On δ-Small Projective Modules

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Abstract
Let R be a commutative ring with unity and let M be a non-zero unitary $R-$module. In this work we present a δ-small projective module concept as a generalization of small projective. Also we generalize some properties of small epimorphism to δ-small epimorphism. We also introduce the notation of δ-small hereditary modules and δ-small projective covers.

Keyword: projective modules, small projective modules, small epimorphism.

المقاسات الاسقاطية من النوع δ الصغيرة

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الخلاصة
لتكن R حلقة ابدالية ذات عنصر محايد وليكن M مقاسا احاديا غير صفري ايسرمعرفا على R . قدمنا في هذا البحث مفهوم المقاس الاسقاطي من النوع δ الصغير بصفته أعماما لمفهوم المقاس الاسقاطي الصغير. و عممنا بعض خصائص المقاسات الاسقاطية الشاملة الصغيرة إلى المقاسات الاسقاطية الشاملة الصغيرة من النوعδ .كذلك سنقدم مفاهيم المقاسات الوراثية الصغيرة من النوع δ، الغطاءات الاسقاطية من النوع δ الصغيرز

1.Introduction

Let R be a commutative ring with identity and M is a non zero unite left R-module. M is called singular module if Z(M)=M where $Z(M) = \{x \in M: ann(x) \subseteq_e R\}$, submodule X of an R- module M is called c-singuler ($X \subseteq_{C.S} M$) if is M/X singular module [1]. A submodule N of an R-module M is called a small submodule of M, denoted by $N << M$, if $N + L \neq M$ for any proper submodule L of M [2].

In [3] Zhou introuced the definition of the concept of δ-small submodule that A submodule N of an R- module M is called a δ-small submodule of M (briefly $N <<_\delta M$) if $N + X = M$ for any proper submodule X of M with M/X singular, we have $X = M$.
An ideal I of a ring R is δ-small ideal if we consider R as R –module.

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In [4] A.K. Tiwary and K.N. Chauby introduse the concept of small projective modules that an R-modules M is called small projective if for any epimorphism $g: B \to A$ whose kernel is small submodule of B. $g \circ Hom(M,B) = Hom(M,A)$.

In this paper we introduce the concept of δ -small projective modules as follows: An R-modules M is called δ -small projective if for any epimorphism $g : B \to A$ whose kernel is δ- small submodule of B $g \circ Hom(M,B) = Hom(M,A)$

In the following we give the properties of the c-singuler submodules and the δ-small submodules.

Remarks1.1[3]

1- Let A be submodule of R-module M if $A \subseteq_e M$, then $A \subseteq_{c.s} M$.

2- Let M and N be R-modules and $f: M \to N$ be an epimorphism if $A \subseteq_{c.s} M$, then $f(A) \subseteq_{c.s} N$.

3- Let M and N be R-modules and $f : M \to N$ be a homomorphism,if $B \subseteq_{c.s} N$, then $f^{-1}(B) \subseteq_{c.s} M$.

4- Let A and B be submodules of an R-module M, if $A \subseteq_{c.s} M$ and $B \subseteq_{c.s} M$, then $A \cap B \subseteq_{c.s} M$.

5- Every submodule of a singuler module is c-singuler .

Lemma 1.2 [3]: Let M be a module,

1) For submodules N, K, L of M with $K \subseteq N$, then

a) $N \ll_\delta M$ if and only if $K \ll_\delta M$ and $N/K \ll_\delta M/K$

b) $N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.

2) If $K \ll_\delta M$ and $f : M \to N$ a homo then $f(K) \ll_\delta N$.

3) If $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$ then $K_1 \oplus K_2 \ll_\delta M_1 \oplus M_2$ if and only if $K_1 \ll_\delta M$ and $K_2 \ll_\delta M_2$.

4)Let $A < B < M$, If $A \ll_\delta M$ and B is a direct summand, then $A \ll_\delta B$.

2. δ-Small epimorphism.

An epimorphism $g: M \to N$ is said to be small, if $\mathrm{Ker}(g)$ is small in M, where M and N are any R- modules [5].

In this section we give the defintion of δ-small epimorphism and some related concept.

Defination 2.1

Let M and N be modules. An epimorphism $g: M \to N$ is said to be δ-small, if $\mathrm{Ker(g)}$ is δ-small in M.

Example 2.2

Consider $\mathbb{Z}_2$ and $\mathbb{Z}_4$ as Z-modules. Define $g: \mathbb{Z}_4 \to \mathbb{Z}_2$ by

$g(\bar{1}) = g(\bar{3}) = \bar{1}, g(\bar{0}) = g(\bar{2}) = \bar{0}$, clearly $\mathrm{Ker(g)} = \{\bar{0}, \bar{2}\} \ll_\delta \mathbb{Z}_4$ .Thus g is δ-small epimorphism .

The following proposition is needed later.

Proposition 2.3

Let M, N and L be modules.If $f: M \to N$ and g: $N \to L$ are two epimorphisms. Then $g \circ f$ is δ-small if and only if f and g are δ-small.

Proof: Let $g \circ f$ is δ-small epimorphism and suppose that $\mathrm{Ker}\ (f) + K = M$,where K is c-singular submodules of M . We claim that $\mathrm{Ker}\ (g \circ f) + K = M. \mathrm{Ker}\ (g \circ f) + K \subseteq M$,we have to show only $M \subseteq \mathrm{Ker}\ (g \circ f) + K$, let $m \in M, f(m) \in N$ and $g(f(m)) \in L$, but $\mathrm{Ker}\ (f) + K = M$, then $m = a + w\ ; a \in \mathrm{Ker}\ (f), w \in K$, then $f(m) = f(w)$, hence $g(f(m)) = g(f(w))$ and $M = \mathrm{Ker}\ (g \circ f) + K$.

But $Ker\ (g \circ f) \ll_\delta M$. therefore, $M = K$. Thus f is δ-small epimorphisim.To show that g is δ-small epimorphisim.

Let $Ker(g) + A = N\ ; A$ is c-singular submodules of N, then $f^{-1}(A)$ is c-singular submodules of M, $f^{-1}(A)$ =B, then $A = f(B)$, $, g(A) = L, g(f(B)) = L = (g \circ f)(M)$.

Similarly we get M= $Ker\ (g \circ f)$+B, then $M = B$ and hence $A = f(M) = N$. Thus g is δ-small epimorphisim. So that both f and g are δ-small epimorphisims.

For the converse, if f and g are δ-small epimorphisims, suppose that $\mathrm{Ker}\ (g \circ f) + K = M$; K is c-singular submodules of M , $g(f(K)) = g(f(M))$, then $f(M) = f(K) + Ker(g)$, but

$\mathrm{Ker}\,(g) <<_\delta f(M) = N$ and $f(K)$ is c-singular submodules of N, hence $f(K) = f(M) = N$, then $M = Ker(f) + K$, since $Ker(f) <<_\delta M$ and K is c-singular submodules of N, then $M = K$.Thus $g \circ f$ is δ-small epimorphism.

Proposition 2.4

Consider the following commutative diagram of modules A_1, B_1, C_1 and A_2, B_2, C_2

$$\begin{array}{ccccccccc} 0 & \to & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \to & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \downarrow\gamma & & \\ 0 & \to & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \to & 0 \end{array}$$

With both rows are exact and α is an epimorphism, if g_1 is a δ-small epimorphism, then so is g_2

.

proof :Let $\mathrm{Ker}\,(g_2) + K = B_2$, where K is c-singular submodules of B_2 , since the sequence is exact, then $f_2\,(A) + K = B_2$, but α is an epimorphism and $f_2\,\alpha\,(A) + K = B_2$, then β $f_1(A)$+K= B_2, but $Im(f_1) = \mathrm{Ker}(g_2)$, so $\beta\big(Ker\,(g_1)\big) + K = B_2$,since $Ker\,(g_1) <<_\delta B_1$, then $\beta(Ker\,(g_1)) <<_\delta B_2$, but $\beta(Ker\,(g_1)) + K = B_2$, then $K = B_2$. Thus $g_2$is δ-small epimorphism.

3. δ-Small projective modules

In this section, we give the defintion of a δ-small projective modules as a generalization to a small projective modules.

Defination 3.1

An R-mdule M is called δ- small projective if for each δ-small epimorphism $g: A \to B$, where A, B are any R-mdules and for each homomorphisim $f: M \to B$, there exist a homomorphisim $h: M \to A$ such that $g \circ h = f$, i.e the following diagram is commute.

$$\begin{array}{ccccc} & & M & & \\ & h\swarrow & \downarrow f & & \\ A & \xrightarrow{g} & B & \to & 0 \end{array}$$

If A is the module in obave definition, then the following remarks are clear.

Remarks 3.2

1. Every small projective is δ-small projective.
2. if A is an indecomposable and M is δ-small projective R- module, then M is small projective.
3. If A is torsion and M is δ-small projective over commutative integral domain R, then M is small projective.
4. If A is uniform and M is δ-small projective R- module, then M is small projective.
5. If A is singular and M is δ-small projective R- module, then M is small projective.

Proposition 3.3

The following are equivalent for an R-module M.

1. M is δ-small projective R- module.
2. For each δ-small epimorphis$g: N \to K$, the homomorphisim $Hom(I, g): Hom(M, N) \to Hom(M, K)$ is an epimorphism
3. For any δ-small epimorphism$g: B \to A$ $g \circ Hom(M, B) = Hom(M, A)$.

Proof:

1→2

Let $g: N \to K$ be a δ-small epimorphism and $f \in Hom(M, K)$. Since M is δ- small projective, then there exist a homomorphisim $h: M \to N$ such that $g \circ h = f$,

$$\begin{array}{ccccc} & & M & & \\ & h\swarrow & \downarrow f & & \\ N & \xrightarrow{g} & K & \to & 0 \end{array}$$

Thus $Hom(I, g) \circ h = f$, then $h \in Hom(M, N)$, therefore, $Hom(I, g)$ is an epimorphism.

2→3

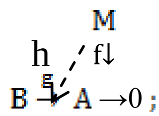
Let $g: B \to A$ be a δ-small epimorphism. By (2) $Hom(I, g)$: $Hom(M, N) \to Hom(M, K)$ is an epimorphism. Now, to show that $g \circ Hom\,(M, B) = Hom(M, A)$.

Let $f \in Hom(M, A)$, so there exist $f_1 \in Hom\,(M, B)$ such that

$Hom(I,g)\circ f_1 = f$, i.e $g\circ f_1 = f$. Thus $f \in g\circ Hom(M,B)$, so $Hom(M,A) \subseteq g\circ Hom(M,B)$, clearly $g\circ Hom(M,B) \subseteq Hom(M,A)$. Thus $g\circ Hom(M,B) = Hom(M,A)$.

3→1

Consider the following diagram



; $Ker(g) \ll_\delta B$

Where A,B are any R-modules, and f is any homomorphism, since $g\circ Hom(M,B) = Hom(M,A)$ and $f \in Hom(M,A)$, so there exist $h \in Hom(M,B)$ such that $g\circ h = f$. Thus M is δ-small projective.

Remark 3.4

Every δ-small epimorphism $N \to M \to 0$, where M is δ-small projective, splits and consequently an isomorphisim.

Proof:

Let $g: N \to M$ be a δ-small epimorphism, where $Ker(g) \ll_\delta N$, since M is δ-small projective, then there exist $h: M \to N$ such that $g\circ h = I$, hence h is one to one and g is onto and $N = Ker(g) \oplus K$,for each co-singular submodules K of M.

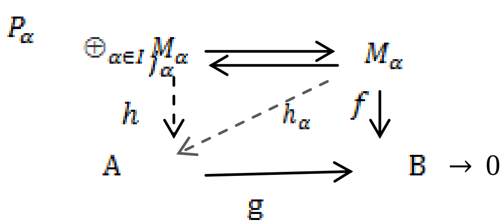
Proposition 3.5

$\oplus_{\alpha\in I} M_\alpha$ is δ-small projective if and only if M_α is δ-small projective for each $\alpha \in I$.

Proof:

Suppose that $\oplus_{\alpha\in I} M_\alpha$ is δ-small projective and let $\alpha \in I$.

Consider the following diagram

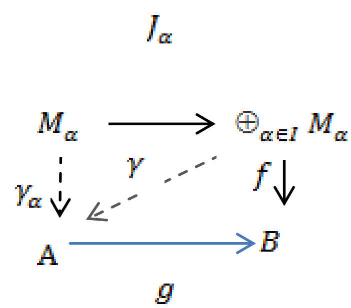


Where $Kerg \ll_\delta A$ and $g: A \to B$ is any epimorphism, $f: M_\alpha \to B$ is any homomorphism, $P_\alpha: \oplus_{\alpha\in I} M_\alpha \to M_\alpha$ is the projection homomorphism and $J_\alpha: M_\alpha \to \oplus_{\alpha\in I} M_\alpha$ is the injection homomorphism, since $\oplus_{\alpha\in I} M_\alpha$ is δ-small projective, there exist a homomorphism $h: \oplus_{\alpha\in I} M_\alpha \to A$ such that $g\circ h = f\circ P_\alpha$. Define $h_\alpha: M_\alpha \to A$ by $h_\alpha = h\circ J_\alpha$

Now,

$g\circ h_\alpha = g\circ h\circ h_\alpha = f\circ P_\alpha \circ J_\alpha = f\circ I = f$.Hence M_α is δ-small projective.

Conversly, Suppose that M_α is δ-small projective for each $\alpha \in I$, and Consider the following diagram



Where $g: A \to B$ is δ-small epimorphism, $f: \oplus_{\alpha\in I} M_\alpha \to B$ is any homomorphism, and $J_\alpha: M_\alpha \to \oplus_{\alpha\in I} M_\alpha$ is the injection homomorphism. Since M_α is δ-small projective for each $\alpha \in I$, ∃ a homomorphim $\gamma_\alpha: M \to P$ such that $g\circ \gamma_\alpha = f\circ J_\alpha$ for each $\alpha \in I$. Define $\gamma: \oplus_{\alpha\in I} M_\alpha \to A$ by $\gamma\circ J_\alpha = \gamma_\alpha$ for each $\alpha \in I$. It remain to show that $g\circ\gamma = f$. Now, $f\circ J_\alpha = g\circ\gamma_\alpha$ and since $\gamma_\alpha = \gamma\circ J_\alpha$. It follows that $f\circ J_\alpha = g\circ\gamma\circ J_\alpha$. Thus by[2, p.82] $f = g\circ\gamma$ i.e., $\oplus_{\alpha\in I} M_\alpha$ is δ-small projective.

Proposition 3.6

Let M be an R-module and $A \le M$, then for each summand B of M, such that $A\cap B \ll_\delta M$ and $A+B$ δ-small projective module, we have $A\cap B = \{0\}$.

Proof:

Consider the following natural epimorphisms $\pi_1: A \to \frac{A}{A\cap B}$; $\pi_2: A+B \to \frac{A+B}{B}$

By the second isomorphism theorem $\frac{A}{A\cap B} \cong \frac{A+B}{B}$, since B is a summand of M, so $M = B \oplus K; K \leq M$, by modular law $M \cap (A+B) = (B \oplus K) \cap (A+B)$, then $(A+B) = B \oplus [K \cap (A+B)]$, so B is a summand of $A+B$ and $K \cap (A+B)$ is a summand of $A+B$, by (3.5) $K \cap (A+B)$ is δ-small projective and hence $\frac{A+B}{B}$ is δ-small projective and so is $\frac{A}{A\cap B}$, since $A \cap B \ll_\delta M$, then $A \cap B \ll_\delta A$,so the epimorphism $\pi_1 : A \to \frac{A}{A\cap B} \to 0$ is δ-small epimorphism, since $\frac{A}{A\cap B}$ is δ-small projective, then by Rem(3.4) π_1 splits and consequently isomorphism, so $Ker\pi_1 = A \cap B$ is a direct summand of A, but $A \cap B \ll_\delta A$, hence $A \cap B = \{0\}$.

Defination 3.7[3]

A pair (P,f) is called a projective δ- cover of M, if P is projective and an epimorphism $f: P \to M$; $Kerf \ll_\delta P$.

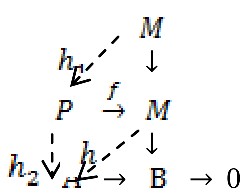
Now, we have the following propotion

Proposition 3.8

A δ-small projective which has a projective δ -cover is projective.

Proof:

Let M be a δ-small projective R-module, let (P,f) be a projective δ- cover for M. Consider the following diagram



Where $g: A \to B$ is an epimorphism, $f: P \to M$ is δ-small epimorphism,

$f_1: M \to B$ is any homomorphim and $I: M \to M$ is the identity. Since M is δ-small projective, then ∃ a homomorphim $h_1: M \to P$ such that $f \circ h_1 = I$. But P is projective, so ∃ a homomorphim $h_2: P \to A$ such that $g \circ h_2 = f_1 \circ f$. Define $h: M \to A$ by $h = h_2 \circ h_1$, then

$g \circ h = g \circ h_2 \circ h_1 = f_1 \circ f \circ h_1 = f_1 \circ I = f_1$.

Thus M is projective.

Proposition 3.9

If M_1 is projective and M_2 is δ-small projective R-modules, then $M_1 \otimes M_2$ is δ-small projective R-module.

Proof:

Let $f: A \to B$ is δ-small epimorphism. Then:

$Hom(I,f): Hom(M_2,A) \to Hom(M_2,B)$ is an epimorphism by (3.3).

Now, since M_1 is projective we have:

$Hom(I, Hom(I,f)): Hom(M_1, Hom(M_2,A)) \to Hom(M_1, Hom(M_2,B))$

is an epimorphism, then

$Hom(M_1 \otimes M_2, A) \to Hom(M_1 \otimes M_2, B)$ is again an epimorphism [5]. Also by (3.3) $M_1 \otimes M_2$ is δ-small projective R-module.

4. δ-cosemisimple Rings

Let M be an R-module It is known that the Jacobson radicalof M, denote $Rad(M)$ is the sum of all small submodules of M [2]. Zhou introduced the defintion of $\delta(M)$ as a generalization of $Rad(M)$ [3].

Definition 4.1 [3]

Let ρ be the class of all singular simple modules. For an R- module M .

$\delta(M) = \cap\{ N \subseteq M : M/N \in \rho\}$

Is the reject M of ρ.

Lemma 4.2 [3]

Let M be an R- module

$\delta(M) = \sum\{ L \subseteq M : L$ is δ - small submodule of M }.

A ring R is called cosemisimple if $Rad(M) = 0$, for each R-module M [4] .

Now, we introduce the following:

Defination 4.3

A ring R is called δ-cosemisimple if $\delta(M) = 0$, for each R-module M.

Proposition 4.3

The following are equivalent for an R-module M.

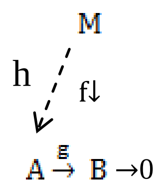
1) R is δ-cosemisimple ring.

2) Every module over R is δ- small projective**.**

Proof:

1)→2)

Let M be an R- module, R is δ-cosemisimple ring**.** Consider the following diagram



Where A, B are any R-modules, $Ker(g) \ll_\delta A$ and g is any epimorphism , since R is δ-cosemisimple, then $\delta(A) = 0$, then $Ker(g) = 0$, hence g is isomorphism.
Let $h = g^{-1} \circ f$, then $(g \circ g^{-1}) \circ f = f$. Thus M is δ- small projective**.**

2)→1)
Let M be any R- module, let $x \in \delta(M)$, then $Rx \ll_\delta M$, therefor the naturale epimorphism $\pi: M \to \frac{M}{Rx}$ splits by Rem(3.4) , since $\frac{M}{Rx}$ is δ-small projective, then $M = Rx \oplus K$, where K any submodule of M
, hence Rx=0, which implise that $x = 0$, so that $\delta(M) = 0$. Thus R is δ-cosemisimple.

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