DOI: 10.24996/ijs.2019.60.12.21

ISSN: 0067-2904

On Lie Structure in Semiprime Inverse Semirings

Rawnaq KH. Ibraheem*, Abdulrahman H. Majeed
Math Department, College of Science, University of Baghdad, , Baghdad, Iraq

Received: 4/5/ 2019 Accepted: 17/ 7/2019

Abstract
In this paper we introduce the definition of Lie ideal on inverse semiring and we generalize some results of Herstein about Lie structure of an associative rings to inverse semirings .

Keywords: Additively inverse Semiring , Lie ideal of an Inverse Semiring, Semiprime Inverse Semirings .

حول بنية لي في أشباه الحلقات المعكوسة شبه الاولية

رونق خليل أبراهيم *، عبد الرحمن حميد مجيد
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق

الخلاصة
قدمنا في هذا البحث مفهوم بنية لي في اشباه الحلقات المعكوسة وعممنا بعض من نتائج الباحث Herstein حول بنية لي في الحلقات التجميعية الى أشباه الحلقات المعكوسة .

1. Introduction

A non-empty set S with two binary operations $+$ and $\cdot$ is said to be a semiring if $(S, +)$ semigroup, $(S, \cdot)$ semigroup and $x. (y + z) = x. y + x. z$ and $(y + z) .x = y. x + z. x$ holds for all $x, y, z \in S$ [1] . A semiring $(S, + ,\cdot)$,with commutative addition and an absorbing zero 0, is called an inverse semiring if for every element $a \in S$ there exists a unique element $a' \in S$ such that $a+a'+a = a$ and $a'+a+a' = a'$ introduced by Bandlet and Petrich [2] and by Karvellas [3]. In previous work [3 , 4], where $(S,+,\cdot)$ be an inverse semiring then the following are valid : $(x')' = x$, $(x+y)'=x'+y'$, $(xy)' = x'y = xy'$, $x'y'= xy$. In inverse semirng, a commutator $[. , .]$ is defined as $[x, y]= xy+yx' = xy+y'x$ and commutator identities are used as $[xy,z]=x[y,z]+[x,z]y$ and $[x,yz]=[x,y]z+y[x,z]$ [5,10]. By Mary *et al.*, [10], if U is a nonempty subset of S, U is called left ideal of S if $x + y \in S$ for all $x, y \in I$, $r. x \in S$ for all $x \in U, r \in S$ and $U\neq S$ (Similarly right ideal). An additive mapping d from S to itself is defined as a derivation if $d(ab)=d(a)b +a\, d(b)$, for all $a,b \in S$. S is prime if whenever $a S b=0$, then $a =0$ or $b =0$, and semiprime whenever $a S a =0$, implies that $a = 0$, or S has no non-zero nilpotetent ideal. We called S as n-tortion free if $na =0$, $a \in S$ implies that $a = 0$. In this paper we will represent S as an inverse semiring with $a+a' \in Z$ (the center of S), and U is a Lie ideal of S, the two subsets U,V of S are however given, then $[V,U]$ will be the additive inverse subgroup of S generated by the all element $vu+uv'$ where $v \in V, u \in U$, and $[S,S]$ be the additive inverse subgroup generated by the all element $ab+ba'$ for all arbitrary element $a,b \in S$.

In 1970, Herstein extended more of general situation results of Lie ideal on associative rings [7]. Our objective in this paper is to introduce the concept of Lie ideal on inverse semirings and to generalize these Herstein's results on inverse semirings .

*Email:rawnaq.kalel@gmail.com

2. Priliminaries

We need some definitions and lemmas in our arguments.

Lemma (2.1) [6]:

Let S be an inverse semiring , for all $a, b \in S$ if $a + b = 0$ then $a = b'$.

Definition (2.2) [8, 9]:

Let *S* be a semiring, let $x, y \in S$, the set
Z(*S*) ={$x \in S,\ xy = yx$, for all $y \in S$ } is called the center of the semiring *S*.

Definition (2,3) [4] :

An additive map $d: S \to S$ where *S* is a semiring, is called a derivation if :

$$d(xy) = d(x)y + xd(y) \qquad \text{for all } x, y \in S$$

Definition (2.4) [10]

Let S be an inverse semiring, let $a \in S$ be a fixed element. A map $d: S \to S$ defined by $d(x) = [a, x]$ for all $x \in S$, is said to be an inner derivation.

Actually *d* is a derivation, since *d* is an additive map and

$$d(xy) = [a, xy] = x[a, y] + [a, x]y = xd(y) + d(x)y, \qquad \text{for all } x,y \in S$$

3. Results

On Lie Structure in Semiprime Inverse Semirings

Lemma (3.1):

Let S be a 2-torsion, free semiprime, inverse semiring , let $a, x \in S$ if a commutes with all commutaters in this form [*x*, *a*] for all $x \in S$, then $a \in Z(S)$.

Proof

For all $x \in S$, define $d: S \to S$ *by*

$$d(x) = [x, a].$$

So, d is defined as a derivation on S that is $d(x + y) = [x + y, a]$ and by commutator identities, we get $[x + y, a] = [x, a] + [y, a] = d(x) + d(y)$.

And $d(xy) = [xy, a]$, again by commutator identities, we get

$[xy, a] = x[y, a] + [x, a]y$
$\quad = xd(y) + d(x)y$, then *d* is a derivation.

Since *a* commutes with all own commutators, we have $a[a, x] + [a, x]a' = 0$

(i.e) $ad(x) + d(x)a' = 0$, and by definition of d we have $d\big(d(x)\big) = 0$, then $d^2(x) = 0$, for all $x \in S$

Now

$$0 = d^2(xy) = d(x)y + xd(y))$$
$$= d^2(x)y + d(x)dy + d(x)d(y) + xd^2(y)$$

Then $2d(x)d(y) = 0$, Since S is 2- torsion free implies that $d(x)d(y) = 0$ for all $x, y \in S$, Now replace y by rx where $r \in S$ then $0 = d(x)d(rx) = d(x)d(r)x + d(x)rd(x) = 0$, then we conclude $d(x)rd(x) = 0$

$$d(x)S\,d(x) = 0$$

And by semiprimeness we get $d(x) = 0$ for all $x \in S$, then $[a, x] = 0$ for all $x \in S$

$$(i.e)\ ax + xa' = 0$$

Then $ax = xa$ thus $a \in Z(S)$.

Lemma (3.2)

Let *S* be an inverse semiring , and let $d: S \to S$ is a derivation , and let

$$M = \{x \in S |\ xd(s) = 0 \text{ and } d(s)x = 0 \text{ for all } s \in S\}$$

Then M is an ideal in *S*.

Proof:

Let $x, y \in M$ then $xd(s) = 0$, $yd(s) = 0$ for all $s \in S$

$$(x + y)d(s) = xd(s) + yd(s) = 0$$

Then $(x + y) \in M$

Now let $x \in M$ and $r \in S$ we get

$xd(s) = 0$ for all $s \in S$ *so* $rxd(s) = 0$, then $rx \in M$

Therefore M is left ideal.

Suppose that $r , s \in S$ and $x \in M$ we have $xd(rs) = 0$

Since *d* is a derivation ,then $xd(r)s + xrd(s) = 0$

$xrd(s) = 0$ so $xr \in M$
Then M is a right ideal, therefore M is an ideal .

Lemma (3.3)

Let S be an inverse semiring, and let
$M = \{x \in S |\ xd(s) = 0,\ d(s)x = 0$ for all $s \in S\}$, then $\bar{S} = S/M$ is an inverse semiring with respect to $\oplus$ and $\odot$ defined as
$(x + M) \oplus (y + M) = (x + y) + M$ and
$(x + M) \odot (y + M) = (x . y) + M$

Proof

We have to show that ($\bar{S}$,$\oplus$, $\odot$) is an inverse semiring:
Let $x + M,\ y + M,\ s + M \in \bar{S}$ such that
$(x + M) \oplus (y + M) = (x + y) + M$, and
$(x + \text{M}) \odot (y + M) = (x . y) + M$
We have to show that (S/M , $\oplus$) is a semigroup:
$((x + M) \oplus (y + M)) \oplus (s + M) = ((x + y) + M)) \oplus (s + M) = (x + y + s) + \text{M} =$
$(x + M) \oplus (y + s) + \text{M} = (x + M) \oplus ((y + M) \oplus (s + M))$.
We have to show that (S/M , $\odot$) is a semigroup:
$((x + M) \odot (y + M)) \odot (s + M) = ((x . y) + M) \odot (s + M) = (\text{x . y . s}) + \text{M} =$
$(\text{x} + \text{M}) \odot ((\text{y . s}) + \text{M}) = (\text{x} + \text{M}) \odot (\text{y} + \text{M}) \odot (s + M)$.
Now we have to show the distributive:
$(\text{x} + \text{M}) \odot ((\text{y} + \text{M}) \oplus (\text{s} + \text{M})) = (\text{x} + \text{M}) \odot (\text{y} + \text{M}) \oplus (\text{x} + \text{M}) \odot (\text{s} + \text{M}) =$
$(\text{x . y}) + \text{M} \oplus (\text{x . s}) + \text{M}$.
Let $x \in S$, since S is an inverse semi ring, then $\exists!\ x' \in S$, such that $x = x + x' + x$, and
$x' = x' + x + x'$.
So,
For all $x + M \in S/M$, there exists $x' + M \in S/M$, such that
$(x + M) \oplus (x' + M) \oplus (x + M) = (x + x' + x) + M = x + M$.
And $(x' + M) \oplus (x + M) \oplus (x' + M) = (x' + x + x') + M = x' + M$
To prove the uniqueness, suppose there exists y+$M \in S/M$, such that
$(x + M) \oplus (y + M) \oplus (x + M) = (x + y + x) + M = x + M$. But $x \in S$
Since x' is unique in S then $x' = y$
Therefore, $(x'$+$M)$ =$(y$+$M)$.
Thus S/M is an inverse semiring .

Lemma (3.4)

Let S be a 2-torsion free inverse semiring , $d: S \to S$, is a derivation. Let $M = \{x \in S |\ xd(s) = 0,\ d(s)x = 0$ for all $s \in S\}$, then $\bar{S} = S/M$ is 2-torion free inverse semiring.

Proof

For all $x \in S,\quad x + M \in \bar{S}$
Let $2(x + M) = M$
$2x + M = M$ that yields $2x \in M$
Thus $2xd(s) = 0$ for all $s \in S$
And since S is 2-torsion free, we get $xd(s) = 0\ so\ x \in M$ and $x + M = M$.
Therefore ($S\ /\ M$) is 2-torsion free inverse semiring.

Lemma (3.5)

Let S be a semiprime inverse semiring, $d: S \to S$ is a derivation. Let $M = \{x \in S |\ xd(s) = 0,$ $d(s)x = 0$ for all $s \in S\}$. Then $\bar{S} = S/M$ is a semiprime inverse semiring.

Proof

Let $\bar{N}$ be an ideal of $\bar{S}$ with $\overline{N^2}$=0, we want to show that $\bar{S} = S/M$ has no non-zero nilpotent ideal.
Let N be the inverse image of $\bar{N}$ in S.
N is an ideal in S and $N^2 \subset M$.
Thus $d(y)N^2 = 0$ for all $y \in S$.
Then $(d(y)N)^2$=$d(y)Nd(y)N \subset d(y)N^2 = 0$
Since $d(y)N$ is a nilpotent right ideal in S, which is a semiprime, then we get $d(y)N = 0$ hence $d(y)N \in M$.

Thus $d(y)N \subset M$, implies that $\bar{N} = 0$
Then $\bar{S}$ has no non-zero nilpotent ideal yields, therefore $\bar{S}$ is a semiprime inverse semiring.

Definition(3.6)

Let *A* be a Lie- inverse sub semiring of an inverse semiring *S*, the additive inverse sub semigroup , $U \subset A$ is said to be a lie ideal of *A* if whenever $u \in U$, $a \in A$, then $[u, a] = ua + au'$, is in U .

Lemma (3.7):

If *S* is a semiprime inverse semiring, then *Z(S)* has no non-zero nilpotent element.

Proof

Let a be a nil potent element such that $a \neq 0,\ a \in Z(S)\ and\ a^n = 0,\ a^n S = 0,\ (i.e)\ a^{n/2}a^{n/2}S = 0$, Since $a \in Z(S)$then $a^{n/2}S\, a^{n/2} = 0$.
And since S is semiprime then $a^{n/2} = 0$, contradiction.
Therefore a=0, thus *Z*(*S*) has no non-zero nilpotent element.

Lemma (3.8):

Let *S* be a 2- torsion free semiprime, inverse semiring, and let *T* be a lie ideal of *S*. Suppose that $[T, T] \subset Z$; then $T \subset Z$.

Proof:

If $[T, T] = 0$ then given $t \in T, tx + xt' \in T$ for any $x \in S$ so commutes with t.
(i.e) $[t, tx + xt'] = 0$ then by **lemma (3.1)** $t \in Z$
Suppose then that we can find $s, t \in T$

$$\text{so that } \lambda = st + ts' \neq 0 \text{ then } \lambda \in Z \text{ for } x \in S$$
$$\text{let } d(x) = xt + tx'; \text{thus } d(s) = st + ts' = \lambda$$
$$d^2(x) = d\big(d(x)\big) = d(\lambda) = \lambda\, t + t\, \lambda' \in Z$$
$$\text{let } \beta = d^2(x); \text{then } d^2(sx) \in Z$$
$$d^2(sx) = d^2(s)x + 2d(s)d(x) + sd^2(x)$$
$$= 2d(s)d(x) + s\beta \text{ since } d^2(s) = 0$$

then $2\lambda d(x) + \beta s \in Z$, so

$$0 = [s, 2\lambda d(x) + \beta s] = [s, 2\lambda d(x)] + [s, \beta s] = 2\lambda[s, d(x)] + [s, 2\lambda]d(x) + [s, \beta s]$$
$$= 2\lambda(sd(x) + d(x)s') = 0, \qquad \text{put } x = st$$
$$d(st) = sd(t) + d(s)t = s(tt + tt') + (st + ts')t = stt + stt' + stt + ts't$$
$$= st(t + t' + t) + ts't = stt + ts't = (st + ts')t = \lambda t$$
$$\text{then } 2\lambda(s\lambda t + \lambda ts') = 2\lambda^2(st + ts') = 2\lambda^3 = 0$$

Since *S* is 2-torsion free then $\lambda^3 = 0$
Since *S* is a semiprime, by **Lemma (3.7)** the center of *S* has no non-zero nilpotent element then we get contradiction, so $\lambda = 0$ thus $st + ts' = 0$ and by **Lemma (2.1)** we have

$$st = ts \text{ for all} t \in T, s \in S .$$

Therefore

$$T \subset Z$$

Lemma (3.9)

Let *S* be a 2- torsion free semiprime inverse semiring and let *U* be a lie ideal of *S*. Suppose that $t \in S$ commutes with every element of [*U*,*U*] then *t* commutes with every element of U .

Proof:

For $x \in S$, let $d(x) = xt + tx'$
Since t commutes with every element of $[U, U]$, then by **Lemma (3.1)** for every $u \in [U, U]$, then $d(u) = 0$
Now$[U, U]$, is a lie ideal of S ,
For $r \in S, ur + ru' \in [U, U]$, where $u \in [U, U]$
Thus $d(ur + ru') = 0$, But $d(ur + ru') = ud(r) + d(u)r + r'd(u) + d(r)u'$

Then $ud(r) + d(r)u' = 0$ ………………….. (1)

Then by **Lemma (2.1)** we get

$ud(r) = d(r)u$, for $u \in [U, U], x \in S$ …………………... (2)

In (2) put $r = x^2$, so $ud(x^2) = d(x^2)u$
Again put $r = x^2$ in (1), we obtain

$$ud(x^2) + d(x^2)u' = 0$$

$$u(d(x)x + xd(x) + \big(d(x)x + xd(x)\big)u' = 0$$
$$ud(x)x + uxd(x) + d(x)xu' + xd(x)u' = 0$$
By using $ud(x) = d(x)u$, we get
$$d(x)ux + uxd(x) + d(x)xu' + xu'd(x) = 0$$
$$then\ d(x)(ux + xu') + (ux + xu')d(x) = 0\ ,$$
Since$(ux + xu') \in [U, U]$, So it commutes with $d(x)$.
$$d(x)(ux + xu') + d(x)(ux + xu') = 0, \text{Then } 2d(x)(ux + xu') = 0$$
And since S is 2- torsion free implies that
$$d(x)(ux + xu') = 0\ for\ u \in [U, U] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$
By linearizing on $x\ in\ (3) where\ v \in [U, U]$,
$$d(x + v)(u(x + v) + (x + v)u' = 0$$
$$d(x)ux + d(x)uv + d(x)xu' + d(x)vu' + d(v)ux + d(v)uv + d(v)xu' + d(v)vu' = 0$$
Since $d(v) = 0\ then\ d(x)ux + d(x)uv + d(x)xu' + d(x)vu' = 0$
$$d(x)(ux + uv + xu' + vu') = 0$$
$$d(x)(ux + xu') + d(x)(uv + vu') = 0$$
then $d(x)(uv + vu') = 0$, for $x \in S, u, v \in [U, U]$ ……………. (4)
Let $M = \{r \in S | d(x)r = 0 \quad \text{for all } x \in S\}$
By **lemma (3.2)** M is an ideal of S .
By (4) we have $\big[[U, U], [U, U]\big] \subset M$
Let $\bar{S} = S/M$, By **Lemma (3.3), Lemma (3.4) , Lemma (3.5)** $\bar{S}$ is 2-torsion free semiprime, inverse semiring.
In $\bar{S}$, $\bar{U}$ is a lie ideal , since $\big[[U, U], [U, U]\big] \subset M$,
$$\text{Then } \big[[U, U], [U, U]\big] = 0\ ,$$
By **Lemma (3.8)** we get $[\bar{U}, \bar{U}] \subset \bar{Z}$, the center of $\bar{S}$, hence by **Lemma (3.8)** again $\bar{U} \subset \bar{Z}$. Thus $[\bar{U}, \bar{S}] = 0$ and $so\ [U, S] \subset M$.
Thus, $d(y)[U, S] = 0$ for all $y \in S$.
Let $M_1 = \{x \in S | \ x[U, S] = 0\ \}$ as above we easily get that M_1 is an ideal of S .
If $u \in U$, for all $y \in S$, since $d(y)[U, S] = 0$
$(i.e)$ $d(y)(ux + xu') = 0$, then $(ux + xu') \in M_1$.
But $d(u) \in [U, S]$
, thus $d(u) \in M_1$ and $d(u)$ is in a right annihilator (rM_1) .
But $M_1 \cap rM_1$ is a nilpotent ideal of S, Hence is (0)
Since $(u) \in M_1 \cap rM_1$, then $d(u) = 0$ for all $u \in U$
$$0 = d(u) = ut + tu' \quad , u \in U, t \in S$$
Then by **Lemma (2.1)** we have $ut = tu$
So t does commute with all element of U.

Definition (3.10)

If A be a subset of S, $C(A) = \{x \in S |\ xa = ax\ for\ all\ a \in A\}$
Then $C(A)$ is said to be the centralizer of A; Clearly that A is an inverse sub semiring of S .

Theorem (3.11)

Let S be 2- torsion free semiprime, inverse semiring, and let U be a lie ideal of S. If $t \in S$ commutes with all $tu + ut'$, $u \in U$ then it commutes with every element of U .

Proof

Let $x \in S$, $d(x) = xt + tx' = [x, t]$, d is a derivation
Since t commutes with all $tu + ut'$, $u \in U$
$$t(tu + ut') + (tu + ut')t' = 0, (i.e) td(u) + d(u)t' = 0$$
$$d\big(d(u)\big) + d^2(u) = 0$$
If $u, v \in U\ then\ d^2(u) = d^2(v) = 0$, since U is a Lie ideal, then $uv + vu' \in U$
Thus $0 = d^2(uv + vu') = d(d(uv + vu') = d(d(u)v + ud(v) + d(v)u' + v'd(u)$
Since $d^2(u) = d^2(v) = 0$, then $0 = 2(d(u)d(v) + d(v)d(u)')$, Since S is 2-torsion free .
Then $d(u)d(v) + d(v)d(u)' = 0$ for all $u, v \in U$ …………… (1)
Now suppose that $u, v \in U$ are such that $uv \in U$.
Hence,$0 = d^2(uv) = d^2(u)v + 2d(u)d(v) + ud^2(v)$

resulting in $2d(u)d(v) = 0$
Since S is 2-torsion free, then $d(u)d(v) = 0$
For any $r \in S,\ u \in U$ the element $v = ur + ru'$, satisfies the criterion $v, uv \in U$
Hence by above

$d(u)d(ur + ru') = 0$ for all $u \in U, r \in S$ …………… (2)

Let $C(t) = \{x \in S |\ xt = tx\}$
Since $d^2(u) = 0$, Then by (2)
We have $d^2(u)(ur + ru') + d(u)d(ur + ru') = 0$,
$d\big(d(u)(ur + ru')\big) = 0$,

hence, $d(u)(ur + ru') \in C(t)$ for $u \in U, r \in S$ ………… (3)

In (3) write $r = tw,\ w \in S$
Then $ur + ru' = utw + twu' = utw + t(w + w' + w)u' = utw + t(w + w')u' + twu' = utw + t(w + w')u' + twu' = utw + tu'(w + w') + twu' = utw + tu'w + tuw + twu' = (ut + tu')w + t(uw + wu' = d(u)w + t(uw + wu')$, Since $d(u)(ur + ru') \in C(t)$,
By (3), $d(u)(d(v)w + t(uw + wu') \in C(t)$,
$d(u)^2w + d(u)t(uw + wu') \in C(t)$.
But $d(u)t = td(u)$, by hypothesis so $d(u)t(uw + wu') = td(u)(uw + wu')$,
Since $t \in C(t)$.
As the same as $d(u)(uw + wu') \in C(t)\ by\ (3)$
The conclusion of this is that $d(u)^2w \in C(t)$ for all $w \in S$.
Hence, $0 = td(u)^2w + d(u)^2wwt' = d(u)^2(tw + wt')$, since $d(u)^2 \in C(t)$.
In this we can replace w by wu, we get :

$$d(u)^2\{twu + wut'\} = d(u)^2\{(t + t' + t)wu + wut'\} = d(u)^2\{twu + wtu + wt'u + wut'\}$$
$$= d(u)^2\{(tw + wt')u + w(tu + ut')\} = 0$$

Thus $d(u)^2(tw + wt')u + d(u)^2w(tu + ut') = 0$
And so $d(u)^2wd(u) = 0$
Hence $d(u)^2wd(u)^2 = 0$, for all $w \in S$, then we obtain :

$d(u)^2Sd(u)^2 = 0$, by semiprimeness of S we have:

$d(u)^2 = 0$ for all $u \in U$ ……………… (4)

By linearizing on u in (4) where $u,v \in U$

$$d(u + v)^2 = 0$$
$$0 = d(u + v)^2 = \big(d(u) + d(v)\big)^2 = d(u)^2 + d(u)d(v) + d(v)d(u) + d(v)^2$$

Since $d(u)d(v) = d(v)d(u)$, Then $2d(u)d(v) = 0$
Since S is 2-torsion free, then:

$d(u)d(v) = 0$ for all $u, v \in U$ …………………….. (5)

We will show that $ud(v)$ and $d(u)v$ are in $C(t)$ for all $u, v \in U$
By (5) $0 = d(u)d(v) = d(u)(vt + tv') =$
$d(u)vt + d(u)tv' = 0$, and by **Lemma (2.1)** we have $d(u)vt = d(u)tv$
Then $d(u)vt = td(u)v$, since $t \in C(t)$.
Thus $d(u)v \in C(t)$ for all $u, v \in U$
And 0= $d(u)d(v) = (ut + tu')d(v)$
So, $utd(v) + tu'd(v) = 0$, and by **Lemma (2.1)** we have $utd(v) = tud(v)$
Then $ud(v)t = tud(v)$, since $t \in C(t)$.
Thus $ud(v) \in C(t)$ for all $u, v \in U$.
Let $u = rd(v)w + wrd(v)$ where $v, u \in U, r \in S$
Since U is a lie ideal of S, $u \in U$
For any $z \in U, ud(z) \in C(t)$
Then $d(v)d(z) = 0$ we see that $rd(v)wd(z) \in C(t)$, so
$Sd(v)wd(z) \subset C(t), for\ all\ u, v, z \in U$,
Hence $d\big(Sd(v)wd(z)\big) = 0$
Now $0 = d\big(Sd(v)wd(z)\big) =$

$$d(S)d(v)wd(z) + Sd^2(v)wd(z) + Sd(v)d\big(wd(z)\big) =$$

$$d(S)d(v)wd(z) + Sd^2(v)wd(z) + Sd(v)d(w)d(z) + Sd(v)wd^2(z) = d(S)d(v)wd(z) = 0$$

If $x, y \in S$, then
$0 = d(xy)d(v)wd(z) = d(x)yd(v)wd(z) + xd(y)d(v)wd(z);$
In particular $d(x)yd(v)wd(z) = 0$.
Let $x = z, then\ d(z)Sd(v)wd(z) = 0,$
Multiply by $d(v)d(w)$ from left we obtain $d(v)wd(z)Sd(v)wd(z) = 0$
By semiprimeness of S we have $d(v)wd(z) = 0$
And so, $d(v)Ud(z) = 0 \quad for\ all\ \ v, z \in U.$ ………… (6)
In particular, for $v \in U\ \ , s \in S\ \ d(v)Ud(vs + sv') = 0$
Now, by (5)
$d(u)d(v) = 0$, since $(vr + rv') \in U\ then\ d(u)(vr + rv') = 0$
Thus $d(u)\big((vr + rv')t + t(vr + rv')\big) = 0,$
Then $d(u)(vr + rv')t + d(u)t(vr + rv')' = 0,\ since\ t \in C(t),$
Then $d(u)(vr + rv')t + td(u)(vr + rv')' = 0$,
$d\big(d(u)(vr + rv')\big) = 0, thus\ d(u)(vr + rv') \in C(t)$, for all $u, v \in U, r \in S$
Replacing *r* by *rs* in above to obtain:

$$d(u)(vrs + rsv') \qquad \in C(t)$$

$$\begin{aligned} d(u)(vrs + rsv') &= d(u)(vrs + r(s + s' + s)v') = d(u)(vrs + rsv' + r(s' + s)v') \\ &= d(u)\big(vrs + rsv' + rv'(s' + s) = d(u)(vrs + rsv' + +rvs + rv's) \\ &= d(u)(vr + rv')s + d(u)r(vs + sv')\big) \qquad \in C(t) \end{aligned}$$

By applying *d* to this

$$\begin{aligned} d\big(d(u)(vr + rv')s + d(u)r(vs + sv')\big) \\ &= d^2(u)(vr + rv')s + d(u)d(vr + rv')s) + d^2(u)r(vs + sv') \\ &\quad + d(u)d\big(r(rs + sv')\big) \\ &= d(u)d(vr + rv')s + d(u)(vr + rv')d(s) + d(u)d(r)(vs + sv') + d(u)rd(vs \\ &\quad + sv') = 0 \end{aligned}$$

Using $d(u)(vr + rv') = 0, d^2(u) = 0, that$

$$0 = d(u)(vr + rv')d(s) + d(u)d(r)(vs + sv') + d(u)rd(vs + sv)$$

In particular if $r \in U$ by using (5) and (6) this give us
$d(u)(vr + rv')d(s) = 0\ that\ is\ d(u)[U, U]d(S) = 0$ …………… (7)
If $x, y \in S, d(u)[U, U]d(xy) = 0$ then $d(u)[U, U]\big(d(x)y + xd(y)\big) = 0$

$$d(u)[U, U]d(x)y + d(u)[U, U]xd(y) = 0 \text{ that is } d(u)[U, U]xd(y) = 0$$

Thus $d(u)[U, U]Sd(S) = 0$, since $Sd(S) \subset S$,
And by multipling by $d(u)[U, U]$ from right side we get
$d(u)[U, U]Sd(u)[U, U] = 0$, then by semiprimeness of inverse semiring we obtain $d(u)[U, U] = 0$
Let $M = \{x \in S |\ x[U, U] = 0\}; M$ is an ideal of S
for $u \in U, \qquad d(U) \subset M.$
Hence $d([U, U]) \subset M.$ But $d([U, U] \subset [U, U]$.
Thus, $d([U, U])$ is in M and its right annihilator, denoted by (M_r)
This $d([U, U]) \in M \cap M_r$, this intersection is a nilpotent ideal, hence $d([U, U]) = 0$, because S is semiprime . In other words, since *t* commutes with all elements of $[U, U]$.
By **Lemma (3.9)** , we get that *t* commutes with all elements of U .

References

1. Petrich, M. **1973**. *Introdiction to semiring*, Charles E Merrill Publishing Company, Ohio.
2. Bandlet, H. J. and Petrich, M. **1982**. *S*ubdirect products of rings and distributive lattics. *Proc. Edin Math. Soc*. **25**: 135-171.
3. Karvaellas, P.H. **1974**. Inverse semirings, *J.Austral. Math Soc.* **18**: 277-288.
4. Chowdhury, K., Sultana, R, A., Mitra, N. K. and Khodadad Khan, A. F. M . **2014**. Some structural properties of Semirings, *Annals of pure and Applied Mathematics*, **5**(2): 158-167.
5. Rawnaq, KH. and Abdulrahman H. Majeed **2019**. *U-R Jordan homomorphisim* of inverse semirings. Iraq Jornal Of Science, **60**(8).

6. Javed M.A., Aslam, M. and Hussain, M. **2012**. On condition $(A_2)$of Bandlet and Petrich for inverse semirings. *Int.Math. Forum*, **7**(59): 2903-2914.

7. Herstein,I. N. **1970**. On the Lie structure of an Associative Rings. *Jornal of Algebra*, **14**: 561-571.

8. Shafiq S. and Aslam, M. and JAVED, M.A. **2016**. Centralizer of Inverse Semiring. *Discussiones Mathematicae General Algebra and Applications*, **36**: 71-84.

9. Mary, D. Murugensan, R. and Namasivayam, P. **2016**. Centralizers on Semiprime Semirings, *IOSR Jornal of Mathematics*, **12**(3): 86-93.

10. Mary, D. Murugensan, R. and Namasivayam, P. **2017**. Some Relations Related to Centralizers on Semiprime Semirings. *Annals of pure and Applied Mathematics*, **37**(1): 119-124.