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Homotopy Perturbation Method and Convergence Analysis for the Linear Mixed Volterra-Fredholm Integral Equations

Pakhshan Mohammed Ameen Hasan*, Nejmaddin Abdulla Sulaiman
Department of Mathematics, Collage of Education, Salahddin University, Erbil, Iraq

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Abstract

In this paper, the homotopy perturbation method is presented for solving the second kind linear mixed Volterra-Fredholm integral equations. Then, Aitken method is used to accelerate the convergence. In this method, a series will be constructed whose sum is the solution of the considered integral equation. Convergence of the constructed series is discussed, and its proof is given; the error estimation is also obtained. For more illustration, the method is applied on several examples and programs, which are written in MATLAB (R2015a) to compute the results. The absolute errors are computed to clarify the efficiency of the method.

Keywords: Aitken method; homotopy perturbation method; second kind linear mixed Volterra-Fredholm integral equations (LMVFIE2nd).

طريقة اضطرابHomotopy وتحليل التقارب لمعادلات فولتيرا – فريدهولم التكاملية الخطية

به خشان محمد امين* ، نجم الدين عبدالله سليمان
قسم الرياضيات، كلية التربيه، جامعه صلاح الدين، اربيل ، العراق

ألخلاصة

في هذه الورقة ، تم تقديم طريقة اضطراب homotopy لحل معادلات Volterra–Fredholm التكاملية المدمجة المختلطة من النوع الثاني ، ثم يتم استخدام طريقة Aitken لتسريع التقارب. تقوم الطريقة ببناء سلسلة يكون مجموعها هو حل المشكلة المدروسة. تمت مناقشة تقارب السلسلة التي تم بناؤها وتقديم دليلها ؛ يتم الحصول على تقدير الخطأ أيضًا. لمزيد من التوضيح ، يتم تطبيق الطريقة على العديد من الأمثلة ويتم كتابة البرامج في MATLAB لحساب النتائج. يتم حساب الأخطاء المطلقة لتوضيح كفاءة الطريقة.

1. Introduction

Volterra-Fredholm Integral equations have received significant meaning in mathematical physics, biology, and contact problems in the theory of elasticity [1, 2]. Also, their solutions can be found analytically in previous investigations [3]. At the same time, the sensing of numerical methods takes an important place in solving these equations [4, 5].

The second kind mixed Volterra-Fredholm integral equation that will be considered in this work has the form

$$u(x) = f(x) + \lambda \int_a^x \int_a^b k(r,t)u(t)dtdr \qquad a \le x \le b \qquad (1)$$

*Email: pakhshansoran@gmail.com

Where the functions $f(x) \in C[a,b]$ and $k(r,t)$ are continuous on $D = \{(r,t): a \leq t \leq b \,\&\, a \leq r \leq x \leq b\}$, while $u(x)$ is the unknown continuous function in $[a,b]$ to be found.

Ibrahim used a new iterative method for solving the mixed Volterra-Fredholm integral equations [6]. Wazwaz treated this problem by using the method of series solutions and the Adomian decomposition method [7]. In addition, Wang used the least square approximation method to solve this type of equations [8]. Ezzati and Najafalizadeh used Cas wavelets for solving Volterra-Fredholm integral equations [9].

Homotopy perturbation method has been used by many authors for different purposes. Mirzaei used it to solve the first kind Fredholhm integral equations [10]. Also, it was used by Biazar to find the exact solution of non-linear Volterra-Fredholm integro-differential equation [11]. In addition, the method was introduced and developed by Ji-Huan He to solve linear and nonlinear problems [12]. Behzazi used it for solving first kind non-linear Volterra- Fredholm integral equations [13], while it was employed by Li for solving non-linear equations [14] and by Vahidi and Isfahani for solving second kind Abel integral equation [15].

In the present work, an approximate solution of equation (1) using the homotopy perturbation method is discussed. The main goal of this paper is to present a convergence condition for the method and the error estimation for the solution. Also, Aitken method is used to accelerate the convergence of the approximation.

2. Basic concepts [16]

This section deals with some important concepts which are used in this work.

Theorem1. Let $\{p_n\}_{n=0}^{\infty}$ be any sequence converging linearly to the limit p with $e_n = p_n - p \neq 0$ for all $n \geq 0$. The sequence $\{\hat{p}_n\}_{n=0}^{\infty}$ converges to p faster than $\{p_n\}_{n=0}^{\infty}$ in the sense $\lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} = 0$, where $\hat{p}_n = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$.

3. Homotopy Perturbation Method (HPM)

Recall equation (1) and define the operator L as follows:

$$L(u^*) = u^*(x) - f(x) - \lambda \int_a^x \int_a^b k(x,t)u^*(t)dtdr = 0 \qquad (2)$$

With the solution is $u^*(x) = u(x)$. The homotopy perturbation method defines a convex homotopy $H(u^*,p): \mathcal{R} \times [0,1] \to \mathcal{R}$ by

$$H(u^*,p) = (1-p)F(u^*) + pL(u^*) = 0 \qquad (3)$$

where $F(u^*) = u^*(x) - f(x)$ is a functional operator, $p \in [0,1]$ is the homotopy parameter, and $u_0(x)$ defines the initial solution of equation (1).

From equation (3), we have

$$H(u^*,0) = F(u^*), \qquad H(u^*,1) = L(u^*) \qquad (4)$$

where the changing of the imbedding parameter p from 0 to 1 is just that of $H(u^*,p)$ from the trivial problem $H(u^*,0) = F(u^*) = 0$ to the original problem $H(u^*,1) = L(u^*) = 0$. In topology, it is called deformation while $H(u^*,0)$ and $H(u^*,1)$ are called homotopic.

By using $L(u^*)$ and $F(u^*)$ defined above, the homotopy operator of the considered equation will be obtained:

$$H(u^*,p) = (1-p)\big(u^*(x) - f(x)\big) + p\left(u^*(x) - f(x) - \lambda \int_a^x \int_a^b k(r,t)u^*(t)dtdr\right)$$

Thus

$$H(u^*,p) = u^*(x) - f(x) + p\left(-\lambda \int_a^x \int_a^b k(r,t)u^*(t)dtdr\right) = 0 \qquad (5)$$

The method admits the use of power series

$$u^*(x) = \sum_{i=0}^{\infty} p^i u^*{}_i(x) \tag{6}$$

If equation (6) has a radius of convergence not less than one, and the series $\sum_{i=0}^{\infty} u^*{}_i(x)$ converges absolutely, then by Abel's theorem the approximate solution of equation (1) is found:

$$u(x) = \lim_{p \to 1^-} \sum_{i=0}^{\infty} p^i u^*{}_i(x) = \sum_{i=0}^{\infty} u^*{}_i(x) \tag{7}$$

Substituting (6) in equation (5) gives

$$\sum_{i=0}^{\infty} p^i u^*{}_i(x) = u_0(x) + p\left(\lambda \int_a^x \int_a^b k(r,t) \sum_{i=0}^{\infty} p^i u^*{}_i(t)\,dt\,dr\right)$$

and equating the same power terms of the embedding parameter p gives the recurrence relations that leads finally to the approximate solution:

$$p^0 \colon u^*{}_0(x) = u_0(x) \tag{8}$$

$$p^i \colon u^*{}_i(x) = \lambda \int_a^x \int_a^b k(r,t) u^*{}_{i-1}(t)\,dt\,dr\,, \qquad i \geq 1 \tag{9}$$

The above relations are obtained with the assumption that the series (6) is convergent. In the following Theorem, the conditions for such convergence will be discussed.

4. Convergence Analysis

Theorem 4.1 Let $k \in C([a,b] \times [a,b])$ and $f \in C[a,b]$, if moreover the following inequality

$$|\lambda| M < \frac{1}{(b-a)^2} \tag{10}$$

is satisfied, and as an initial solution $u_0 \in C[a,b]$ is chosen, then for each $p \in [0,1]$, the series (6) converges uniformly in the interval $[a,b]$ in which the functions $u^*{}_i$ are found by equations (8) and (9).

Proof: Since *k* and *f* are bounded, then there exist positive numbers M and L such that

$$|k(x,t)| \leq M \text{ and } |f(x)| \leq L \qquad \forall x,t \in [a,b] \tag{11}$$

Let $u_0(x) \in C[a,b]$. Therefore, there exists a positive number L_0 such that

$|u_0(x)| \leq L_0 \quad \forall x \in [a,b]$

The above assumptions imply the estimations below:

$$|u^*{}_0(x)| = |u_0(x)| \leq L_0$$

$$|u^*{}_1(x)| = \left|\lambda \int_a^x \int_a^b k(r,t) u^*{}_0(t)\,dt\,dr\right| \leq |\lambda| \int_a^x \int_a^b |k(r,t)||u^*{}_0(t)|\,dt\,dr$$

$$\leq |\lambda| M L_0 (b-a)^2$$

In general, we have

$$|u^*{}_i(x)| = \left|\lambda \int_a^x \int_a^b k(r,t) u^*{}_{i-1}(t)\,dt\,dr\right| \leq |\lambda| \int_a^x \int_a^b |k(r,t)||u^*{}_{i-1}(t)|\,dt\,dr$$

$$\leq \left|\lambda^i\right| M^i L_0 ((b-a)^2)^i\,, \quad \forall x \in [a,b], \qquad i \geq 1$$

In this way, for series (6), we have, for $p \in [0,1]$,

$$\sum_{i=0}^{\infty} p^i u^*{}_i(x) \leq \sum_{i=0}^{\infty} |u^*{}_i(x)| \leq L_0 \left(\sum_{i=0}^{\infty} \left|\lambda^i\right| M^i ((b-a)^2)^i\right)$$

The above last series is a geometric series with the common ratio $r = |\lambda| M (b-a)^2 < 1$ which is convergent (by the assumption in equation (10)). Hence, series (7) converges uniformly in the interval $[a,b]$ for each $p \in [0,1]$. ■

If it is impossible or difficult to calculate the sum of series (7), for *p*=1, then the partial sum of it can be accepted as an approximate solution of equation (1). In the limit for $p \to 1$, the first *n*+1 terms of series (7) produce the *n*th-order approximate solution in the form

$$\hat{u}_n(x) = \sum_{i=0}^{n} u^*{}_i(x) \qquad (12)$$

The level of the error of the solution $\hat{u}_n(x)$ can be estimated by the following theorem:

Theorem 4.2 The estimated error of the *n*th-order approximate solution can be determined as follows:

$\varepsilon_n \leq L_0 \dfrac{|\lambda^{n+1}| M^{n+1}((b-a)^2)^{n+1}}{1-|\lambda| M(b-a)^2}$,

Where $\varepsilon_n = \sup_{x\in[a,b]}|u(x)-\hat{u}_n(x)|$, *M* and L_0 are determined in theorem (4.1).

Proof: For any $x \in [a,b]$, the use of estimations of functions $u^*{}_i(x)$ gives

$$|u(x)-\hat{u}_n(x)| = \left|\sum_{i=0}^{\infty} u^*{}_i(x) - \sum_{i=0}^{n} u^*{}_i(x)\right| = \left|\sum_{i=n+1}^{\infty} u^*{}_i(x)\right|$$

$$\leq \sum_{i=n+1}^{\infty} |u^*{}_i(x)| \leq L_0\left(\sum_{i=n+1}^{\infty} |\lambda^i| M^i((b-a)^2)^i\right)$$

$$= L_0|\lambda^{n+1}|M^{n+1}((b-a)^2)^{n+1}\{1+|\lambda|M(b-a)^2+\cdots\}$$

$$= L_0 \frac{|\lambda^{n+1}|M^{n+1}((b-a)^2)^{n+1}}{1-|\lambda|M(b-a)^2}. \quad \blacksquare$$

5. The HPM Algorithm

To find an approximate solution of (LMVFIE2nd) by HPM, perform the following steps:

Step 1: select positive integers *a*, *b*, and *n*.

Step 2: put $u_0(x) = f(x)$ as an initial approximation.

Step 3: calculate $u_i(x)$ in equations (8-9) for all $i = 1, 2, \ldots, n$.

Step 4: compute the partial sum $\hat{u}_n(x) = \sum_{i=0}^{n} u^*{}_i(x)$ from equation (12).

Step 5: find $\hat{u}_n(x_j)$, for some $x_j \in [a,b]$.

Step 6: compute the absolute error of each root $|u(x_j) - \hat{u}_n(x_j)|$.

6. Acceleration of the Approximation (Aitken)

In this section, the Aitken's method has been applied successfully on the homotopy perturbation method to find the solution of our integral equation, where the first three approximations are computed by HPM as discussed in section 3 and then substituted in the definition (2.3) to get the following procedure:

$$U_i(x) \simeq \frac{\hat{u}_{i+1}(x)\hat{u}_{i-1}(x) - \hat{u}_i^2(x)}{\hat{u}_{i+1}(x) - 2\,\hat{u}_i(x) + \hat{u}_{i-1}(x)}; \quad i = 1,2,\ldots,n \qquad (13)$$

where good estimations and sometimes exact solutions will be found.

7. Numerical examples

Several examples will be solved in this section to show the accuracy of our approach.

Example 1 Consider the following LMVFIE2nd

$$u(x) = cos(x) + sin(x) - \frac{1}{4}x^2 + \frac{\pi}{8}x + \frac{1}{4}\int_0^x \int_0^{\frac{\pi}{2}} (r-t)u(t)dtdr, \qquad 0 \leq x \leq \frac{\pi}{2}$$

The exact solution is $u(x) = \cos(x) + sin\,(x)$.

First, it will be verified whether the described method can be used for solving this problem. Since $k \in C\left(\left[0,\frac{\pi}{2}\right] \times \left[0,\frac{\pi}{2}\right]\right)$ and $f \in C\left(\left[0,\frac{\pi}{2}\right]\right)$, then we check the satisfying of inequality (10). In this example $|\lambda| = \frac{1}{4}$, and $M = \max_{r,t\in[0,\frac{\pi}{2}]}|k(r,t)| = \frac{\pi}{2}$

$$0.392699082 = \frac{\pi}{8} = |\lambda| M < \frac{1}{(b-a)^2} = \frac{1}{(\frac{\pi}{2}-0)^2} = \frac{1}{2.4674011003} = 0.405284735$$

This means that HPM can be applied if a continuous function u_0 is selected in the interval $[0, \frac{\pi}{2}]$. Let $u_0(x) = f(x)$. Applying the algorithm of the HPM with different values of *n*, the following *n*th-order approximate solutions resulted:

$$u_0{}^*(x) = u_0(x) = cos(x) + sin(x) - \frac{1}{4}x^2 + \frac{\pi}{8}x$$

$$\hat{u}_6(x) = \cos(x) + \sin(x) - \frac{x(2x-\pi)}{8} + \frac{x(\pi^3+384)(2x-\pi)}{3072} - \frac{x\pi^3(\pi^3+384)(2x-\pi)}{1179648} + \frac{(x\pi^6(\pi^3+384)(2x-\pi))}{452984832} - \frac{x\pi^9(\pi^3+384)(2x-\pi)}{173946175488} + \frac{x\pi^{12}(\pi^3+384)(2x-\pi)}{66795331387392} - \frac{(x\pi^{15}(\pi^3+384)(2x-\pi))}{25649407252758528}$$

$$\hat{u}_9(x) = \cos(x) + \sin(x) - \frac{x(2x-\pi)}{8} + \frac{x(\pi^3+384)(2x-\pi)}{3072} - \frac{x\pi^3(\pi^3+384)(2x-\pi)}{1179648} + \frac{(x\pi^6(\pi^3+384)(2x-\pi))}{452984832} - \frac{x\pi^9(\pi^3+384)(2x-\pi)}{173946175488} + \frac{x\pi^{12}(\pi^3+384)(2x-\pi)}{66795331387392} - \frac{(x\pi^{15}(\pi^3+384)(2x-\pi))}{25649407252758528} + \frac{x\pi^{18}(\pi^3+384)(2x-\pi)}{9849372385059274752} - \frac{x\pi^{21}(\pi^3+384)(2x-\pi)}{3782158995862761504768} + \frac{x\pi^{24}(\pi^3+384)(2x-\pi)}{1452349054411300417830912}$$

$$\hat{u}_{12}(x) = \cos(x) + \sin(x) - \frac{x(2x-\pi)}{8} + \frac{x(\pi^3+384)(2x-\pi)}{3072} - \frac{x\pi^3(\pi^3+384)(2x-\pi)}{1179648} + \frac{(x\pi^6(\pi^3+384)(2x-\pi))}{452984832} - \frac{x\pi^9(\pi^3+384)(2x-\pi)}{173946175488} + \frac{x\pi^{12}(\pi^3+384)(2x-\pi)}{66795331387392} - \frac{(x\pi^{15}(\pi^3+384)(2x-\pi))}{25649407252758528} + \frac{x\pi^{18}(\pi^3+384)(2x-\pi)}{9849372385059274752} - \frac{x\pi^{21}(\pi^3+384)(2x-\pi)}{3782158995862761504768} + \frac{x\pi^{24}(\pi^3+384)(2x-\pi)}{1452349054411300417830912} - \frac{x\pi^{27}(\pi^3+384)(2x-\pi)}{557702036893939360447070208} + \frac{x\pi^{30}(\pi^3+384)(2x-\pi)}{2141575821672727144116749598 72} - \frac{x\pi^{33}(\pi^3+384)(2x-\pi)}{8223651155223272233408318459 0848}$$

The absolute error of each of them is presented in Table-1.

Now to apply Aitken method, only the first three approximate solutions $\hat{u}_1(x)$, $\hat{u}_2(x)$, and $\hat{u}_3(x)$ will be found by HPM and then substituted in the Aitken procedure of equation (13) to get

$$U_1(x) \simeq \frac{\hat{u}_3(x)\hat{u}_1(x) - \hat{u}_2{}^2(x)}{\hat{u}_{3(x)} - 2\,\hat{u}_2(x) + \hat{u}_1(x)}$$

which gives the exact solution by only one iteration. The outcomes are tabulated in Table-1.

Table 1-The results of Example 1 using HPM with $n = 6, 9, 12$ and Aitken with only one iteration

x_i	exact solution $u(x_i)$	absolute error for $u(x_i)$ using			
		HPM			Aitken
		$\lvert u - \hat{u}_6(x)\rvert$	$\lvert u - \hat{u}_9(x)\rvert$	$\lvert u - \hat{u}_{12}(x)\rvert$	$\lvert u - U_1(x)\rvert$
0					
0.05π	1	0	0	0	0
0.10π	1.1441228056	1.5386245e-08	8.0999651e-12	4.4408921e-15	0
0.15π	1.2600735107	2.7353324e-08	1.4400259e-11	7.5495166e-15	0
0.20π	1.3449970239	3.5901238e-08	1.8900215e-11	9.7699626e-15	0
0.25π	1.3968022467	4.1029986e-08	2.1600278e-11	1.1324275e-14	0
0.30π	1.4142135624	4.2739569e-08	2.2500224e-11	1.1768364e-14	0
0.35π	1.3968022467	4.1029986e-08	2.1600277e-11	1.1324275e-14	0
0.40π	1.3449970239	3.5901238e-08	1.8900215e-11	9.7699626e-15	0
0.45π	1.2600735107	2.7353324e-08	1.4400259e-11	7.5495166e-15	0
0.50π	1.1441228056	1.5386245e-08	8.0999651e-12	4.4408921e-15	0
	1	0	0	0	

Example 2. Consider the following mixed V-F integral equation

$$u(x) = xe^x - \frac{x^2}{4} + \frac{1}{2}\int_0^x \int_0^1 ru(t)dt\ , \qquad 0 \le x \le 1$$

The exact solution is $u(x) = xe^x$.

In this example, we have $\frac{1}{2} = |\lambda|M < \frac{1}{(b-a)^2} = 1$.

As described in section 3,

Let $u_0(x) = f(x)$

$$u_0{}^*(x) = u_0(x) = xe^x - \frac{x^2}{4}$$

By choosing different values of (*n*), different values of the *n*th-order approximate solutions will be found and the results are listed in Table 1.

Now to accelerate the convergence, the approximate solutions $\hat{u}_1(x)$, $\hat{u}_2(x)$, and $\hat{u}_3(x)$ will be found by HPM as follows:

$$\hat{u}_1(x) = \sum_{i=0}^{1} u^*{}_i(x) = xe^x - \frac{x^2}{48}\ ,\ \hat{u}_2(x) \sum_{i=0}^{2} u^*{}_i(x) = xe^x - \frac{x^2}{576}$$

$$\hat{u}_3(x) = \sum_{i=0}^{3} u^*{}_i(x) = xe^x - \frac{x^2}{6912}$$

then they are substituted in the Aitken procedure to get $U_1(x)$ which gives the exact solution with only one iteration. The outcomes are compared with HPM in Table-2.

Table 2-The results of Example 2 using HPM with $n = 3, 6, 9$ and Aitken with $n = 1$

x_i	exact solution $u(x_i)$	absolute error for $u(x_i)$ using			
		HPM			Aitken
		$\lvert u - \hat{u}_3(x)\rvert$	$\lvert u - \hat{u}_6(x)\rvert$	$\lvert u - \hat{u}_9(x)\rvert$	$\lvert u - U_1(x)\rvert$
0.0	0	0	0 8.3724494e-	0 4.8451897e-	0
0.1	0.1105170918	1.4467592e-06	10	13	0
0.2	0.2442805516	5.7870370e-06	3.3489798e-09	1.9380759e-12	0
0.3	0.4049576423	1.3020833e-05	7.5352045e-09	4.3606707e-12	0
0.4	0.5967298791	2.3148148e-05	1.3395919e-08	7.7521925e-12	0
0.5	0.8243606354	3.6168981e-05	2.0931124e-08	1.2112919e-11	1.0 e-15
0.6	1.0932712802	5.2083333e-05	3.0140818e-08	1.7442683e-11	0

0.7	1.4096268952	7.0891204e-05	4.1025002e-08	2.3741374e-11	0
0.8	1.7804327428	9.2592593e-05	5.3583676e-08	3.1009214e-11	0
0.9	2.2136428000	1.1718750e-04	6.7816840e-08	3.9245981e-11	1.0 e-15
1.0	2.7182818285	1.4467593e-04	8.3724494e-08	4.8451675e-11	0

8. Conclusion

In this paper, the sufficient condition of the convergence of the homotopy perturbation method for the second kind linear mixed Volterra–Fredholm integral equations is formulated and proved. Also, the estimation of the error is given. Moreover, HPM is used to solve the presented equation and then the solution is accelerated by Aitken formula. Two examples illustrated the accuracy by obtaining good approximate results. The results from HPM and Aitken are compared with the exact solutions to demonstrate the implementation of the method. Also, it should be considered that better results could have been obtained in HPM by increasing the number of components of the partial sum (n). The given numerical examples and the outcomes in tables 1 and 2 supported these claims.

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