Fuzzy Soft Modules Over Fuzzy Soft Rings

Amjad Hamead Alhusiny
Department of computer science, College of education, University of Kufa,Alnajaf,Iraq

Abstract.
Let M be an $R-$ module, and let $A \neq \emptyset$ be a set, let (F,A) be a soft set over M. Then (F,A) is said to be a fuzzy soft module over M iff $\forall a \in A$, $F(a)$ is a fuzzy submodule of M. In this paper, we introduce the concept of fuzzy soft modules over fuzzy soft rings and some of its properties and we define the concepts of quotient module, product and coproduct operations in the category of $FSFS$ modules.
Keyords: Fuzzy Soft Modules, Fuzzy Soft Rings, quotient module

المقاسات الضبابية الملساء على الحلقات الضبابية الملساء

أمجاد حميد الحسيني
قسم علوم الحاسوب، كلية التربية، جامعة الكوفة،النجف ،العراق.

الخلاصة
ليكن M مقاساً على R ، ولتكن $A \neq \emptyset$ مجموعة ، و (F,A) مجموعة ملساء على M . فإن (F,A) يسمى مقاساً ضبابياً أملساً على M إذذا $F(a)$ مقاساً ضبابياً جزئياً من M, $\forall a \ni A$. في بحثنا هذا قدمنا مفهوم المقاس الضبابي الأملس على الحلقة الضبابية الملساء وعرفنا مفهوم مقاس القسمة ، عمليات الضرب ، الضرب المشترك ، وبعض الخواص الأساسية لصنف المقاسات الضبابية الملساء .

1. Introduction

Most of our traditional mathematical tools are deterministic and precise in character. In the other hand many practical problems in economics, engineering, environment, social science, medical science etc. cannot be dealt with by classical methods, because classical methods have inherent difficulties. Molodtsov [1] initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties. Later other authors Maji et al. [2-4] have further studied the theory of soft sets and also introduced the concept of fuzzy soft set, which is a combination of fuzzy set [5] and soft set. Thereafter, Aktas and Cagman [6] have introduced the notion of soft groups. Aygunoglu and Aygun [7] have generalized the concept of Aktas and Cagman [6] and introduce fuzzy soft group. F.Feng et al. [8] gave soft semirings and U.Acar et al. [9] introduced initial concepts of

E.mail. amjad.alhusseini@uokufa.edu.iq

soft rings. Ghosh et al. [10] gave the notion of fuzzy soft rings. The definition of fuzzy modules is given by some authors [11-13] Qiu-Mei Sun et al. [14] defined soft modules and investigated their basic properties. Gunduz and Bayramov [15] introduced a basic version of fuzzy soft module theory. In this paper the main purpose is to introduce a basic version of fuzzy soft module over fuzzy soft ring, which extends the notion of module by including some algebraic structures in soft sets. And finally some basic properties of fuzzy soft module over fuzzy soft ring has been investigated.

2. Preliminaries:

Definition (2.1) [1] Let U be an initial universe set and E be the set of parameters. Let $P(U)$ denotes the power set of U. A pair $(\mathcal{F},E)$ is called a soft set over U, where $\mathcal{F}$ is a mapping given by $\mathcal{F}: E \to P(U)$.

Definition (2.2) [2] Let U be an initial universe set and E be the set of parameters. Let $A \subseteq E$. A pair $(\mathcal{F},A)$ is called fuzzy soft set over U, where $\mathcal{F}$ is a mapping given by $\mathcal{F}: A \to I^U$, where I^U denotes the collection of all fuzzy subsets of U.

Definition (2.3) [16] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative,

(ii) $*$ is continuous,

(iii) $a * 1 = a$ for all $a \in [0,1]$,

(iv) $a * b \leq c * d$ whenever $a \leq c, b \leq d$, and $a,b,c,d \in [0,1]$.

Definition (2.4) [16] A binary operation $\diamond: [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\diamond$ satisfies the following conditions:

(i) $\diamond$ is commutative and associative,

(ii) $\diamond$ is continuous,

(iii) $a \diamond 0 = a$ for all $a \in [0,1]$,

(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c, b \leq d$ and $a,b,c,d \in [0,1]$.

Definition (2.5) [6] Let X be a group and $(\mathcal{F},A)$ be a soft set over X. Then $(\mathcal{F},A)$ is said to be a soft group over X if and only if $\mathcal{F}(a)$ is a subgroup of X for each $a \in A$.

Definition (2.6) [7] Let X be a group and $(\mathcal{F},A)$ be a fuzzy soft set over X. Then $(\mathcal{F},A)$ is said to be a fuzzy soft group over X if and only if for each $a \in A$ and $x,y \in X$,

(1) $\mathcal{F}_a(x \cdot y) \geq \mathcal{F}_a(x) * \mathcal{F}_a(y)$

(2) $\mathcal{F}_a(x^{-1}) \geq \mathcal{F}_a(x)$

Where $\mathcal{F}_a$ is the fuzzy subset of X corresponding to the parameter $a \in A$.

Remark (2.7) $\mathcal{F}_a(x^{-1}) = \mathcal{F}_a(x)$.

Proof: Let $x,y \in X$, such that $x^{-1} = y$ which means that $y^{-1} = x$, since $\mathcal{F}_a(x^{-1}) \geq \mathcal{F}_a(x)$ then $\mathcal{F}_a(y^{-1}) \geq \mathcal{F}_a(y)$, and so $\mathcal{F}_a(x) \geq \mathcal{F}_a(y) = \mathcal{F}_a(x^{-1})$, and thus $\mathcal{F}_a(x^{-1}) = \mathcal{F}_a(x)$.

Definition (2.8) [10] Let f and g be any two fuzzy subset of a ring R. Then $f \circ g$ is a fuzzy subset of R defined by

$$(f \circ g)(z) = \begin{cases} sup_{z=x\cdot y}\{min\{f(x), g(y)\}\} \\ 0 \quad otherwise \end{cases}, \text{ if } z$$

is expressed as $z = x \cdot y$, where $x, y\, z \in R$.

Definition (2.9) [10] The intersection of two fuzzy soft sets $(\mathcal{F},A)$ and $(\mathcal{G},B)$ over the same universe U is denoted by $(\mathcal{F},A)\tilde{\sqcap}(\mathcal{G},B)$ and defined by a fuzzy soft set $(\mathcal{H},C)$ where $C = A \cap B$ and $\mathcal{H}: C \to [0,1]^U$ such that for each $e \in C$,

$$\mathcal{H}(e) = \{(x, \mathcal{H}_e(x)) : x \in U\}$$

Where $\mathcal{H}_e(x) = \mathcal{F}_e(x) * \mathcal{G}_e(x)$ and $\mathcal{H}_e(x), \mathcal{F}_e(x), \mathcal{G}_e(x)$ are the fuzzy subset of U corresponding to the parameter $e \in C$.

Definition (2.10) [10] The union of two fuzzy soft sets $(\mathcal{F},A)$ and $(\mathcal{G},B)$ over the same universe U is denoted by $(\mathcal{F},A) \breve{\cup} (\mathcal{G},B)$ and defined by a fuzzy soft set $(\mathcal{H},C)$ where

$C = A \cup B$ and $\mathcal{H}: C \to [0,1]^U$such that for each $e \in C$,

$$\mathcal{H}(e) = \{(x, \mathcal{F}_e(x)) : x \in U\}, if\ e \in A - B$$
$$= \{(x, \mathcal{G}_e(x)) : x \in U\}, if\ e \in B - A$$
$$= \{(x, \mathcal{H}_e(x)) : x \in U\}, if\ e \in A \cap B.$$

Where $\mathcal{H}_e(x) = \mathcal{F}_e(x) \circ \mathcal{G}_e(x)$ and $\mathcal{H}_e(x), \mathcal{F}_e(x), \mathcal{G}_e(x)$ are the fuzzy subset of U corresponding to the parameter $e \in C$.

Definition (2.11) [4] If (f, A) and $(g, B)$are two soft sets, then (f, A) **AND** (g, B) is denoted as $(f, A) \wedge (g, B)$ defined as $(h, A \times B)$, where $h(a, b) = h_{a,b} = f_a \wedge g_b$, $\forall (a, b) \in A \times B$.

Definition (2.12) [10] Let $(R, +, \cdot)$ be a ring and E be a parameter set and $A \subset E$. Let $\mathcal{R}$ be a mapping given by $\mathcal{R} : A \to P(R)$. Then $(\mathcal{R}, A)$ is called a soft ring over R if and only if for each $a \in A, \mathcal{R}(a)$ is a subring of R i.e.

$(i)\ x, y \in \mathcal{R}(a) \Rightarrow x + y \in \mathcal{R}(a)$,
$(ii)\ x \in \mathcal{R}(a) \Rightarrow -x \in \mathcal{R}(a)$,
$(iii)\ x, y \in \mathcal{R}(a) \Rightarrow x \cdot y \in \mathcal{R}(a)$.

Definition (2.13) [10] Let $(R, +, \cdot)$ be a ring and E be a parameter set and $A \subset E$. Let $\mathcal{R}$ be a mapping given by $\mathcal{R} : A \to [0,1]^R$, where $[0,1]^R$ denotes the collection of all fuzzy subsets of R. Then $(\mathcal{R}, A)$ is called a fuzzy soft ring over R if and only if for each $a \in A$ the corresponding fuzzy subset $\mathcal{R}_a$ of R is a fuzzy subring of R i.e. $\forall x, y \in R$,

$(i)\ \mathcal{R}_a(x + y) \geq \mathcal{R}_a(x) * \mathcal{R}_a(y)$
$(ii)\ \mathcal{R}_a(-x) \geq \mathcal{R}_a(x)$
$(iii)\ \mathcal{R}_a(x \cdot y) \geq \mathcal{R}_a(x) * \mathcal{R}_a(y)$.

Theorem (2.14) [10] Let $(\mathcal{R}, A)$ be a fuzzy soft set over R, then $(\mathcal{R}, A)$ is a fuzzy soft ring over R if and only if for each $a \in A$, $x, y \in R$ the following conditions hold:

$(i)\ \mathcal{R}_a(x - y) \geq \mathcal{R}_a(x) * \mathcal{R}_a(y)$
$(ii)\ \mathcal{R}_a(x \cdot y) \geq \mathcal{R}_a(x) * \mathcal{R}_a(y)$.

Definition (2.15) [14] Let (F, A) be a soft set over M. (F, A) is said to be a soft module over M if and only if $F(x) < M$ for all $x \in A$.

Definition (2.16) [14] Let (F, A) and (G, B) be two soft modules over M and N respectively. Then $(F, A) \times (G, B) = (H, A \times B)$ is defined as $H(x, y) = F(x) \times G(y)$ for all $(x, y) \in A \times B$.

Proposition (2.17) [14] Let (F, A) and (G, B) be two soft modules over M and N respectively. Then $(F, A) \times (G, B)$ is soft module over $M \times N$.

Definition (2.18) [14] Let (F, A) and (G, B) be two soft modules over M and N respectively, $f: M \to N, g: A \to B$ be two functions. Then we say that (f, g) is a soft homomorphism if the following conditions are satisfied:

(1) f is a homomorphism from M onto N,
(2) g is a mapping from A onto B, and
(3) $f(F(x)) = G(g(x))$ for all $x \in A$.

Definition (2.19) [17] Let (F, A) be a fuzzy soft set over G. Then (F, A) is said to be a fuzzy soft group over G if and only if $F(x)$ is a fuzzy subgroup of G, for all $x \in A$.

Definition (2.20) [15] Let (F, A) be a fuzzy soft set over M . Then (F, A) is said to be a fuzzy soft module over M if and only if $\forall a \in A, F(a)$ is a fuzzy submodule of M and denoted as F_a.

Definition (2.21) [15] Let (F, A) and (H, B) be two fuzzy soft modules over M . Then (F, A) is called a fuzzy soft submodule of (H, B) if

$(i)\ A \subset B$
(ii) For all $a \in A$, F_a is a fuzzy submodule of H_a.

Definition (2.22) [15] Let (F, A) and (H, B) be two fuzzy soft modules over M and N respectively, and let $f: M \to N$ be a homomorphism of modules, and let $g: A \to B$ be a mapping of sets. Then we say that

$(f,g):(F,A)\to(H,B)$ is a fuzzy soft homomorphism of fuzzy soft modules, if the following condition is satisfied:

$$f(F_a)=H(g(a))=H_{g(a)}.$$

We say that (F,A) is a fuzzy soft homomorphic to (H,B).

Note that for $\forall a\in A, f:(M,F_a)\to(N,H_{g(a)})$ is a fuzzy homomorphism of fuzzy modules.

Theorem (2.23) [15] Let (F,A) and (H,B) be two fuzzy soft modules over M. Then their intersection $(F,A)\cap(H,B)$ is a fuzzy soft module over M .

Theorem (2.24) [15] Let (F,A) and (H,B) be two fuzzy soft modules over M . Then $(F,A)\wedge(H,B)$ is a fuzzy soft module over M .

Definition (2.25) [15] $\prod_{i\in I}(F_i,A_i)$ is said to be direct product of soft modules.

Proposition (2.26) [15] Let $\{(F_i,A_i)\}_{i\in I}$, be a family of soft modules over $\{M_i\}_{i\in I}$ and $\{(H_i,B_i)\}_{i\in I}$, be a family of soft modules over $\{N_i\}_{i\in I}$ and $(f_i,g_i):(F_i,A_i)\to(H_i,B_i)$ be a soft homomorphism of soft modules for each $i\in I$. Then

$$\left(\prod_{i\in I}F_i,\prod_{i\in I}g_i\right):\prod_{i\in I}(F_i,A_i)\to\prod_{i\in I}(H_i,B_i)$$

is a soft hommorphism of soft modules.

Proposition (2.27) [15] $\Pi:\Pi FSM\to FSM$ is a functor.

Now, let parameter set of $\{(F_i,A_i)\}_{i\in I}$, be fixed point. We denote fixed point of A_i as a_{0i} and let $F_i(a_{0i})=0$. For $A=\prod_{i\in I}A_i$ and $M=\bigoplus_{i\in I}M_i$, we define the mapping $F:A\to M$ by $F(a)=\bigoplus_{i\in I}F(a_i)$, for all $a=\{a_i\}\in A$. Then, (F,A) is a soft module over M .

Definition (2.28) [15] (F,A) is said to be direct sum of $\{(F_i,A_i)\}_{i\in I}$, and denoted as $\bigoplus_{i\in I}(F_i,A_i)$.

The mapping : $\varphi_j:A_j\to\prod_{i\in I}A_i$ is defined by $\varphi_j(a_j)=\{a_i\}=\begin{cases}a_{0i} & if\ i\neq j\\ a & if\ i=j\end{cases}$.

Also for embedding mapping $q_j:M_j\to\bigoplus_{i\in I}M_i$, $(q_j,\varphi_j):(F_j,A_j)\to(F,A)$ is a soft homomorphism of soft modules.

Proposition (2.29) [15] Let $\{(F_i,A_i)\}_{i\in I}$, and $\{(H_i,B_i)\}_{i\in I}$ be family of soft modules over $\{M_i\}_{i\in I}$ and $\{N_i\}_{i\in I}$, respectively, and let $(f_i,g_i):(F_i,A_i)\to(H_i,B_i)$ be a soft homomorphism of soft modules. Then

$$\left(\bigoplus_{i\in I}f_i,\prod_{i\in I}g_i\right):\bigoplus_{i\in I}(F_i,A_i)\to\bigoplus_{i\in I}(H_i,B_i)$$

is a soft homomorphism of soft modules.

Proposition (2.30) [15] $\bigoplus:\Pi FSM\to FSM$ is a functor.

Let M and N be ,respectively, right and left modules over R (ring). Let (F,A) and (G,B) be two soft modules over M and N , respectively. We consider tensor product of modules as $M\otimes N$. The maping $F\otimes G:A\times B\to M\otimes N$ is defined by $(F\otimes G)(a,b)=F(a)\otimes G(b), for\ \forall(a,b)\in A\times B$.

Proposition (2.31) [15] $(F\otimes G,A\times B)$ is a soft module over $M\otimes N$.

Definition (2.32) [15] $(F\otimes G,A\times B)$ is said to be tensor product of (F,A) and (G,B) and denoted by $(F,A)\otimes(G,B)$.

3. Fuzzy soft modules over fuzzy soft rings:

Definition (3.1): Let M be an $R-$ module, (F,A) be a fuzzy soft module over M, and $(\mathcal{R},A)$ be a fuzzy soft ring over R, then $(F,\mathcal{R},A)$ is called fuzzy soft module over fuzzy soft ring if and only if (F,A) is an $(\mathcal{R},A)-$ module.

Remark (3.2): 1) We shall denote the category of fuzzy soft modules over fuzzy soft rings by *FSFS* modules .

2) For convenience we denote the fuzzy soft module over fuzzy soft ring $(F,\mathcal{R},A)$ by (F,A) , wherever there is no risk of confusion.

Example (3.3): Let $M_{n\times n}(\mathbb{R})$ be the set of all $n\times n$ matrices over $\mathbb{R}$, and

$R = A = M_{n\times n}(\mathbb{R})$, define the function $\mathcal{R}: A \to [0,1]^R$ by $\mathcal{R}(B) = \{C \cdot B \mid C \in M_{n\times n}(\mathbb{R})\}$ for all $B \in A$. Then $(\mathcal{R}, A)$ is a fuzzy soft ring over R, now consider $M = M_{n\times n}(\mathbb{R})$ as an $R-$ module and $F: A \to FP(M)$, defined by $F(\mathcal{M}) = \{\mathcal{N} \mid \mathcal{M} \cdot \mathcal{N} = \mathcal{N} \cdot \mathcal{M}\}$ for all $\mathcal{M} \in M$. Then (F, A) is a fuzzy soft module over M, and (F, A) is an $(\mathcal{R}, A)-$module, which means $(F, \mathcal{R}, A)$ is an $FSFS$ module.

Definition (3.4): Let $(F, \mathcal{R}, A)$ and $(H, \mathcal{R}, B)$ be two $R-FSFS$ modules over M. Then $(F, \mathcal{R}, A)$ is called $FSFS$ submodule of $(H, \mathcal{R}, B)$ if

(i) $A \subset B$

(ii) For all $a \in A$, F_a is a fuzzy submodule of H_a.

Theorem (3.5): Let $(F, \mathcal{R}, A)$ and $(H, \mathcal{R}, B)$ be two $R-FSFS$ modules over M. If $F_a \leq H_a$ for all $a \in A$, then $(F, \mathcal{R}, A)$ is an $FSFS$ submodule of $(H, \mathcal{R}, B)$.

Proof. The proof of the theorem is straightforward.

Theorem (3.6): Let (F, A) and (H, B) be two $FSFS$ modules over M. Then their intersection $(F, A) \cap (H, B)$ is an $FSFS$ module over M.

Proof. Let $(F, A) \cap (H, B) = (G, C)$, where $C = A \cap B$. Since the fuzzy soft set $G_c = F_c \wedge H_c$ is a fuzzy submodule, for $\forall c \in C$, (G, C) is a $FSFS$ module over M.

Theorem (3.7): Let (F, A) and (H, B) be two $FSFS$ modules over M. Then $(F, A) \wedge (H, B)$ is an $FSFS$ module over M.

Proof. By Definition 2.7, we can write $(F, A) \wedge (H, B) = (G, A \times B)$. Since F_a and H_b are fuzzy submodules of M, $F_a \wedge H_b$ is a fuzzy submodule of M. Thus, $G(a, b) = F_a \wedge H_b$ is a fuzzy submodule of M, for all $(a, b) \in A \times B$. Hence, we find that $(F, A) \wedge (H, B)$ is an $FSFS$ module over M.

Theorem (3.8): Let (F, A) and (H, B) be two $FSFS$ modules over M. If $A \cap B = \emptyset$, then $(F, A) \cup (H, B)$ is an $FSFS$ module over M.

Proof. By Definition 2.5, we can write $(F, A) \cup (H, B) = (G, C)$. Since $A \cap B = \emptyset$, it follows that either $c \in A - B$ or $c \in B - A$ for all $c \in C$. If $c \in A - B$, then $G_b = F_b$ is a fuzzy submodule of M, and if $c \in B - A$, then $G_b = H_b$ is a fuzzy submodule of M. Hence, $(F, A) \cup (H, B)$ is an $FSFS$ module over M.

The following theorem is a generalized of theorems **3.6, 3.7, 3.8**.

Theorem (3.9): Let (F, A) be an $FSFS$ module over M, and let $\{(F_i, A_i)\}_{i\in I}$, be nonempty family of $FSFS$ submodules of (F, A). Then

(i) $\bigcap_{i\in I}(F_i, A_i)$ is an $FSFS$ submodule of (F, A),

(ii) $\bigwedge_{i\in I}(F_i, A_i)$ is an $FSFS$ submodule of (F, A),

(iii) If $A_i \cap A_j = \phi$, for all $i, j \in I$, then $\bigcup_{i\in I}(F_i, A_i)$ is an $FSFS$ submodule of (F, A).

Definition (3.10): Let $(F, \mathcal{R}_M, A)$ and $(H, \mathcal{R}_N B)$ be two $R-FSFS$ modules over M and N respectively, and let $f: M \to N$ be a homomorphism of modules, and let $g: A \to B$ be a mapping of sets. Then we say that $(f, g): (F, \mathcal{R}_M, A) \to (H, \mathcal{R}_N B)$ is an $FSFS$ homomorphism of $FSFS$ modules, if the following condition is satisfied:

$$f(F_a) = H(g(a)) = H_{g(a)}.$$

We say that (F, A) is an $FSFS$ homomorphic to (H, B).

Note that for $\forall a \in A, f: (M, F_a) \to (N, H_{g(a)})$ is a fuzzy homomorphism of fuzzy modules.

To introduce the kernel and image of $FSFS$ homomorphism of $FSFS$ modules, let $\tilde{M} = \ker f$. Define $\tilde{F}: A \to PF(\tilde{M})$ by $\tilde{F}_a = F_a|_{\tilde{M}}$. Then $(\tilde{F}, A)$ is an $FSFS$ module over $\tilde{M}$. It is clear that this module is an $FSFS$ submodule of (F,A).

Definition (3.11): $(\tilde{F},A)$ is said to be kernel of (f,g) and denoted by $ker(f,g)$.

And, let $\tilde{B} = g(A)$. Then for all $b \in \tilde{B}$, there exists $a \in A$ such that $g(a) = b$. Let $\tilde{N} = Im\, f < N$. We define the mapping $\tilde{H}: \tilde{B} \to PF(\tilde{N})$ as $\tilde{H}(\tilde{b}) = \tilde{B}(g(a))|_{\tilde{N}}$. Since (f,g) is an $FSFS$ homomorphism, $f(F_a) = H_{g(a)}$ is satisfied for all $a \in A$. Then the pair $(\tilde{H},\tilde{B})$ is an $FSFS$ module over $\tilde{N}$ and $(\tilde{H},\tilde{B})$ is an $FSFS$ submodule of (H,B).

Definition (3.12): $(\tilde{H},\tilde{B})$ is said to be image of (f,g) and denoted by $Im(f,g)$.

Proposition (3.13): Let (F,A) be an $FSFS$ module over M and N be an R −module and $f: M \to N$ be a homomorphism of $R-$ modules. Then $(f(F),A)$ is an $FSFS$ module over N.

Proof. If the mapping $f(F): A \to PF(N)$ is defined by $f(F)_a(y) = sup\{F_a(x): f(x) = y\}$, the proof is completed.

Proposition (3.14): If M is an $R-$ module, (H,A) is an $FSFS$ module over N and $f: M \to N$ is a homomorphism of $R-$ modules, then $(f^{-1}(H),A)$ is an $FSFS$ module over M .

Proof. If the mapping $f^{-1}(H): A \to PF(M)$ is defined by $\left(f^{-1}(H)\right)_a(x) = H_a(f(x))$, the proof is completed.

It is clear that $(f,I_A): (f^{-1}(H),A) \to (H,A)$, is an $FSFS$ homomorphism of $FSFS$ modules.

We now introduce the following lemma:

Lemma (3.15): Let M and N be an R −modules and $f: M \to N$ be a homomorphism and (F,A) and (H,A) are two $FSFS$ modules over M and N respectively.

(i) $(f,I_A): (F,A) \to (H,A)$, is an $FSFS$ homomorphism if and only if $\forall a \in A$, $H_a \geq f(F_a)$.

(ii) $(f,I_A): (F,A) \to (H,A)$, is an $FSFS$ homomorphism if and only if $\forall a \in A$, $F_a \leq f^{-1}(H_a)$.

Proof. (i): (f,I_A) is an FSFS homomorphism which means

$f(F_a) = H(I_A(a)) = H((a)) \leq H_a$.

Conversely $f(F_a) = H_a$, then (f,I_A) is an FSFS homomorphism.

(ii) similarly as in (i).

Now we define other algebraic operations over $FSFS$ modules.

Definition (3.16):

$$\prod_{i\in I}(F_i,A_i)$$

is said to be direct product of $FSFS$ modules.

Proposition (3.17): Let $\{(F_i,A_i)\}_{i\in I}$, be a family of $FSFS$ modules over $\{M_i\}_{i\in I}$ and $\{(H_i,B_i)\}_{i\in I}$, be a family of $FSFS$ modules over $\{N_i\}_{i\in I}$ and $(f_i,g_i): (F_i,A_i) \to (H_i,B_i)$ be a $FSFS$ homomorphism of $FSFS$ modules for each $i \in I$. Then

$$\left(\prod_{i\in I} F_i, \prod_{i\in I} g_i\right): \prod_{i\in I}(F_i,A_i) \to \prod_{i\in I}(H_i,B_i)$$

is an $FSFS$ hommorphism of $FSFS$ modules.

Proof. Since

$$\left(\prod_i f_i\right)\circ\left(\prod_i F_i\right) = \prod_i (f_i \circ F_i) = \prod_i (K_i \circ g_i) = \left(\prod_i K_i\right)\circ\left(\prod_i g_i\right),$$

the proof is completed.

Proposition (3.18): $\Pi: \Pi FSFSM \to FSFSM$ is a functor.

Proof. since by proposition (2.27), $\Pi: \Pi FSM \to FSM$ is a functer, and every $FSFS$ module is an FS module, the proof is completed.

Definition (3.19): (F,A) is said to be direct sum of $\{(F_i,A_i)\}_{i\in I}$, and denoted by $\oplus_{i\in I}(F_i,A_i)$.

The mapping :

$$\varphi_j: A_j \to \prod_{i\in I} A_i$$

is defined as $\varphi_j(a_j) = \{a_i\} = \begin{cases} a_{0i} & if\ i \neq j \\ a & if\ i = j \end{cases}$.

Also for embedding mapping $q_j: M_j \to \oplus_{i\in I} M_i$, $(q_j,\varphi_j): (F_j,A_j) \to (F,A)$ is an $FSFS$ homomorphism of $FSFS$ modules.

Proposition (3.20): Let $\{(F_i, A_i)\}_{i\in I}$, and $\{(H_i, B_i)\}_{i\in I}$ be family of $FSFS$ modules over $\{M_i\}_{i\in I}$ and $\{N_i\}_{i\in I}$, respectively, and let $(f_i, g_i): (F_i, A_i) \to (H_i, B_i)$ be an $FSFS$ homomorphism of $FSFS$ modules. Then

$$\left(\underset{i\in I}{\oplus} f_i, \prod_{i\in I} g_i\right): \underset{i\in I}{\oplus}(F_i, A_i) \to \underset{i\in I}{\oplus}(H_i, B_i)$$

is an $FSFS$ homomorphism of $FSFS$ modules.

Proposition (3.21): $\oplus: \Pi FSFSM \to FSFSM$ is a functor.

Proof. since by proposition (2.30), $\oplus: \Pi FSM \to FSM$ is a functor, and every $FSFS$ module is an FS module, the proof is completed.

Theorem (3.22): If $\{(F_i, A_i)\}_{i\in I}$ is a family of $FSFS$ modules over $\{M_i\}_{i\in I}$, then

$$\prod_{i\in I}(F_i, A_i),$$

is an $FSFS$ module over

$$\prod_{i\in I} M_i.$$

Proof. Define

$$F: \prod_{i\in I} A_i \to \prod_{i\in I} M_i$$

by $F(\{a_i\}) = \underset{i\in I}{\vee} p_i^{-1}(F_i)_{a_i}$, where

$$p_i: \prod_{i\in I} M_i \to M_i$$

is a projection mapping. Since

$$p_i^{-1}(F_i)_{a_i}: \prod_{i\in I} M_i \to [0,1]$$

is an $FSFS$ module over

$$\prod_{i\in I} M_i,$$

for all $i \in I$,

$$\bigvee_{i\in I} p^{-1}(F_i)_{a_i}$$

is also an $FSFS$ module over

$$\prod_{i\in I} M_i.$$

Theorem (3.23): If $\{(F_i, A_i)\}_{i\in I}$ is a family of $FSFS$ modules over $\{M_i\}_{i\in I}$, then

$$\underset{i\in I}{\oplus}(F_i, A_i)$$

, is an $FSFS$ module over

$$\underset{i\in I}{\oplus} M_i.$$

Proof. Define

$$F: \prod_{i\in I} A_i \to \underset{i\in I}{\oplus} M_i$$

for all

$$\{a_i\} \in \prod_{i\in I} A_i$$

by

$$F(\{a_i\}) = \bigwedge_{i\in I} j_i(F_i)_{a_i}$$

, where $j_i: M_i \to \underset{i\in I}{\oplus} M_i$ is a embedding mapping. Since $j_i(F_i)_{a_i}$ is an $FSFS$ submodule over $\underset{i\in I}{\oplus} M_i$, for all $i \in I$, $F(\{a_i\})$ is also an $FSFS$ module over $\underset{i\in I}{\oplus} M_i$.

Lemma (3.24): 1) Given modules $\{M_i\}_{i\in I}$ and N and a family of R –homorphisms $A = \{f_i: M_i \to N\}_{i\in I}$. If $\{(F_i, A_i)\}_{i\in I}$ are $FSFS$ modules over $\{M_i\}_{i\in I}$, then there exist an $FSFS$ module ,

$$\left(H, \prod_{i\in I} A_i\right)$$

over N such that for all $i \in I$,

$$f_i: (F_i, A_i) \to \left(H, \prod_{i\in I} A_i\right)$$

is an $FSFS$ homomorphism of $FSFS$ modules.

2) Given modules M and $\{N_i\}_{i\in I}$ and a family of R –homorphisms $B = \{g_i: M \to N_i\}_{i\in I}$. If $\{(H_i, B_i)\}_{i\in I}$ are $FSFS$ modules over $\{N_i\}_{i\in I}$, then there exist an $FSFS$ module , $(F, \prod_{i\in I} A_i)$ over M such that for all $i \in I$,

$$g_i: \left(F, \prod_{i\in I} A_i\right) \to (H_i, B_i)$$

is an $FSFS$ homomorphism of $FSFS$ modules.

Proof. 1) Define : $H: \prod_{i\in I} A_i \to N$ by $F(\{a_i\}) = \underset{i}{\vee} f_i(F_i)_{a_i}$.

2) Define : : $F: \prod_{i\in I} A_i \to M$ by $F(\{a_i\}) = \underset{i\in I}{\wedge} g^{-1}{}_i(F_i)_{a_i}$ for all $\{a_i\} \in \prod_{i\in I} A_i$.

We define the concepts of quotient module, product and coproduct operations in the category of $FSFS$ modules.

Corallary (3.25): If (F, A) is a $FSFS$ module over M and N is a submodule of M, $i: N \to M$

is an embedding mapping, then $(i^{-1}(F), A)$ is an $FSFS$ module over N.

Corallary (3.26): If (F, A) is an $FSFS$ module over M and $p: M \rightarrow M/\mathcal{N}$ is a canonical projection, then $(p(F), A)$ is an $FSFS$ module over quotient module $M/\mathcal{N}$.
If $\{(F_i, A_i)\}_{i\in I}$ is a family of $FSFS$ modules over the family of modules $\{M_i\}_{i\in I}$, then we can define the product and coproduct of these families by $\prod_{i\in I}(F_i, A_i)$ and $\bigoplus_{i\in I}(F_i, A_i)$ respectively.

Theorem (3.27): The category of $FSFS$ modules has zero objects, sums, product, kernel and cokernel.

Theorem (3.28): Let (F, A) and (G, B) be two $FSFS$ modules over M and N , respectively, and $F \otimes G: A \times B \rightarrow M \otimes N$. Then $(F \otimes G, A \times B)$ is an $FSFS$ module over $M \otimes N$.

Proof. Let $(F \otimes G)(a,b) = F_a \otimes G_b$, $\forall(a,b) \in A \times B$ [7], $\forall(a,b) \in A \times B, (M, F_a)$ and (N, G_b) are $FSFS$ modules. From [12], $F_a \otimes G_a$ is a fuzzy submodule over $M \otimes N$. Then $(F \otimes G, A \times B)$ is an $FSFS$ module over $M \otimes N$.

Definition (3.29): $(F \otimes G, A \times B)$ is said to be tensor product of (F, A) and (G, B) , and denoted by $(F, A) \otimes (G, B)$.

Refrences:

1. Molodtsov, D.**1999** Soft set theory-First results, *Coumpt. Math. Appl.*, 37 , 19-31.
2. Maji, P.K. and Biswas, R. and Roy, A.R. **2001**. Fuzzy soft sets, *The Journal of Fuzzy Mathematics*, 9(3) 589-602.
3. Maji, P.K. and Biswas, R. and Roy, A.R. **2001**. Intuitionistic fuzzy soft sets, *The Journal of Fuzzy Mathematics*, 9(3) , 677-692.
4. Maji, P.K. and Biswas, R. and Roy, A.R. **2004**. On intuitionistic fuzzy soft sets, *The Journal of Fuzzy Mathematics*, 12(3), 669-683.
5. Zadeh L.A.**1965**. Fuzzy sets, *Information and control*, 8, 338-353.
6. Aktas H. and Cagman N. **2007**. Soft sets and soft groups, *Information Science*, 177, 2726-2735.
7. Aygunoglu A. and Aygun H. **2009**.ntroduction to fuzzy soft group, *Coumpt.Math. Appl.*, 58 1279-1286.
8. Feng F. and Jun Y.B. and Zhao X. **2008**.Soft semirings, Comput. Math. Appl.56 2621-2628.
9. Acar U. and Koyuncu F. and Tanay B. **2010**. Soft sets and soft rings, Comput. Math. Appl.59 3458-3463.
10. Ghosh J. and Dinda B. and Samanta T.K. **2011**.Fuzzy Soft Rings and Fuzzy Soft Ideals *Int. J. Pure Appl. Sci. Technol.*, 2(2) ,pp. 66-74.
11. Lopez-Permouth S.R. and Malik D.S. **1990** On categories of fuzzy modules, Information Sciences 52, 211-220.
12. Lopez-Permouth S.R. and Morita L. **1992**. Equivalence to Categories of Fuzzy Modules,Information Sciences 64, 191-201.
13. Zahedi M. and Ameri R. **1995.** On Fuzzy Projective and Injective Modules, The journal of Fuzzy Mathematics, Vol.3, No.1, 181-190.
14. Sun Q.M. and Zhang Z.L. and Liu J. **2008**,Soft sets and soft modules, Lecture Notes in Comput. Sci. 5009 403-409.
15. Gunduz C. and Bayramov S**. 2011**. Fuzzy Soft Modules International Mathematical Forum, 6, 11, 517 – 527.
16. Schweizer B. and Sklar A.**1960**. Statistical metric space, *Pacific Journal of Mathematics*, 10, 314-334.
17. Jin-liang L. and Rui-xia Y. and Bing-xue Y. Fuzzy soft sets anf fuzzy soft groups, Chinese Control and Decision Conference,**2008**,2626-2629