On D- Compact Topological Groups

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Abstract

In the present paper, we have introduced some new definitions On D-compact topological group and D-L. compact topological group for the compactification in topological spaces and groups, we obtain some results related to D- compact topological group and D-L. compact topological group.

Keywords: groups , cyclic groups , topological group , D- cover topological groups , isomorphism , direct product , D- compact topological group .

التراص من نوع D للزمر التبلوجية

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الخلاصة

في هذا البحث قدمنا بعض التعاريف الجديدة عن التراص من نوع D ونوع D-L. للزمر التبلوجية كنوع من التراص للفضاءات التبلوجية والزمر . وقدمنا بعض النتائج المتعلقة بهذه الأنواع إضافة إلى النتائج التي تبين علاقتهم ببعض.

1. Introduction

A compact (topological , often understood) group $(G,*,\tau)$ is a topological group whose topology is compact, [1]. D.G. Salih in [2] gave the concept of D - cover groups as follows : For an index set I, a family G_i of proper subgroups of $(G,*)$ is called D - Cover if $G = \bigcup_{i \in I} G_i$. A group $(G,*)$ is said to be D - compact group if for every D - cover groups of $(G,*)$ there is a finite sub D - cover groups of $(G,*)$ see also [2]. In this paper we investigated On D - compact, D-L.compact and weakly D- compact Topological group for the compactification in topolgical Spaces and groups. In Particular case we introduce weakly D - compact cyclic , D - compact cyclic and D-L. compact cyclic topological group. we obtain Some good result related these concepts above .

Note : we mean throughout this paper a topological group is a just group as a set with topology

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2. Definitions and Examples

Definition 1

Let $(G,*,\tau)$ be a topological group and I be an indexed set , we say that;

1. The family $\{G_i \in \tau : (G_i ,*)$ is a proper subgroup of $(G ,*)$, $\forall i \in I\}$, is a D – cover topological groups of $(G ,*, \tau)$ if $G = \bigcup_{i\in I} G_i$.
2. The family $\{G_i \in \tau : (G_i ,*)$ is a proper cyclic subgroup of $(G,*)$, $\forall i\in I\}$, is a D – cover cyclic topological groups of $(G,*,\tau)$ if $G = \bigcup_{i\in I} G_i$.

Definition 2

Let $(G,*,\tau)$ be a topological group , we say that;

1. $(G,*,\tau)$ is weakly D- compact topological group if there exists a finite D – cover topological groups of $(G,*,\tau)$.
2. $(G,*,\tau)$ is D- compact topological group if for any D – cover topological groups of $(G,*,\tau)$, there is a finite sub - D – cover topological groups of $(G,*,\tau)$.
3. $(G,*,\tau)$ is weakly D-L. compact topological group if there exists a countable D – cover topological groups of $(G,*,\tau)$.
4. $(G,*,\tau)$ is D-L. compact topological group if for any D – cover topological groups of $(G,*,\tau)$, there is a countable sub - D – cover topological groups of $(G,*,\tau)$.

Definition 3

Let $(G,*,\tau)$ be a topological group , we say that;

1. $(G,*,\tau)$ is weakly D- compact cyclic topological group if there exists a finite D – cover cyclic topological groups of $(G,*,\tau)$.
2. $(G,*,\tau)$ is D- compact cyclic topological group if for any D – cover cyclic topological groups of $(G,*,\tau)$, there is a finite sub - D – cover cyclic topological groups of $(G,*,\tau)$.
3. $(G,*,\tau)$ is weakly D-L. compact cyclic topological group if there exists a countable D – cover cyclic topological groups of $(G,*,\tau)$.
4. $(G,*,\tau)$ is D-L. compact cyclic topological group if for any D – cover cyclic topological groups of $(G,*,\tau)$, there is a countable sub - D – cover cyclic topological groups of $(G,*,\tau)$.

Definition 4

Let $(G,*,\tau)$ be a topological group and $(H,*)$ be a subgroup of $(G,*)$. The topological subgroup $(H,* \tau_H)$ $[\tau_H = \tau \cap H]$ is said to be :

1. D- compact topological subgroup (weakly D- compact topological subgroup , D-L. compact topological subgroup, weakly D-L. compact topological subgroup), if $(H,*,\tau_H)$ is a D-compact topological group (weakly D-compact topological group), respectively.
2. D- compact cyclic topological subgroup (weakly D- compact cyclic topological subgroup , D-L. compact cyclic topological subgroup, weakly D-L. compact cyclic topological subgroup), if $(H,*,\tau_H)$ is a D-compact cyclic topological group (weakly D – compact cyclic topological group), respectively.

Definition 5[3]

1. Let $(G,*,\tau)$ and $(\bar{G},\bar{*},\bar{\tau})$ be two topological groups then ,
   i. $f : (G,*,\tau) \to (\bar{G},\bar{*},\bar{\tau})$ is a topological homomorphism if $f : (G,\tau) \to (\bar{G},\bar{\tau})$ is continuous and $f(x * y) = f(x) \bar{*} f(y)$ $\forall x,y \in G$.
   ii. $f : (G,*,\tau) \to (\bar{G},\bar{*},\bar{\tau})$ is an isomorphism if it is a topological homeomorphism and $f(x * y) = f(x) \bar{*} f(y)$ $\forall x,y \in G$.
2. Suppose $\wedge$ is non – empty set and $(G_\lambda,*_\lambda,\tau_\lambda)$ is a topological group for each $\lambda\in\wedge$. Their product is $\prod_{\lambda\in\wedge} G_\lambda$ equipped with the usual product topology $\tau_{\prod_{\lambda\in\wedge} G_\lambda}$ and with multiplication given by $(x \otimes y) = x_\lambda *_\lambda y_\lambda$ for each $x_\lambda, y_\lambda \in G_\lambda$ and $\lambda\in\wedge$.
3. If $G_\lambda = G$ and $\tau_\lambda = \tau$, $\forall\lambda\in\wedge$, then we denoted that $G^\wedge = \prod_{\lambda\in\wedge} G_\lambda$ and $\tau^\wedge = \tau_{\prod_{\lambda\in\wedge} G_\lambda}$.

Example 1.

Let $(S_3, 0)$ be the symmetric groupof degree 3 , and
T = {Ø,{e}, {e,(12)}, {e,(13)}, {e,(23)} , {e,(123),(132)} , {e,(12),(13)} , {e,(12),(23)} , {e,(13),(23)}, {e,(12),(123),(132)} , {e,(13),(123), (132)}, { e, (23),(123),(132) } , {e

,(12),(13),(23) } , { e ,(12) , (13) ,(123) ,(132) } , {e,(13),(23),(123),(132)}, {e, (12) , (23) , (123) , (132)} , S_3}.

Where (12) = $\binom{123}{213}$, (13) = $\binom{123}{321}$, (23) = $\binom{123}{132}$, (123) = $\binom{123}{231}$ and (132) = $\binom{123}{312}$.

The topological group (S_3,o,τ) is a D-compact cyclic topological group , since {{e,(12)} , {e,(13)}, {e,(23)}, {e,(123) ,(132)}} is a D-cover cyclic topological groups of (S_3,o,τ) .

In general , the symmetric group (S_n,o) of degree n ≥ 4, with suitable topology is D – compact topological group.

Example 2.

Let G = {0 , 1 , 2 ,….}, defined a binary operation ∗ as follows :

$$a * b = \begin{cases} \max\{a,b\} & a \neq b \\ 0 & a = b \end{cases}, \forall a,b \in G, \text{and } \tau = \{\{0,1,2,\dots,n\}: n \in Z^+\} \cup \emptyset$$

, it is easy to show that (G,∗,τ) is an actually topological i.e. the two operation g : G × G → G , g(a,b) = a ∗ b and h : G → G , h(a) = a^{-1} are continuous a , b ∈ G.

Now let $\{G_n : G_n \in \tau, n \in z^+\}$ be any family of subsets of τ , it is easy to show that $\{G_n\}_{n\in z^+}$ is a D- cover topological groups of (G,∗ ,τ) which have a countable sub– D– cover topological group $\{G_n\}_{n\in z^+}$, such that $G=\bigcup_{n\in z^+} G_n$ and $(G_n,*)$ is a group $\forall n \in Z^+$. And hence (G,∗ ,τ) D-L. compact topological group which is not D - compact topological group , Since there is no finite Sub - D - cover topological groups of (G ,∗ ,τ). Also (G,∗ ,τ) is not weakly D - compact topological group.

3. main results

It is easy to prove direct from definitions the following lemmas,

Lemma 1.

1. Any D – compact topological group is weakly D – compact topological group.
2. Any D – compact topological group is D –L. compact topological group.

Lemma2.

1. Any D – compact cyclic topological group is weakly D – compact cyclic topological group.
2. Any D – compact cyclic topological group is D–L. compact cyclic topological group.

Lemma 3.

Any D –compact topological group is D-compact cyclic topological group.

We can prove directly , by order the group and Lemma1, the following theorem.

Theorem 1.

Let (G,∗ ,τ) be a topologicalgroup , such that G is a finite set. Then the following are equivalents:

1. (G,∗ ,τ) is a D – compact topological group.
2. (G,∗ ,τ) is a D-L. compact topological group.

If we replace D-(D-L.) compact topological group with D-(D-L.) compact cyclic topological group, respectively, the result is true , too.

One can show easily by definition that Any cyclic group (finite or infinite) cannot be D-compact topological group. Thus we have the following theorem .

Theorem 2.

Any infinite group (not cyclic) can be a D-compact topological group.

Proof.

Let (G,∗) be any infinite not cyclic group, *I* be a set (finite or infinite), defined τ = $\{A_i \subseteq G: A_i^c$ is a finite set , $(A_i,*)$ group $\forall_i \in I \,\&\, A_{i_1} \subseteq A_{i_2}$ for $i_1 \leq i_2\} \cup \emptyset$.

It is clear that τ ≠ Ø , since every finite group G, (O (G) ≥ 4), has a nontrivial subgroups unless it is a cyclic of prime order, but G is an infinite so G has a nontrivial subgroups [4].

(G,τ) is a topological space since,

1) Ø ∈ τ and G^c = Ø is a finite implies G ∈ τ.
2) Let A_1 , $A_2 \in \tau$ so A_1^c, A_2^c are finite , but $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$ which is clear finite hence $(A_1 \cap A_2)^c$is finite and we know that $(A_1 \cap A_2, *)$ is a group implies $A_1 \cap A_2 \in \tau$.
3) Let $A_s \in \tau$, ∀s ∈ S i.e. A_s^c is a finite ∀s ∈ S hence $\bigcap_{s\in S} A_s^c$ is a finite on the other hand $(\bigcup_{s\in S} A_s)^c = \bigcap_{s\in S} A_s^c$, but $\bigcup_{s\in S} A_s = A_t$ for some t where s ≤ t ∀s ∈ S so $(\bigcup_{s\in S} A_s, *)$ is a group and hence $\bigcup_{s\in S} A_i \in \tau$.

Therefore (G,τ) is a topological space, implies (G,∗ ,τ) is a topological group. Now let $\{A_\lambda: A_\lambda \in \tau , \lambda \in \wedge\}$, indexed by ∧, be any D-cover topological groups of $(G,*,\tau)$, that is $G = \cup_{\lambda\in\Lambda} A_\lambda$. If $A_\circ \in \{A_\lambda\}_{\lambda\in\Lambda} \Rightarrow (A_\circ,*)$ is a group and $A_\circ^c$ is a finite set , i.e. $A_\circ^c = \{a_1, a_2, \ldots, a_n\}$, where $a_j \in G\ \forall j \in J$. For each $j \in J$ there $A_{\lambda_j} \in \{A_\lambda\}_{\lambda\in\Lambda}$ such that $a_j \in A_{\lambda_j}$ implies $A_\circ^c \subseteq \cup_{j\in J} A_{\lambda_j}$ .But $G = A_\circ \cup A_\circ^c$ implies $G \subseteq A_\circ \cup \left(\cup_{j\in J} A_{\lambda_j}\right)$ which gives there is a finite sub – D- cover topological groups $\{A_\circ, A_{\lambda_1}, A_{\lambda_2}, \ldots A_{\lambda_n}\}$. Therefore $(G,*,\tau)$ is D- compact topological group.

The following corollary is direct from Lemma 1 and Theorem 2:

Corollary 1.

Any infinite group (not cyclic) can be a weakly D- (D-L.) compact topological group.

Theorem 3.

Let $(G,*,\tau)$ and $(\bar{G},\bar{*},\bar{\tau})$ be two topological groups , if $(G,*)$ is a group and $(\bar{G},\bar{*},\bar{\tau})$ is a D- compact topological group. Then $(G\times\bar{G}, \otimes, \tau\times\bar{\tau})$ is a D- compact topological group.

Proof

Let $\{(G\times\bar{G}_i, \otimes)$; $\bar{G}_i \in \bar{\tau}$ and $(\bar{G}_i, \bar{*})$ group $\forall i \in I\}$ be any D-cover topological groups of $G\times\bar{G} \Rightarrow G\times\bar{G} = \cup_{i\in I}(G\times\bar{G}_i) = G\times(\cup_{i\in I}\bar{G}_i) \Rightarrow \bar{G} = \cup_{i\in I}\bar{G}_i$ but $(\bar{G},\bar{*},\bar{\tau})$ is a D- compact topological group, so there is a finite subset $J \subseteq I$ such that $\bar{G} = \cup_{j\in J}\bar{G}_i$ and $(\bar{G}_i, \bar{*})$ is a group for each $j \in J \Rightarrow G\times\bar{G} = G\times(\cup_{i\in I}\bar{G}_i) = \cup_{j\in J}(G\times\bar{G}_i)$, where $G\times\bar{G}_i \in \tau\times\bar{\tau}$ and $(G\times\bar{G}_j, \otimes)$ is a group for each $j \in J$.

Therefore $(G\times\bar{G}, \otimes, \tau, \bar{\tau})$ is a D- compact topological group.

Theorem 4.

Let $(G,*,\tau)$ and $(\bar{G},\bar{*},\bar{\tau})$ be two D- compact topological groups . Then $(G\times\bar{G}, \otimes, \tau\times\bar{\tau})$ is a D- compact topological group.

proof:

Let $(G,*,\tau)$ and $(\bar{G},\bar{*},\bar{\tau})$ be any two D- compact topological groups.

Then there exists a D- cover topological groups $\{G_a\}_{a\in A}$ and $\{\bar{G}_b\}_{b\in B}$ of G and $\bar{G}$ respectively , where $G\times\bar{G} = \cup_{a\in A, b\in B}(G_a\times\bar{G}_b)$ implies $\{G_a\times\bar{G}_b\}_{a\in A, b\in B}$ is a D- cover topological groups of $(G\times\bar{G}, \otimes, \tau\times\bar{\tau})$.

Let $\{W_i\}_{i\in I}$ be any D – cover topological groups of $(G\times\bar{G}, \otimes, \tau\times\bar{\tau})$, that means $G\times\bar{G} = \cup_{i\in I} W_i$ such that $W_i = U_i\times V_i$ where $U_i\in\tau$ and $V_i\in\bar{\tau}$, $\forall i \in I$.

But $(G,*,\tau)$ is a D- compact topological group, so there is a finite subset $J \subseteq I$ such that $G = \cup_{j\in J} U_j$ and $(U_j,*)$ is a group for each $j \in J$.

Let $U_{j_1} \in \{U_j\}_{j\in J}$ implies $\{U_{j_1}\times V_i\}_{i\in I}$ is a D- cover topological groups of $(U_{j_1}\times\bar{G}, \otimes)$ hence $U_{j_1}\times\bar{G} = \cup_{i\in I}(U_{j_1}\times V_i)$, but $U_{j_1}\times\bar{G}$ is a D- compact topological group from Theorem 3 since $(U_{j_1},*)$ is a group and $(\bar{G},\bar{*},\bar{\tau})$ is a D- compact topological group , so there is a finite set $S \subseteq I$ such that is a group $\forall s \in S$ and $\{U_{j_1}\times V_s\}_{s\in S}$ $U_{j_1}\times\bar{G} = \cup_{s\in S}(U_{j_1}\times V_s)$ =

[see 5] hence $G\times\bar{G} = (\cup_{j\in J}U_j)\times(\cup_{s\in S}V_s) = \cup_{j\in J, s\in S}(U_j\times V_s)$ where $(U_j\times V_s, \otimes)$ are gr $U_{j_1}\times(\cup_{s\in S}V_s)$ but $\cup_{j_1\in J}(U_{j_1}\times(\cup_{s\in S}V_s)) = (\cup_{j\in J}U_j)\times(\cup_{s\in S}V_s) = G\times\bar{G}$

Theorem 5.[1]

Let $\{G_i : i \in I\}$ be a family of topological group. Then the direct $G = \prod_{i\in I} G_i$, equipped with the product topology is a topological group.

From Theorem 4 and Theroem 5, respectively , and by induction we can prove the following theorem:

Theorem 6.

The product of any finite collection of D- compact topological groups is a D- compact topological group.

If we replace D- compact topological group with D-L. compact topological group, the result is true.

Corollary 2.

If $(G,*,\tau)$ is a D- compact topological group. Then $(G^n, \otimes, \tau^n)$ is D- compact topological group , where $G^n = \frac{G\times G\times\ldots\times G}{n-time}$ and $\tau^n = \frac{\tau\times\tau\times\ldots\times\tau}{n-time}$.

Theorem 7.

Let (G,∗ ,τ) and $(\bar{G},\bar{*},\bar{\tau})$ be two topological groups , f : (G,∗ ,τ) → $(\bar{G},\bar{*},\bar{\tau})$ be an isomorphism . Then

1. If S is a D- compact topological subgroup in (G,∗ ,τ) , then f(S) is a D- compact topological subgroup in $(\bar{G},\bar{*},\bar{\tau})$.
2. If T is a D- compact topological subgroup in $(\bar{G},\bar{*},\bar{\tau})$ and f is a open map then f^{-1}(T) is a D- compact topological subgroup in (G,∗ ,τ).

Proof:

1. Let $\{\bar{G}_i\}_{i\in I}$ be any D- cover topological groups of f(S) in $(\bar{G},\bar{*},\bar{\tau})$ that is f (S) = $\cup_{i\in I}\bar{G}_i$ implies S = f^{-1} ($\cup_{i\in I}\bar{G}_i$) =$\cup_{i\in I}$ f^{-1} ($\bar{G}_i$) [see 5] . It is clear that f^{-1} ($\bar{G}_i$) ∈ τ , ∀i ∈ I since $\bar{G}_i \in\bar{\tau}$, ∀i ∈ I and f is continuous , but S is a D-compact topological subgroup in (G,∗ ,τ) , so there is a finite subset J ⊆ I such that S = $\cup_{j\in J}$ f^{-1} ($\bar{G}_i$) and (f^{-1} ($\bar{G}_i$),∗) is a group ∀j ∈ J, hence S = f^{-1}($\cup_{j\in J}\bar{G}_j$) ⇒ f(S) = f (f^{-1}($\cup_{j\in J}\bar{G}_j$)) = $\cup_{j\in J}\bar{G}_j$ [see 5], where $(\bar{G}_j,\bar{*})$ is a group ∀j ∈ J since f is an isomorphism. Thus f (S) is a D- compact topological subgroup in $(\bar{G},\bar{*},\bar{\tau})$.
2. Let $\{ G_i\}_{i\in I}$ be any D- cover topological groups of f^{-1}(T) in (G,∗ ,τ) that is f^{-1}(T) = $\cup_{i\in I} G_i$, (G_i ∈ τ, ∀i ∈I) implies T = f ($\cup_{i\in I} G_i$) =$\cup_{i\in I}$ f(G_i) , it is clear that f (G_i) ∈$\bar{\tau}$, ∀i ∈ I since f is an open map, but T is a D – compact topological subgroup $(\bar{G},\bar{*},\bar{\tau})$, so there is a finite subset J ⊆ I such that T = $\cup_{j\in J}$ f (G_j) where (f (G_j),$\bar{*}$) is a group ∀j ∈ J then T = f ($\cup_{j\in J} G_j$) hence f^{-1}(T) = $\cup_{j\in J}G_j$, where (G_j,∗) is a group ∀j ∈ J since f is an isomorphism and hence f^{-1} (T) is a D- compact topological subgroup in (G,∗ ,τ).

Theorem 8.

Let (G,∗ ,τ) and $(\bar{G},\bar{*},\bar{\tau})$ be two topological groups , f : (G,∗ ,τ) → $(\bar{G},\bar{*},\bar{\tau})$ is an isomorphism . Then the following are equivalents :

1. (G,∗ ,τ) is a D- compact topological group.
2. $(\bar{G},\bar{*},\bar{\tau})$ is a D- compact topological group.

Proof.

(⇒) Suppose that (G,∗ ,τ) is a D- compact topological group, let $\{\bar{G}_i\}_{i\in I}$ be any D- cover topological groups of $(\bar{G},\bar{*},\bar{\tau})$ that is $\bar{G}$ = $\cup_{i\in I}\bar{G}_i$ gives G = f^{-1} $(\bar{G}) = f^{-1}(\cup_{i\in I}\bar{G}_i)$ =$\cup_{i\in I}$ f^{-1} ($\bar{G}_i$), but (G,∗ ,τ) is a D- compact topological group , so there is a finite subset J ∈ I such that G =$\cup_{j\in J}$ f^{-1}($\bar{G}_i$) and (f^{-1} ($\bar{G}_i$),∗) is a group ∀j ∈ J hence G = f^{-1}($\cup_{j\in J}\bar{G}_j$) and hence $\bar{G}$ = f(G) = f (f^{-1}($\cup_{j\in J}\bar{G}_j$)) = $\cup_{j\in J}\bar{G}_j$, where $(\bar{G}_j,\bar{*})$ is a group ∀j ∈ J .

Therefore $(\bar{G},\bar{*},\bar{\tau})$ is a D – compact Topological group.

(⇐) Suppose that $(\bar{G},\bar{*},\bar{\tau})$ is a D – compact Topological group, let $\{ G_i\}_{i\in I}$ be any D - cover topological groups of (G,∗ ,τ) i.e. G = $\cup_{i\in I} G_i$. Now $\bar{G}$ = f $(G) = f(\cup_{i\in I} G_i)$ =$\cup_{i\in I}$ f (G_i) , but$(\bar{G},\bar{*},\bar{\tau})$ is a D – compact topological group , so there is a finite subset J ⊆ I such that $\bar{G}$ = $\cup_{j\in J}$ f (G_j) and (f (G_j),$\bar{*}$) is a group ∀j ∈ J implies $\bar{G}$ = f ($\cup_{j\in J} G_j$) hence G = $f^{-1}(\bar{G}) = f^{-1}\left(f\left(\cup_{j\in J} G_j\right)\right) = \cup_{j\in J} G_j$ where (G_j ,∗) is a group ∀j ∈ J.

Therefore (G,∗ ,τ) is a D-compact topological group.

We can prove by the similar way the following theorem.

Theorem 9.

Let (G,∗ ,τ) and $(\bar{G},\bar{*},\bar{\tau})$ be two topological groups and f : (G,∗ ,τ) → $(\bar{G},\bar{*},\bar{\tau})$ be an isomorphism. Then the following are equivalents:

1. (G,∗ ,τ) is a D- compact cyclic topological group.
2. $(\bar{G},\bar{*},\bar{\tau})$ is a D- compact cyclic topological group.

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