alpha \varphi(\left[\lambda_j^\beta\right]_N)$. Then φ is a homomorphism from $M/\,ker(f) \to Im(f)$. Thus;

$\varphi\left(\left[\lambda_i^\beta\right]_N\right) = T_{Im(f)}$, that implies $f\left(\lambda_i^\beta\right) = T_{Im(f)}$,

$\Rightarrow \lambda_i^\beta \in ker(f)$, implies $\lambda_i^\beta \in N$.

Therefore, $\left[\lambda_i^\beta\right]_N = [T]_N$:

Therefore, φ is one-one. Hence, φ is an isomorphism from $M/\,ker(f)\ onto\ Im(f)$.

5. Conclusions

In this paper, we extended the concepts of $(PBCK - A)$ to $\Gamma - (PBCK - A)$, and quotient $\Gamma - (PBCK - A)$. We studied $\Gamma - (PBCK - CI)$, $\Gamma - (PNBCK - I)$ and some of properties were investigated. We proved that a $\Gamma - (PNBCK - I)$ of a $\Gamma - (PBCK - A)$ is a $\Gamma - (PBCK - SA)$, if $f : M \to N$ is a homomorphism of $\Gamma - (PBCK - A)$, then $ker(f)$ is a $\Gamma - (PBCK - CI)$. We further study fuzzy implicative ideals of $\Gamma - (PBCK - A)$.

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