Bimodal Transitive Maps with Zero Topological Entropy

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Abstract
Our goal in this work is to describe the structure of a class of bimodal self maps on the compact real interval I with zero topological entropy and transitive.

Keyword :bimodal map , transitive map , topological entropy.

الدوال المتعدية ذات النسقين صفرية الانتروبي

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الخلاصة
في هذا العمل درسنا صف من الدوال المتعدية ذات النسقين على فترة مغلقة عندما الانتروبي مساويا صفرا.

1 -Introduction

One of the central questions in the theory of dynamical systems is how to recognize chaos and how to see how large is it? One of the best known methods of measuring chaoticity is by means of topological entropy of the system (see [1] for definition). Then we can restate our question as: how can one get estimates for topological entropy form other properties of the system?

Many authors introduced various definitions of chaos . The notion of chaos that was introduced by Li and Yorke [2] is weaker than other definitions [3]. It turns out that chaos in the sense of Li and Yorke is a weaker property than positive topological entropy (and is equivalent to the property that the map has a trajectory which is not approximable by cycles [4]. An example showing this is given in [5].

M.Misiurewicz and J.Smital in [6] find a class of C^{∞} maps of the interval with zero topological entropy and chaotic in the sense of Li and Yorke , there maps are unimodal . In this work we study an analogous class of bimodal maps of the interval with zero entropy and transitive.

2-Preliminary Definitions

Let I be a compact real interval , $f: I \to I$ denotes a continuous map of I into itself , we use the symbol f^n to denote $f^\circ f^\circ f \dots {}^\circ f$(n-times) f^0 denotes the identity map on I. For a compact real interval I and a continuous map $f: I \to I$,there are several ways of describing the behavior which has chaotic properties, one of these properties is transitivity

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Definition 2.1 [7]

A map $f: I \to I$ is called transitive if f is onto and there is a point with a dense orbit.

Definition 2.2 [4]

We recall that the w-limit set of $x \in I$, denoted by $w_f(x)$ is the set of limit points of the sequence $\{f^n(x)\}_{n=0}^{\infty}$.

Definition 2.3 [4]

F is chaotic (in the sense of Li and Yorke) if there is some $\epsilon > 0$ and a non-empty perfect set $S \subset I$ such that for any $x, y \in S$,$x \neq y$ and any periodic point p of f ,

$\lim_{n\to\infty} \sup|f^n(x) - f^n(y)| \geq \epsilon$ …(2.1)

$\lim_{n\to\infty} inf|f^n(x) - f^n(y)| = 0$ …(2.2)

$\lim_{n\to\infty} \sup|f^n(x) - f^n(p)| \geq \epsilon$ …(2.3)

Theorem 2.4

If there is a point $x \in I$ such that the set $w_f(x)$ is infinite and f is not injective on $w_f(x)$ then f is transitive.

Proof: By[theorem 1,10], f is chaotic in the sense of Li and Yorke. Let U,V be open sets . We may assume $U \cap V = \varphi$. By the definition of Li and Yorke , there is a perfect set S with the properties listed easlier section thus $U \cap S \neq \varphi$. $\forall x \in U \cap S$ and $y \in U \cap S, x \neq y$, $inf|f^n(x) - f^n(y)| = 0$. This implies, for all open neighbourhood $M_x, N_y, f^n(M_x) \cap f^n(N_y) \neq \varphi$.Then by [theorem2.2]there is a positive integer m such that $f^m(M_x) \cap N_y \neq \varphi$ this implies f is transitive.■

3- Preliminary Constructions

Let $f: I \to I$ be a continuous bimodal map , we will define an equivalence relation on I and study the maps from the quotient space into itself , we start by the following :

Definition 3.1

Let $f: I \to I$ be a continuous map where I=[a,b] f is called bimodal if there exists $c_1, c_2 \in (a, b)$ such that f is strictly increasing on $[a, c_1]$ and$[c_2, b]$ and decreasing on $[c_1, c_2]$.

We shall call f weakly bimodal if there exists $c_1, c_2 \in (a, b)$ such that f is non-decreasing on $[a, c_1]$ and$[c_2, b]$ and non-increasing on $[c_1, c_2]$.

Let $f: I \to I$ be weakly bimodal , we say that $x, y \in I$ are equivalent (denoted by $x \sim y$) if there exists $n \geq 1$ such that f^n is constant on $[x, y] or [y, x]$ (x may be equal to y). The definition of this relation is similar to the relation given for unimodal map in [5].

Proposition 3.2

Let $I = [a, b]$ and $f: I \to I$ be a continuous map. Define a relation $\sim$ on I as above . Then this relation is an equivalence relation.

Proof: Since for all ≥ 1 , f^n is constant on the closed degenerate interval {x} then the relation $\sim$ is reflexive . Let $x, y \in I$ such that $\sim y$, then by definition of the relation $y \sim x$, hence the relation is symmetric . Let $x, y, z \in I$ such that $x \sim y \; and \; y \sim z$ then there is $n_1 \in N$ such that f^{n_1} is constant on $[x, y]$ and there is n_2 such that f^{n_2} is constant on $[y, z]$. Then $f^{n_1 n_2}$ is constant on $[x, z]$. Thus the relation $\sim$ is transitive. ■

Each equivalence class is a closed interval , possibly degenerate to a point .

Let $\bar{I} = I/\sim$ be the quotient space , $\bar{I}$ can be identified with a closed interval . This interval may degenerate to a point , $\bar{I}$ is equipped with the quotient topology. We will define order in $\bar{I}$ as follows , for all $[x_1], [x_2] \in \bar{I}, [x_1] < [x_2]$ means for all $y_1 \in [x_1]$ and$y_2 \in [x_2]$,$y_1 < y_2$ …(1).Let $\pi: I \to \bar{I}$ be the natural projection , then π is continuous .

Remark 3.3

π is non-decreasing.

Proof: Let $x_1, x_2 \in I$such that $x_1 < x_2$, we consider two cases:

Case (1): If there is $n \in N$ .Such that f^n is constant on $[x_1, x_2]$ or $[x_2, x_1]$ then $[x_1] = [x_2]$ (by the definition). This implies $\pi(x_1) = \pi(x_2)$.

Case (2) : There is no such n , then $[x_1] \neq [x_2]$. Thus either $[x_1] < [x_2]$ or $[x_2] < [x_1]$, if $[x_2] < [x_1]$ then by definition (1) above , for all

$y_2 \in [x_2]$ and for all $y_1 \in [x_1]$, $y_2 < y_1$ in particular $x_2 < x_1$. But this is a contradiction , hence $[x_1] < [x_2]$, that is ,$\pi(x_1) < \pi(x_2)$. ■

Proposition 3.4

Let $f: I \to I$ and $x, y \in I$ such that $x \sim y$ then $f(x) \sim f(y)$

Proof :Since $x \sim y$ then there is $n \in N$ such that f^n is constant on $[x, y]$ or $[y, x]$ thus f^{n+1} is also constant on $[x, y]$ or $[y, x]$.Hence f^n is constant on $[f(x), f(y)]$ or $[f(y), f(x)]$,this implies $f(x) \sim f(y)$. ■

We now define $F: \bar{I} \to \bar{I}$ by $F[y] = [f(y)]$.It is clear that this is a good definition.Note that $F\pi = \pi f$. The proof of the following proposition is simple:

Proposition 3.5

Let $F: \bar{I} \to \bar{I}$ be the function define above then F is continuous.

Proposition 3.6

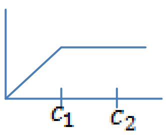
Let $f: I \to I$ be a weakly bimodal map and let $F: \bar{I} \to \bar{I}$ be the function defined above . Then F has one of the following properties:

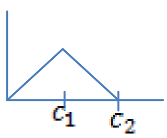
1)monotone

2)unimodal

3)bimodal

Proof: Let f be weakly bimodal map then there is $c_1, c_2 \in I$ two critical points such that f is non-decreasing on $[a, c_1]$ and $[c_2, b]$ …(2) and f is non increasing on $[c_1, c_2]$…(3). Thus f has one of the three forms given in Figures (a),(b) and (c).





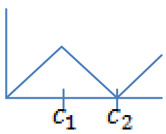


Figure (a) Figure (b) Figure (c)

Proposition 3.7

Let $f: I \to I$ be a weakly bimodal and transitive ,let $F: \bar{I} \to \bar{I}$ be the function define above then F is tranitive.

Proof :Since f is transitive then by definition 2.1 , f is onto and there is a point $x \in I$ such that the orbit of x is dence in I . That is , $\{f^n(x): n \in N\}$ is dence in I, hence $f^i(x) \neq f^j(x)$, hence f is nont constant on $[f^i(x), f^j(x)]$ or $[f^j(x), f^i(x)]$ for all $i, j \in N, i \neq j$. Thus $[f^i(x)] \neq [f^j(x)]$ this implies $F^i[x] \neq F^j[x]$, for all $i \neq j$. Moreover , it is easily seen that $\{F^n[x]: n \in N\}$ dense in $\bar{I}$ and F is onto , thus , by definition 2.1 , F is transitive on $\bar{I}$. ■

Lemma 3.8

a) Let $y \in \bar{I}$ be a periodic point of F of period k then there exists a unique periodic point $x \in I$ of f for which $\pi(x) = y$ and the period of x is k.

b) Let $x \in I$ be a periodic point of f period k then $\pi(x)$ is a periodic point for F of period k.

Proof(a): Let $y \in \bar{I}$ be a periodic point of period k for F , that is , $F^k(y) = y$, then $\pi^{-1}(y)$ is one 0f two cases:

i) $\pi^{-1}(y) = [a_1, b_1] \subseteq I$

ii) $\pi^{-1}(y) = \{a\}$

Case(i) Since $\pi^{-1}(y) = [a_1, b_1]$ then $f^k[a_1, b_1] \subseteq [a_1, b_1]$.Since f is continuous then by the intermediate value theorem there is $x \in [a_1, b_1]$ such that $f^k(x) = x$ and $\pi(x) = y$.

To show that x is unique , assume $z \in [a_1, b_1]$,this implies $x \sim y$ then by definition there is $r \in N$,such that $f^{tk}(z) = f^{rk}(x) = f^r(x) = x$.Then z=x and hence x is the unqiue periodic point of f in $[a_1, b_1]$.

Case (ii) Let $\pi^{-1}(y) = \{a\}$ then $\pi(a) = y$, since y is a periodic point of F then $F^k(y) = y = F^k\pi(a) = \pi f^k(a)$ which implies $a = f^k(a)$, since $\pi^{-1}(a)$ consists of one point then this periodic point is unique.

Proof (b):Let $x \in I$ be a periodic point of f such that $f^k(x) = x.\pi f^k(x) = \pi(x) = F^k\pi(x)$, then $\pi(x)$ is a periodic point of F of period $\leq k$. To show that the period of $\pi(x)$ is not less than k .

Assume $\pi(x) = F^i\pi(x)$ such that i<k. Then $\pi(x)$ is a periodic point of period I for F . thus by (a) there exists a unique periodic point $x_1 \in I$ of f for which $\pi(x_1) = \pi(x)$ and the period of x_1 is i<k, but this is a contradiction. ■

4. Main Results

In [6] , Misiurewiz,M. and Smital,J.found a class of C^∞ maps of an interval which are unimodal with zero topological entropy and chaotic in the sense of Li and Yorke . In this section , we will use the construction which was given in section three to get results similar to the results of [6] except that we replace chaotic in sense of Li and Yorke by transitivity.

Let H be the class of all weakly bimodal maps f for which the sets:

- $L_f = \{x \in I: f(x) \le f(y); \forall y \in I\}$ consists of more than one point…(4)
- $J_f = \{x \in I: f(x) \ge f(y); \forall y \in I\}$ consists of more than one point…(5)
- For all $n \ge 0$, f has a periodic point of period 2^n ... (6)
- F has no periodic point of other periods…(7)

Lemma 4.1

Let H be the class of all weakly bimodal maps f that satisfy (4),(5),(6) and(7) above , and let $F: \bar{I} \to \bar{I}$ be the induced map by f . Then F is bimodal and for all $n \ge 0$,F has a periodic point of period 2^n and F has no periodic point of other periods.

Proof:Since each L_f and J_f consists of more than one point ,that is , these sets are not empty. Then F has maximal and minimal points then by proposition 3.6, F is bimodal.By lemma 3.8, and condition (6), F has a periodic point of all period 2^n , for all $n \ge 0$,and by condition (7), F has no periodic point of other period. ■

Theorem 4.2

Let H be the class of maps defined above , then any map $f \in H$ is transitive and has zero topological entropy.

Proof:Let $f \in H$ then f is weakly bimodal and each of L_f , J_f has more than one element , and f has periodic points only of the form 2^n , for all $n \ge 0$Then by [5],$h(f) = 0$.

Let ω be the map from I into the set of all kneading sequences .By Lemma 4.1, F is bimodal and has a periodic point of period 2^n , for all $n \ge 0$, and there are the only periodic points, therefore, it has the same kneading sequences (see[8]).Hence the realation position of turning points, its images and the periodic points are the same for ω and F since π is non-decreasing by remark 3.4 .By Lemma 3.8, it is the same also for If. However for ω this relation position is well known (see[4]).

Let c_1', c_2' be the critical points of ω and let a_n be the periodic point of ω of period 2^n with the largest image under ω. Then from[2] we get:

$$\omega^2(c_1') < a_1 < \omega^{2^3}(c_1') < a_3 < \cdots < c_1' < \cdots < a_4 < \omega^{2^4}(c_1') < a_2 < \omega^{2^2}(c_1') < a_0 < \omega^{2^0}(c_1')$$

And

$$\omega^2(c_2') < a_1 < \omega^{2^3}(c_2') < a_3 < \cdots < c_2' < \cdots < a_4 < \omega^{2^4}(c_2') < a_2 < \omega^{2^2}(c_2') < a_0 < \omega^{2^0}(c_2')$$

Assume $c_1' < c_2'$ since ω is increasing (see[9]) then $\omega^k(c_1') < \omega^k(c_2')$, for all $k \in N$, then $\omega^2(c_1') < \omega^2(c_2') < a_1 < \cdots < c_1' < c_2' < \cdots < a_0 < \omega^{2^0}(c_2')$

. Therefore , if b_n is the periodic point of f of period 2^n with the largest image under f and $d_1, d_2 \in J_f$ such that$d_1 < d_2$ then $f^2(d_1) < f^2(d_2) < b_1 < f^{2^3}(d_1) < f^{2^3}(d_2) < \cdots < d_1 < d_2 < \cdots < b_2 < f^{2^2}(d_1) < f^{2^2}(d_2) < b_0 < f^{2^0}(d_1) < f^{2^0}(d_2)$.

Let $d_1' = \lim_{n\to\infty} b_{2n+1}$ and $d_2' = \lim_{n\to\infty} b_{2n}$, since $\omega(a_0) < \omega(a_1) < \cdots < \omega(c_1) < \omega(c_2)$. We have also:$f(b_0) < f(b_1) < \cdots < f(d_1) < f(d_2)$.

Therefore $\lim_{n\to\infty} f(b_i) = f(d_1') = f(d_2') \le f(d_1) \le f(d_2)$.

Since $\lim_{n\to\infty} \omega(a_n) = \lim_{n\to\infty} \omega^{2^n+1}(c_1) = \omega(c_1) = \lim_{n\to\infty} \omega^{2^n+1}(c_2) = \omega(c_2)$

and c_1, c_2 are not periodic points of ω , the itineraries of all points $y_1 \in [\lim_{n\to\infty} F\pi(b_n), F\pi(d_1)]$ that is, $y_1 \in [\lim_{n\to\infty} \pi f(b_n), \pi f(d_1)]$ and in the same way $y_2 \in [\lim_{n\to\infty} \pi f(d_1), \pi f(d_2)]$ hence $d_i' \in w_f(f(d_i)), i = 1,2$. Since d_i are arbitrary elements of J_f then $J_f \subset [d_1', d_2']$ such that $d_1' < d_2'$ then by above , we get $f(d_1') = f(d_2')$ that is f is not injective on $w_f(f(d_i))$ .by Lemma 2.4, f is transitive. ∎

To show that there exist smooth maps satisfying Theorem 4.2 in H we give the following:

Theorem 4.3

Let H be the class which is defined in Lemma 4.1 then H contains a C^∞ map.

Proof:Let $g: [0,1] \to [0,1]$ be a C^∞ map which is weakly bimodal such that each of J_f and L_f has more than element and $g(0) = 0$ and $g(1) = 1$. Set $g_\lambda(x) = \lambda g(x)$ for all λ ,for all $x \in [0,1]$. then g_λ is of classC^∞ , for all λ . let =

$\{\lambda: g_\lambda\, has\, periodic\, points\, of\, period\, 2^n, for\, all\, n \geq 0\}$

. it is easily seen that A is closed , $g_1 = g$ then g_1 has a periodic point of period 2^n , for all $n \geq 0$. $g_0(x) = 0$, for all $x \in [0,1]$,thus g_0 has no periodic point of period 2^n . hence if $\mu = \inf A$, then $\mu > 0$, g_μ has a periodic point of period 2^n , for all $n \geq 0$. Suppose $g_\mu$has a periodic point different form 2^n , then by [6], if λ is sufficiently close to μ then g_μ has a periodic point of period 2^n , but this is a contradiction. Then g_λ has no periodic point other than 2^n. Clearly g_μ is such that each of J_{g_μ} and L_{g_μ} has more than one point. ∎

Example: We will give an example satisfy theorem 4.1 and 4.3 : Let g be the map from [0,5] into itself defined by $g(x) = \frac{1}{5}x^3 - \frac{8}{2}x^2 + 2x + 1$, g is a bimodal map and $h(g) > 0, g_\lambda(x) = \lambda x$, for all $\lambda \in [0,1]$ and $x \in [0,5]$ then the map g_λ is of class C^∞ for each λ. In such away we obtain another simple example of transitive bimodal smooth map zero or positive topological entropy.

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