Coregular Modules

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Abstract
In this paper we study the concepts of copure submodules and coregular modules. Many results related with these concepts are obtained.

Keywords: Pure submodule, copure submodule, regular module, coregular module, multiplication module, strongly comultiplication module, completely distributive module.

المقاسات المنتظمة المضادة

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الخلاصة
في هذا البحث درسنا المفهومين المقاسات الجزئية النقية المضادة والمقاسات المنتظمة المضادة. العديد من النتائج المتعلقة بهاذين المفهومين قد أعطيت.

Introduction

Throughout this paper, R will denote a commutative ring with identity and M be a left R-module. A submodule N of M is called pure if for any ideal I of R, $IM \cap N = IN$, [1]. An ideal I of R is called pure if for each $x \in I$, there exists y in I, such that $x = xy$ [2]. Equivalently I is a pure ideal in R if $IJ = I \cap J$ for each ideal J of R; that is I is a pure ideal of R if and only if I is a pure R-submodule of R.

It is known that a ring R is regular if every element in R is regular (in sense of Von Neumann). Equivalently R is regular if every ideal of R is pure. M is called a regular R-module if every submodule of M is pure [3]. Equivalently M is regular if $R/(0 \underset{R}{:} x)$ is a regular ring [4].

H.Ansari and F.Farshadifar in [5] introduced the concept of copure submodules, where a submodule N of M is called copure if $(N \underset{M}{:} I) = N + (0 \underset{M}{:} I)$ for each ideal I of R.

Also they in [6], present the concept of fully copure modules, where M is called fuly copure if each submodule of M is copure we see it is more convientnt to use the name coregular module for fully copure module.

In this paper we continuo the studying of copure submodules and coregular modules. In S.1 of this paper, we give new results about copure submodules.

Among other results, it is shown that every direct summand of a module is copure submodule (see Prop. 1.4). Also it is shown that (Prop. 1.5) Let $M = \overset{n}{\underset{i=1}{\oplus}} M_i$, where M_i is R-module, for each i = 1, …,n. If $N_i \leq M_i$ for each i = 1, …,n. Then $\overset{n}{\underset{i=1}{\oplus}} N_i$ is copure in M if and only if N_i is copure in M_i, for each i = 1, …,n. Moreover we study the hereditary property of copure submodules (see Th. 1.6).

Next we give a characterization of copure submodules in the class of completely distributive modules (see Th. 1.9).

In S.2, we study coregular modules. We give some relationships between coregular modules (rings) and regular modules (rings) (see Prop.2.7, 2.8, 2.9, 2.12, 2.13), Also, we study the hereditary property of the class of coregular modules (see Th.2.14).

Moreover we study the direct summand of coregular module (see Cor.2.6) and give certain conditions such that the direct sum of coregular modules is coregular (see Prop.2.15, Prop. 2.16).

S.1 Copure Submodules

Following [5], a submodule N of an R-module M is called copure if $(N \underset{M}{:} I) = N + (0 \underset{M}{:} I)$ for each ideal I of R. We present new properties of copure submodules. Furthermore these results will be needed in S.2.

Proposition 1.1:

Let M and M′ be two R-modules. If $f : M \longrightarrow M'$ be an epimorphism and N is a copure submodule of M such that ker $f \subseteq N$. Then $f(N)$ is a copure submodule in M′.

Proof: Let I be an ideal of R. To prove $(f(N) \underset{M'}{:} I) = f(N) + (0 \underset{M'}{:} I)$. Let $m' \in (f(N) \underset{M'}{:} I)$. Hence $I m' \subseteq f(N)$. Since f is onto, $m' = f(m)$ for some $m \in M$, so that $I f(m) \subseteq f(N)$; that is $f(Im) \subseteq f(N)$. Hence for each $i \in I$, there exists $n \in N$ such that $i m - n \in \ker f$. It follows that $Im \subseteq N$; i.e. $m \in (N \underset{M}{:} I)$. But N is copure in M, so that $(N \underset{M}{:} I) = N + (0 \underset{M}{:} I)$ and hence $m = n_1 + x$ for some $n_1 \in N$, $x \in (0 \underset{M}{:} I)$. It follows $m'=f(m) = f(n_1) + f(x)$, $f(n_1) \in f(N)$ and $f(x) \in (0 \underset{M'}{:} I)$. Thus $m' \in f(N) + (0 \underset{M'}{:} I)$.

Hence $(f(N) \underset{M'}{:} I) \subseteq f(N) + (0 \underset{M'}{:} I)$. The reverse inclusion is clear. Therefore $(f(N) \underset{M'}{:} I) = f(N) + (0 \underset{M'}{:} I)$; that is f (N) is copure.

Corollary 1.2: [5, Th.2.9 (c)]

Let M be an R-module and let N and K be submodules of M with $N \subseteq K$. If K is copure in M, then K/N is copure in M/N.

Proof: It follows directly by Prop.1.1, by taking the natural projection $\pi: M \longrightarrow M/N$.

Proposition 1.3:

Let $N_1 \subseteq N_2 \subseteq \ldots$ be an ascending chain of copure submodules of an R-module M. Then $\overset{\infty}{\underset{i=1}{\cup}} N_i$ is copure submodule in M.

Proof: It is easy, so it is omitted.

Proposition 1.4:

Every direct summand of an R-module M is copure.

Proof: Let N be a direct summand of M. Then $M = N \oplus K$ for some $K \leq M$. Let I be an ideal of R, and let $m \in (N \underset{M}{:} I)$. Hence $I m \subseteq N$. But $m \in M$, so $m = n + k$ for some $n \in N$, $k \in K$, then $I(m + k) \subseteq N$. It follows that $Ik \subseteq N$ and so $I k \subseteq N \cap K = (0)$. Hence $k \in (0 \underset{M}{:} I)$ and $m = n + k \in N + (0 \underset{M}{:} I)$. Thus $(N \underset{M}{:} I) \subseteq N + (0 \underset{M}{:} I)$ and the reverse inclusion is clear, so $(N \underset{M}{:} I) = N + (0 \underset{M}{:} I)$.

Proposition 1.5:

Let M be a direct sum of R-modules M_1, M_2, …, M_n. If, $N_i \leq M_i$ for each i = 1, …, n. Then $N = \overset{n}{\underset{i=1}{\oplus}} N_i$ is a copure submodule in M if and only if N_i is copure in M_i, for each i= 1,…,n.

Proof: if n = 2

$\Rightarrow$ Let $\rho_1: M_1 \oplus M_2 \longrightarrow M_1$, $\rho_2: M_1 \oplus M_2 \longrightarrow M_2$ be projections. Then ker $\rho_1 = (0) \oplus N_2 \subseteq N$, $\rho_2 = N_1 \oplus (0) \subseteq N$. Hence by Prop. 1.1, $\rho_1(N)=N_1$ and $\rho_2(N) = N_2$ are copure in M_1, M_2 respectively.

$\Leftarrow$ To prove N is copure, it is enough to show that $(N \underset{M}{:} I) = N + (0 \underset{M}{:} I)$. But it is easy to see that $(N \underset{M}{:} I) = (N_1 \underset{M_1}{:} I) \oplus (N_2 \underset{M_2}{:} I)$. On the other hand $(N_1 \underset{M_1}{:} I) = N_1 + (0 \underset{M_1}{:} I)$, $(N_2 \underset{M_2}{:} I) = N_2 + (0 \underset{M_2}{:} I)$, since N_1 and N_2 are copure in M_1 and M_2 respectively. Thus

$$(N \underset{M}{:} I) = (N_1 + (0 \underset{M_1}{:} I)) \oplus (N_2 + (0 \underset{M_2}{:} I))$$
$$= (N_1 \oplus N_2) + ((0 \underset{M_1}{:} I) + (0 \underset{M_2}{:} I))$$
$$= (N_1 \oplus N_2) + (0 \underset{M}{:} I)$$

Thus N is copure in M.

A similar proof for n > 2.

Recall that an R-module M is called a multiplication module if for any N ≤ M, N = IM for some ideal I of R.

Equivalently, M is multiplication if for any N ≤ M, N = (N:M)M [5].

Theorem 1.6:

Let M be a faithful finitely generated multiplication R-module, let N ≤ M. Then N is copure in M if and only if $(N \underset{R}{:} M)$ is a copure ideal of R. (where $(N \underset{R}{:} M)$ is a copure ideal if it is a copure R-submodule of R).

Proof:

(⇒) Let J be an ideal of R. So we must prove $((N \underset{R}{:} M) \underset{R}{:} J) = (N \underset{R}{:} M) + (0 \underset{R}{:} J)$.

Let $a \in ((N \underset{R}{:} M) \underset{R}{:} J)$. Hence aJM ⊆ N, so that aM ⊆ $(N \underset{M}{:} J)$. But N is copure in M, hence a M ⊆ N + $(0 \underset{M}{:} J)$. Since M is a multiplication R-module, we have:

$$aM \subseteq (N \underset{R}{:} M)M + ((0 \underset{R}{:} J) \underset{R}{:} M)M$$
$$= [(N \underset{R}{:} M) + ((0 \underset{R}{:} J) : M)]M$$

But M is faithful finitely generated multiplication, so $(a) \subseteq (N \underset{R}{:} M) + ((0 \underset{M}{:} J) \underset{R}{:} M)$ (see [6,Th.3.1]). Beside this, it is easy to see that $((0 \underset{M}{:} J) \underset{R}{:} M) = (0 \underset{R}{:} J)$ therefore $a \in (N \underset{R}{:} M) + (0 \underset{R}{:} J)$; hence $((N \underset{R}{:} M) \underset{R}{:} J) \subseteq (N \underset{R}{:} M) + (0 \underset{R}{:} J)$. Thus $(N \underset{R}{:} M)$ is a copure ideal of R.

(⇐) Let I be an ideal of R. To prove $(N \underset{M}{:} I) = N + (0 \underset{M}{:} I)$. Since $(N \underset{R}{:} M)$ is a copure ideal of R.

We have $((N \underset{R}{:} M) \underset{R}{:} I) \subseteq (N \underset{R}{:} M) + (0 \underset{R}{:} I)$ and hence

$$((N \underset{R}{:} M) \underset{R}{:} I)M \subseteq (N \underset{R}{:} M)M + (0 \underset{R}{:} I)M$$
$$= N + (0 \underset{R}{:} I)$$, since M is multiplication.

However we can show that $(N \underset{M}{:} I) \subseteq ((N \underset{R}{:} M) \underset{R}{:} I)M$. To see this:

let $x \in (N \underset{M}{:} I) \subseteq ((N \underset{M}{:} I) \underset{R}{:} M)M$, so that $x = \sum_{i=1}^{n} a_i M_i$ for some n ∈ Z_+, $a_i \in ((N \underset{M}{:} I) \underset{R}{:} M)$, m_i ∈ M. Hence $a_i M \subseteq (N \underset{M}{:} I)$, and so a_iIM ⊆ N for each i = 1, …, n, thus $a_i \in ((N \underset{R}{:} M) \underset{R}{:} I)$.

This implies $x \in ((N \underset{R}{:} M) \underset{R}{:} I)M$, that is $(N : I) \subseteq ((N \underset{M}{:} M) \underset{R}{:} I)M$. It follows that $(N \underset{M}{:} I) \subseteq N + (0 \underset{M}{:} I)$. Thus N is copure in M.

Corollary 1.7:

Let M be a faithful finitely generated multiplication R-module, let N ≤ M. Then the following statements are equivalent:

(1) N is a copure submodule in M.

(2) $(N \underset{R}{:} M)$ is a copure ideal of R.

(3) N = IM for some copur ideal in R.

Proof: (1) ⇔ (2) (follows by Th. 1.6).

(2) ⇒ (3) is clear.

(3) ⇒ (2) if N = IM and I is a copure ideal of R. Since M is multiplication, $N = (N \underset{R}{:} M)M$ and by [7,Th.3.1]. $I = (N \underset{R}{:} M)$. Thus $(N \underset{R}{:} M)$ is a copure in R.

Proposition 1.8:

Let M be a multiplication R-module with $\underset{R}{\text{ann}} M$ is a pure ideal in R. If N is a multiplication copure submodule of M, then N is a pure submodule of M.

Proof: It is clear that $M = (N \underset{M}{:} (N \underset{R}{:} M))$ and since N is copure in M, $M = N + (0 \underset{M}{:} (N \underset{R}{:} M))$.

Hence $(N : M)M = (N : M)N + (N \underset{R}{:} M)(0 \underset{M}{:} (N \underset{R}{:} M))$. It follows that $(N \underset{R}{:} M)M = (N \underset{R}{:} M)N$ and hence N = (N:M)N, since M is multiplication. Then by [8, Th.1.1 (1)⇔(2)], N is pure.

Recall that an R-module M is called completely distributive if for each L, K_λ(λ ∈ Λ), submodules of M $\bigcap_{\lambda \in \Lambda} (L + K_\lambda) = L + \bigcap_{\lambda \in \Lambda} K_\lambda$, [7].

The last result in this section is the following:

Proposition 1.9:

Let M be a completely distributive R-module and let N ≤ M. Then N is copure M if and only if $(N \underset{M}{:} I) = N + (0 \underset{M}{:} I)$ for each principal ideal I of R.

Proof: $\Rightarrow$ It is clear.

$\Leftarrow$ Let I ≤ R. Then $I = \sum_{a_i \in I} Ra_i$. If m ∈ $(N \underset{M}{:} I)$, then Im ⊆ N; hence (a_i)m ⊆N for each a_i ∈ I. Then m ∈ $(N \underset{M}{:} (a_i))$ for each a_i ∈ I and so by hypothesis, m ∈ N + $(0 \underset{M}{:} (a_i))$ for each a_i ∈ I. This implies m ∈ $\bigcap_{a_i \in I} (N + (0 \underset{M}{:} (a_i)))$. But M is completely distributive, so m ∈ N + $\cap (0 \underset{M}{:} (a_i))$. But $(0 \underset{M}{:} I) = \bigcap_{a_i \in I} (0 \underset{M}{:} (a_i))$. Thus m ∈ N + $(0 \underset{M}{:} I)$. Therefore N is copure in M.

S.2 Coregular Modules

In this section we study the concept of coregular modules (which is appeared in [5], under the name fully copure modules). Ansari and Farshadifar, in [7] gave relations between this concept and idempotent (coidempotent) modules.

However we give some basic results about this concept, also we give some relationships between coregular modules (rings) and regular modules (rings) and other related modules. Beside these we study the direct summand of coregular module and the direct sum of coregular modules.

Definition 2.1:

An R-module M is called coregular if every submodule of M is copure.

A ring R is coregular if every ideal of R is copure.

Remarks and Examples 2.2:

(1) By using Prop.1.4, it is clear that every semisimple module is coregular. For example the Z-module Z_6 is coregular.

(2) By using [3, Th.2.12(a)], a module M over a P.I.R R is coregular iff R is regular. In particular, each of the Z-module Z, Q, Z_4, Z_{p^∞} is not coregular (also not regular).

(3) It is well-known that every vector space over a field F is regular, so it is coregular.

Proposition 2.3:

Let M be a coregular module and let N ≤ M. Then M/N is a coregular module.

Proof: It follows directly by Cor.1.2.

Corollary 2.4:

If M, M′ are isomorphic R-modules, then M is coregular if and only if M′ is coregular.

Proof: Since M ≅ M′, there exists f :M ⟶ M′ an isomorphism. If M is coregular. Let W ≤ M′ then W = $f f^{-1}$(W), but f^{-1}(W) is copure submodule in M, hence by prop. 1.1, W is coprime in M′.

Similarly if M′ is coregular module then M is coregular module.

Corollary 2.5:

Let f : M ⟶ M′ be an epimorphism. If M is regular, then M′ is coregular.

Proof: By 1st fundamental theorem, M/ker$f$ ≅M′. But M/ker f is coregular by Prop.2.3. Hence M′ is coregular by Cor.2.4.

Corollary 2.6:

A direct summand of coregular module M is coregular.

Proof: Let N be a direct summand of M. Then M = N ⊕ K for some K ≤ N. Hence M/K ≡ N. But M/K is coregular by Prop.2.3. Hence N is coregular by Cor.2.4.

Proposition 2.7:

Every coregular ring R is regular.

Proof: Let a be any element of R. Then (a) is copure in R, hence $((a) \underset{R}{:} I) = (a) + (0 \underset{R}{:} I)$ for each ideal I of R. It follows that $((a) \underset{R}{:} (a)) = (a) + (0 \underset{R}{:} (a))$ and hence $R = (a) + (0 \underset{R}{:} (a))$. This implies 1 = r$a$ + b for some r ∈ R and b ∈ $(0 \underset{R}{:} (a))$, and so $a = ra^2 + ab$. But ab = 0. Hence $a = ra^2$. Thus R is regular.

We claim that regular ring may not be coregular, but we have no example to ensure this. However we have the following:

Proposition 2.8:

Every Noetherian regular ring is coregular.

Proof: Since R is Noetherian, every ideal I of R is finitely generated and since R is regular, I is a direct summand of R. Hence by Prop. 1.4, I is copure. Thus R is coregular.

It is well-known that an ideal I of a ring R is an annihilator ideal if I = $\mathrm{ann}_R \mathrm{ann}_R I$, i.e. $I = (0 \underset{R}{:} (0 \underset{R}{:} I))$.

Proposition 2.9:
Let R be a ring with every ideal is an annihilator ideal. Then R is coregular if and only if R is regular.
Proof: $\Rightarrow$ It follows by Prop.2.7.
$\Leftarrow$ Let I be an ideal of R. Then for any ideal J of R, it is easy to check that

$(I \underset{R}{:} J) = \underset{R}{\text{ann}}(\underset{R}{\text{ann}}\, I \underset{R}{:} J)$

$= \underset{R}{\text{ann}}(\underset{R}{\text{ann}}\, I \cap J)$ (since every ideal of R is pure)

$= \underset{R}{\text{ann}}\,\underset{R}{\text{ann}}\, I + \underset{R}{\text{ann}}\, J$ (since every ideal of R is an annihilator ideal)

$= I + (0 \underset{R}{:} I)$

Thus $(I \underset{R}{:} J) = I + (0 \underset{R}{:} J)$, so that I is copure in R. Hence R is coregular.

Proposition 2.10:
Every module M over coregular ring R is regular.
Proof: Since R is coregular, then by Prop.2.7, R is regular, hence R/ $\underset{R}{\text{ann}}(x)$ is a regular ring for each $x \in M$, that is M is regular.

Now, we will prove that in the class of completely distributive modules (rings) the two concepts regular and coregular modules (rings) are equivalent.
First we prove the following lemma:

Lemma 2.11:
Let R be a ring. Then R is regular if and only if for each ideal J of R,
$(J \underset{R}{:} (b)) = J + (0 \underset{R}{:} (b))$ for any $b \in R$.
Proof: $\Rightarrow$ Let $a \in (J \underset{R}{:} (b))$. Then $a\,(b) \subseteq J$. Since R is regular (b) is a direct summand of R and so $R = (b) \oplus K$ for some $K \leq R$. It follows that $1 = rb + k$ for some $r \in R$, $k \in K$, and so $a = rab + ak$. But $ab \in J$ and $(ak)\, b \in K \cap (b) = 0$; that $ak \in (0 \underset{R}{:} b)$. Then $a \in J + (0 \underset{R}{:} b)$; that $(J \underset{R}{:} (b)) \subseteq J + (0 \underset{R}{:} b)$ and reverse inclusion is clear. Hence we get the result.
$\Leftarrow$ By hypothesis, for any $a \in R$, $((a) \underset{R}{:} (a)) = (a) + (0 \underset{R}{:} (a))$.
Hence $R = (a) + (0:(a))$ and this implies R is regular.

Corollary 2.12:
Let R be a completely distributive ring. Then R is regular if and only if R is coregular.
Proof: $\Rightarrow$ It follows by Prop.2.7.
$\Leftarrow$ By Lemma 2.11 and Prop.1.9, the result follows.

Proposition 2.13:
Let M be a completely distributive R-module. Then M is coregular if M is regular.
Proof: Let $N \leq M$. To prove N is copure, it is enough to show that $(N \underset{M}{:} (a)) = N + (0 \underset{M}{:} (a))$ for each $a \in R$ (by Prop.1.9). Let $m \in (N \underset{M}{:} (a))$. Then $a\,m \in N$ and so that $a\,m \in (a)M \cap N$. But N is pure in M, so that $(a)M \cap N = (a)N$. Thus $a\,m \in (a)N$ and $a\,m = a\,n$ for some $n \in N$. Hence $a\,(m - n) = 0$; that is $m - n \in (0 \underset{M}{:} (a))$. It follows that $m = n + (m - n) \in N + (0 \underset{M}{:} (a))$. Thus N is copure.

Theorem 2.14:
Let M be a faithful finitely generated multiplication R-module. For the following statements:
(1) M is a coregular R-module.
(2) R is a coregular ring.
(3) M is a regular ring.
(4) R is a regular ring.
(1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4). Furthermore the statements (1) to (4) are equivalent if every ideal of R is an annihilator ideal.
Proof: (1) $\Rightarrow$ (2), Let I be an ideal of R. Hence N = IM is a copure submodule of M and by Th.1.6, $(N \underset{R}{:} M)$ is a copure ideal of R. But N = IM = $(N \underset{R}{:} M)M$, so that by [7, Th.3.1], I = (N:M), thus I is copure.
(2) $\Rightarrow$ (1) Let $N \leq M$. Since R is coregular, (N:M) is a copure ideal of R. Hence by Th.1.6, N is copure. Thus M is coregular.
(2) $\Rightarrow$ (3) It follows by Prop. 2.10.
(3) $\Rightarrow$ (4) Since M is finitely generated and regular, then R/ann M $\cong$ R is regular by [9, Th.1.10].
(4) $\Rightarrow$ (3) It is clear.
Furthermore (2) $\Leftrightarrow$ (4) if every ideal is an annihilator ideal by Prop.2.9.

Recall that an R-module M is called strongly comultiplication if for every submodule N of M, there exists an ideal I of R such that $N = (0 :_M I)$ and for each ideal J of R, $J = \text{ann}_R(0 :_M J)$ (i.e. $J = (0 :_R (0 :_M J))$, [5].

Proposition 2.15:

Let M be a strongly comultiplication R-module. For the following statements:

(1) M is a coregular R-module.

(2) R is a regular ring.

(3) M is a regular R-module.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) if M is finitely generated faithful R-module.

Proof: (1) $\Rightarrow$ (2) Let I ≤ R. Since M is strongly comultiplication R-module, $I = \text{ann}_R(0 :_M I)$. But M is coregular, so $(0 :_M I)$ is copure in M. Hence by [5, Th.2.13 (2)], $\text{ann}_R(0 :_M I)$ is a pure ideal of R. Thus I is pure and R is a regular ring.

(2) $\Rightarrow$ (1) Let N ≤ M. Then $\text{ann}_R N$ is an ideal of R, so it is pure, since R is regular. Hence by [10, Th.2.13 (1)] N is a copure submodule of M. Thus M is coregular.

(2) $\Rightarrow$ (3) It is clear.

Now, if M is finitely generated. To prove (3) $\Rightarrow$ (2):

Since M is finitely generated regular R-module, then $R / \text{ann}_R M$ (which is isomorphic to R) is regular, by [11, Th.1.10].

Next we turn our attention to direct sum of coregular modules.

Proposition 2.16:

Let M = $M_1 \oplus M_2$ where M_1 and M_2 are R-modules. If M is coregular; then M_1 and M_2 are coregular and the converse is true if $\text{ann}_R M_1 + \text{ann}_R M_2 = R$.

Proof: Let $\rho_1: M_1 \oplus M_2 \longrightarrow M_1$, $\rho_2: M_1 \oplus M_2 \longrightarrow M_2$, ρ_1 and ρ_2 are epimorphism. Hence by Cor.2.5, M_1 and M_2 are coregular.

For the converse. Let N ≤ M. Since $\text{ann}_R M_1 + \text{ann}_R M_2 = R$, then by the proof of Prop.4.2 in [5], N = $N_1 \oplus N_2$ for some $N_1 \leq M_1$, $N_2 \leq M_2$. But M_1 and M_2 are coregular, so N_1 copure in M_1, N_2 copure in M_2 and hence by Prop.1.5, N is copure in M. Thus M is is coregular.

To give our next result, we need to recall that:

A submodule N of an R-module is fully invariant if for each $f \in \text{End}(M)$, $f(N) \subseteq N$.

Lemma 2.17:

Let $M = \oplus_{i \in I} M_i$, with $M_i \leq M$, for each i ∈ I and let N be a fully invariant submodule of M. Then $N = \oplus_{i \in I}(N \cap M_i)$, [5].

Theorem 2.18:

Let $M = \oplus_{i=1}^{n} M_i$ with $M_i \leq M$, for each i = 1, n. If M is coregular, then M_i is coregular, for each i = 1, , n. The converse is true if each submodule of M is fully invariant.

Proof: If M is coregular, then M_i is coregular for each i = 1, …, n by Cor.2.6.

The converse, let N ≤ M. Hence by lemma 2.17, $N = \oplus_{i=1}^{n}(N \cap M_i)$. But for each i = 1, …, n, M_i is coregular, hence N ∩ M_i is copure in M_i for each i = 1, …, n. Then by Prop.1.5, N is copure in M. Thus M is coregular.

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