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Comments on Copula Functions and Their Relationship to Probability Density Functions

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Abstract
Copulas are very efficient functions in the field of statistics and specially in statistical inference. They are fundamental tools in the study of dependence structures and deriving their properties. These reasons motivated us to examine and show various types of copula functions and their families. Also, we separately explain each method that is used to construct each copula in detail with different examples. There are various outcomes that show the copulas and their densities with respect to the joint distribution functions. The aim is to make copulas available to new researchers and readers who are interested in the modern phenomenon of statistical inferences.

Keywords: Copulas, Distribution Functions, Constructing Methods of Copula, Copula Densities, Dependences.

تعليقات حول دوال الكوبلة وعلاقاتها بدوال الكثافة الاحتمالية

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الخلاصة
الكوبله هي وظائف فعالة جدا في المجال الإحصائي. فهي أداة أساسية في دراسة هياكل الاعتماد، واستنباط خصائصها. كانت هذه الأسباب هي الدافع لفحص وبناء أنواع مختلفة من وظائف الكوبله وعائلاتهم. علاوة على ذلك ، نوضح بشكل منفصل كل طريقة يتم استخدامها لإنشاء كل نسخة مفصلة بأمثلة مختلفة. هناك العديد من النتائج التي تبين دوال الكوبله و دوال كثافتها بالنسبة الى دوال التوزيع. والهدف من ذلك هو جعل الكوبله في يد الباحثين والقراء الجدد الذين يرغبون في ظاهرة حديثة من الاستدلالات الإحصائية

1. Introduction

A general form of copula $(C(p,q))$ corresponds to the joint distribution function $(H(s,t))$. In general, copula function was firstly presented to describe the dependence structure between bivariate or multivariate random variables. They are useful in the analysis of data, whether the data follows normal distribution or not (elliptical and non-elliptical distributions). In other words, copulas are very effective tools for linear and nonlinear inferences [1].

In this paper, we review the methods that are utilized to illustrate copulas and we interpret them in details. Afterwards, we derive each density function from each desired copula and explain the

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relationships that gather them to each other. The aim beyond this investigation is to figure out and understand the whole construction of such important functions so that people who are interested in such subject can share their contributions in using proper approaches.

From a historical point of view, we refer to some names that have made impressive impact to copula function, which have been implemented in many fields of science. As a first step, it is essential to know that copulas are effective tools in the studies of financial analysis [1, 2]. Also, copulas have appeared in many other important fields such as biomedical studies, insurance, physics, and even in mechanical approaches [2].

Furthermore, the word copula was firstly used in 1959 by Sklar to describe the relationship between the joint distribution and its marginal distribution functions. Other authors who have studied copula functions and made an impact in this field include Hoeffding (1948)[3], Dall'Aglio (1991) [2], Kimeldorf and Sampson (1975)[4], Galambos (1978), and many others[5]. Also, it is important to refer to the many contributions of "Nelsen"[6,7] related to copulas and their concepts. He collected most of copula concepts, theorems, definitions, properties, applications and examples in his very well-known book entitled "an introduction to copulas" in 1998. Hoeffding and Frechet proved that each copula lies between the extreme functions that are well-known by maximum and minimum functions and are also copulas. In 1975, Kimeldorf and Sampson referred to copulas as uniform representations, while, in 1978, Galambos and Deheuvels called them the dependence functions [1].

We have divided this study to four parts: The first part represents the introduction. The second part deals with the basic concepts of copulas and their structures. The third part is devoted to explain the methods of constructing copulas, derive their densities, and show the relationships between copulas and the joint distribution functions. The last part is dedicated to present the conclusions and some related future works.

2. Basic concepts

In this section, we present some basic concepts like definitions and important propositions of copulas. First, we begin with definitions of bivariate copula and bivariate co-copula [1].

Definition 2.1 Let $I = [0,1]$, a two-dimensional copula is a function $C: I^2 \to I$, that satisfies the following conditions :

1- C is grounded;

2- $C(p,1) = p$ and $C(1,q) = q$, $\forall\ p, q \in I$;

3- C is 2-increasing.

Definition 2.2 A two-dimensional Co-copula is a function $\varphi: I^2 \to I$ such that $\varphi = 1 - C(1-p, 1-q)$ and holds the following conditions

1- $\varphi(p, 0) = p, \varphi(0, q) = q$, for all $p, q \in I$;

2- $\varphi(p, 1) = \varphi(1, q) = 1$, for all $p, q \in I$;

3- If $p_1 \le p_2$ and $q_1 \le q_2$, then $\varphi(p_1, q_1) + \varphi(p_2, q_2) \le \varphi(p_1, q_2) + \varphi(p_2, q_1)$, for all $p_1, p_2, q_1, q_2 \in I$.

The copula in ***Definition 2.1*** is basically related to Sklar's theorem, that has a main role in the whole theory of copula [6-8,]. That is

Theorem *2.*1 Let H be a joint distribution function with the marginal distribution functions A and B. Then there exists a copula C such that for all $(s,t) \in \bar{R} = [-\infty, \infty]$

$$H(s,t) = C(A(s), B(t)) \tag{2.1}$$

Note that when A and B are continuous, the copula C is unique [1]. Also, it is easy to present the copula and its joint distribution function in an inverse form, that is:

$$C(p,q) = H(A^{-1}(p), B^{-1}(q)) \tag{2.2}$$

Next, it is important to show some properties of C and its density, through the propositions below [5,8].

Proposition 2.1 Let C be a copula. Then the partial derivative of C with respect to p and q exists and satisfies that, for all $p, q \in [0,1]$,

$0 \le \frac{\partial C(p,q)}{\partial p} \le 1,$

Similarly, $0 \le \frac{\partial C(p,q)}{\partial q} \le 1.$

More details are provided in a previous study [2]. Note that a bivariate copula also has the following property

$$0 \leq C(p,q) \leq 1.$$

Further, we need to present a very important and essential property related to what is known as the extreme copulas. There are two well-known copulas that were derived by Frechet and Hoeffding in 1948, and are called the maximum and minimum copulas. They are also known as Frechet-Hoeffding conditions [3,4]. These two copulas represent the upper and lower bounds of each copula, as we can see in ***Proposition 2.2*** [2].

Proposition 2.2 Let $C, \mathfrak{R}$, and $\aleph$ be bivariate copulas. Then $\mathfrak{R}(\mathrm{p,q}) \leq \mathrm{C(p,q)} \leq \aleph(\mathrm{p,q})$, where $\mathfrak{R}(\mathrm{p,q}) = \max(\mathrm{p+q-1,0})$ and $\aleph(\mathrm{p,q}) = \min(\mathrm{p,q})$.

The notion in ***Proposition 2.2*** means that each copula cannot be out of $\mathfrak{R}$ neither $\aleph$. The proof is shown in a previous report [3].

Methods of constructing copulas, such as the inversion method and the Archimedean method are also reviewed within this part. Each method has its characteristics for obtaining the appropriate copula. Via Sklar's theorem, one can produce copulas directly by using the formula in equation (2.2). Whereas, the Archimedean copula can be recalled to construct non-elliptical copulas [2]. We will show and explain these two methods in detail in the next part of this study.

3. Methods of Constructing Copulas

The methods of constructing copulas were presented in many different works. This section focuses on some of the common methods that explain and illustrate several constructions of various copulas. First, we present a method of inversion about which more details were published earlier [2].

3.1Interpretation of Inversion Method

This method is widely used to construct copulas via the inverse function technique. Basically, it considers that $\mathrm{H(s,t)} = \mathrm{C(A(s),B(t))}$ with quasi inverse $\mathrm{p} = A^{-1}(s)$ and $\mathrm{q} = B^{-1}(\mathrm{t})$. The general form of this method is shown in equation (2.2).

In particular, and in order to understand how copulas can be constructed through this method, we present the following examples that explain the methodology of constructing copulas [4, 9].

Example 3.1.1 Suppose the following joint distribution function

$H(\mathrm{s,t}) = (1+e^{-s}+e^{-t})^{-1}$, $0 < s,t < \infty$.

In order to build the corresponding copula of H, we firstly need to find the marginal distribution functions of that H. Let $A(s)$ and $B(t)$ be the marginal distribution functions of H. Then

$$A(s) = H(s,\infty) = (1+e^{-s})^{-1}$$
$$B(t) = H(\infty,\mathrm{t}) = (1+e^{-t})^{-1}$$

Further, we need to find the inverse of A(s) and B(t). Then, let $\mathrm{p = A(s)}$ and $\mathrm{q = B(t)}$, implies that $p = (1+e^{-s})^{-1}$ and $\mathrm{q} = (1+e^{-t})^{-1}$. Now, we solve for s and , respectively, and obtain that

$$\mathrm{s} = -\ln(\frac{1}{p}-1)$$

Similarly,

$$\mathrm{t} = -\ln(\frac{1}{q}-1)$$

According to equation (2.2)

$$\mathrm{C(p,q)} = (1+e^{\ln(\frac{1}{p}-1)}+e^{\ln(\frac{1}{q}-1)})^{-1} = (1+\frac{1}{p}-1+\frac{1}{q}-1)^{-1}$$

and with some simple arrangements of C, we obtain that :

$$C(p,q) = (\frac{p+q-pq}{pq})^{-1} = \frac{pq}{p+q-pq}$$

Example 3.1.*2* Let the joint distribution function $\mathrm{H(s,t)} = (1+e^{-s}+e^{-t}+e^{-(s+t)})^{-1}$, $0 < s,t < \infty$.

Again, let us repeat the same steps of Example 3.1.1. Then, the marginal functions are the followings

$$A(s) = H(s,\infty) = (1+e^{-s})^{-1}$$
$$B(t) = H(\infty,\mathrm{t}) = (1+e^{-t})^{-1}$$

While their inverses, respectively, are $\mathrm{p = A(s)}$, and $\mathrm{q = B(t)}$. It means that $p = (1+e^{-s})^{-1}$ and $\mathrm{q} = (1+e^{-t})^{-1}$. Now, we solve for s and t, respectively, and the inverse transforms yield the following forms

$$\mathrm{s} = -\ln(\frac{1}{p}-1)$$

Similarly,

$$t = -\ln(\frac{1}{q} - 1)$$

According to equation (2.2)

$$C(p,q) = (1 + e^{\ln(\frac{1}{p}-1)} + e^{\ln(\frac{1}{q}-1)} + e^{(\ln\left(\frac{1}{p}-1\right)+\ln\left(\frac{1}{p}-1\right))})^{-1}$$

$$= (1 + \frac{1}{p} - 1 + \frac{1}{q} - 1 + (\frac{1}{p} - 1)(\frac{1}{q} - 1))^{-1}$$

$$= \left(\frac{1}{pq}\right)^{-1}$$

$$C(p,q) = pq$$

Table 3.1.1 shows other different relationships between copulas and the joint distribution function [1].

Table 3.1.1 Other copulas that are using the inversion method.

$H(s,t)$	$A(s)$	$B(t)$	$C(p,q)$
$\frac{(s+1)(e^t-1)}{s+2e^t-1}$	$\frac{(s+1)}{2}$	1-e^{-t}	$\frac{pq}{p+q-pq}$
$e^{-(e^{-\theta s}+e^{-\theta t})^{\frac{1}{\theta}}}$	$e^{-e^{-s}}$	$e^{-e^{-t}}$	$e^{-((lnp)^{-\theta}+(lnq)^{-\theta})^{\frac{1}{\theta}}}$
$(1+e^{-s}+e^{-t}+(1-\theta)e^{-s-t})^{-1}$	$(1+e^{-s})^{-1}$	$(1+e^{-t})^{-1}$	$\frac{pq}{1-\theta(1-p)(1-q)}$

3.2. Archimedean Generator Technique of Constructing Copulas

This method is one of the essential methods that is widely utilized to build copulas. Indeed, there are several reasons that motivated us to explain this method. One of these reasonable causes is the easy way of constructing copulas. A second reason is that there are several families of copulas that occur within this method [2, 9]. This method depends on the following general form, which is known by the Archimedean copula

$$C(p,q) = \psi^{-1}[\psi(p), \psi(q)] \quad (3.1)$$

where ψ is called a generator function of the Archimedean copula function.

To show how copulas are built via this method, we present the following examples. Frist, we begin with a generator of negative logarithmic function [1].

Example 3.2.1 Let the generator $\psi(s) = -\ln(1 - (1-s)^{\theta})$. Then, we firstly need to find the inverse of ψ in order to substitute in the general formula of equation (3.1). Hence, suppose that $s = \psi(s)$, implies that $s = -\ln(1-(1-s)^{\theta})$. Simplifying the term that is equal to s yields the following inverse form $\psi^{-1}(s) = 1 - (1 - \frac{1}{e^s})^{\frac{1}{\theta}}$. Finally, we substitute $\psi(p) = -\ln(1 - (1-p)^{\theta})$ and $\psi(q) = -\ln(1-(1-q)^{\theta})$ in equation (3.1). Thus, we obtain the following copula

$$C(p,q) = 1 - (1 - \frac{1}{e^{-\ln(1-(1-p)^{\theta})\ -\ln(1-(1-q)^{\theta})}})^{\frac{1}{\theta}}$$

$$= 1 - (1 - \frac{1}{(1-(1-p)^{\theta})^{-1}\ \ (1-(1-q)^{\theta})^{-1}})^{\frac{1}{\theta}}$$

$$= 1 - (1 - (1-(1-q)^{\theta} - (1-p)^{\theta} - (1-p)^{\theta}(1-q)^{\theta})))^{\frac{1}{\theta}}$$

$$= 1 - ((1-q)^{\theta} - (1-p)^{\theta} - (1-p)^{\theta}(1-q)^{\theta})^{\frac{1}{\theta}}.$$

Furthermore, one can also explain a construction of another copula function, for example when the generator of Archimedean formula depends on the exponential function [1].

Example 3.2.2 Let the generator $\psi(s) = e^{\frac{\theta}{s}} - e^{\theta}$. Then, a quasi-inverse of the generator is $\psi^{-1}(s) = \frac{\theta}{\ln(s+e^{\theta})}$. By replacing p, q instead of s, we obtain that $\psi(p) = e^{\frac{\theta}{p}} - e^{\theta}$ and $\psi(q) = e^{\frac{\theta}{q}} - e^{\theta}$, respectively. Once, substitute all the obtained forms above in equation (3.1) to obtain the desired copula below

$$C(p,q) = \frac{\theta}{\ln\left(e^{\frac{\theta}{p}} - e^{\theta} + e^{\frac{\theta}{q}} - e^{\theta} + e^{\theta}\right)} = \frac{\theta}{\ln(e^{\frac{\theta}{p}} + e^{\frac{\theta}{q}} - e^{\theta})}$$

One could explain more examples that have different situations and more complicated level [1].

Example 3.2.3 Let the generator $\psi\,(s) = -\ln\left[\frac{(1+s)^{-\theta}-1}{2^{-\theta}-1}\right]$. Once again, we need to determine the quasi inverse of the generator, as we have shown in the previous examples. So, we can see that $\psi^{-1}\,(s) = \left[e^{-s}\left(2^{-\theta}-1\right)+1\right]^{\frac{-1}{\theta}} - 1$, $\psi\,(p) = -\ln\frac{(1+p)^{-\theta}-1}{2^{-\theta}-1}$, and $\psi\,(q) = -\ln\frac{(1+q)^{-\theta}-1}{2^{-\theta}-1}$. Finally, we substitute $\psi^{-1}, \psi(p)$, and $\psi(q)$ in our general form of equation (3.1) , and simplify the implemented forms to yield the following copula

$$C(p,q) = (e^{-(-\ln\frac{(1+p)^{-\theta}-1}{2^{-\theta}-1}-\ln\frac{(1+q)^{-\theta}-1}{2^{-\theta}-1})}\left(2^{-\theta}-1\right)+1)^{\frac{-1}{\theta}}-1$$

$$=((\frac{(1+p)^{-\theta}-1}{2^{-\theta}-1})(\frac{(1+q)^{-\theta}-1}{2^{-\theta}-1})\left(2^{-\theta}-1\right)+1)^{\frac{-1}{\theta}}-1$$

$$=\left[1+\frac{\left((1+p)^{-\theta}-1\right)\left((1+q)^{-\theta}-1\right)}{2^{-\theta}-1}\right]^{\frac{-1}{\theta}}-1$$

Indeed, some other types of copulas are present and are shown in Table 3.2.1, since they are very popular and essential in many real applications [1, 6, 9].

Table 3.2.1-Archimedean families of copulas with one parameter

Family of copula	$\psi\,(s)$	$\psi^{-1}\,(s)$	$C(p,q)$
Gumble	$(-\ln s)^{\theta}$	$e^{-s^{\frac{1}{\theta}}}$	$e^{-[(-\ln p)^{\theta}+(-\ln q)^{\theta}]^{\frac{1}{\theta}}}$
Clayton	$\frac{1}{\theta}(s^{-\theta}-1)$	$(\theta s+1)^{-\frac{1}{\theta}}$	$\left[\max\left(p^{-\theta}+q^{-\theta}-1,0\right)\right]^{\frac{-1}{\theta}}$
Frank	$-\ln\frac{e^{-\theta s}-1}{e^{-\theta}-1}$	$\frac{-\ln(e^{-\theta-s}-e^{-s}+1)}{\theta}$	$\frac{-1}{\theta}\ln\left[1+\frac{\left(e^{-\theta p}-1\right)\left(e^{-\theta q}-1\right)}{(e^{-\theta}-1)}\right]$
Gumble Hougaard	$\ln(1-\theta\ln s)$	$e^{\frac{1-e^{s}}{\theta}}$	$pqe^{-\theta\ln p\ln q}$

4.Copula Density Functions and Their Relationships to Copulas

In order to examine that our copulas fulfill the conditions of being copulas, we need to prove that each density copula obtained from its relevant copula by using ***proposition 4.1*** satisfies the two conditions of being density. That is $\forall\, p,q \in [0,1]$

1- $c(p,q) \geq 0,$ (4.1)

2- $\int_0^1\int_0^1 c(p,q)\,dp\,dq = 1,$ (4.2)

In fact, we need to examine each density with respect to its parameter when it exists.

Proposition 4.1 Let C be a copula. Then the copula density function c exists and nonnegative.

The proof is trivial.

Notice that c can be defined by the following two different ways.

1- $c(p,q) = \frac{\partial^2 C(p,q)}{\partial p\,\partial q}$ (4.3)

2- $c(A(s),B(t)) = \frac{h(s,t)}{a(s)b(t)}$ (4.4)

Let us investigate the density of the derived copula in ***Example 3.1.1*** by using equations (4.3), (4.4) of ***Proposition 4.1,*** respectively. Then, we firstly differentiate C partially with respect to p, q, respectively, and apply equation (4.3). Thus

$$\frac{\partial C(p,q)}{\partial p} = \frac{q(p+q-pq)-pq(1-q)}{(p+q-pq)^2} = \frac{q^2}{(p+q-pq)^2}$$

$$\frac{\partial^2 C(p,q)}{\partial p\,\partial q} = \frac{2q(p+q-pq)^2-q^2(2(1-p)(p+q-pq))}{(p+q-pq)^4}$$

Therefore, and with some simplifications, our differentiation yields the following density copula

$$c(p,q) = \frac{2pq}{(p+q-pq)^3}$$

The second method can be achieved by using equation (4.4). We know that the marginal distribution functions are $A(s) = (1+e^{-s})^{-1}$, and $B(t) = (1+e^{-t})^{-1}$. Then, the marginal probability density functions follow from a differentiation of $A(s)$, and $B(t)$, respectively. Hence

$$a(s) = \frac{\partial A}{\partial s} = e^{-s}(1 + e^{-s})^{-2}$$
$$b(t) = \frac{\partial B}{\partial t} = e^{-t}(1 + e^{-t})^{-2}$$

Also, we have to find the joint density function $h(t,s)$ in order to substitute it in the numinator of equation (4.4). That is

$$\frac{\partial H(s,t)}{\partial s} = e^{-s}(1 + e^{-s} + e^{-t})^{-2}$$
$$\frac{\partial^2 H(s,t)}{\partial s\, \partial t} = 2e^{-(s+t)}(1 + e^{-s} + e^{-t})^{-3} = h(s,t)$$

Substituting the equivalent forms of h(s,t), a(s), and b(t), respectively, in equation (4.4) yields the following copula density.

$$c(p,q) = \frac{2e^{-(s+t)}(1 + e^{-s} + e^{-t})^{-3}}{e^{-(s+t)}(1 + e^{-s})^{-2}(1 + e^{-t})^{-2}} = \frac{2(1 + e^{-s})^2(1 + e^{-t})^2}{(1 + e^{-s} + e^{-t})^3}$$

But $s = -\ln(\frac{1}{p} - 1)$ and $t = -\ln(\frac{1}{q} - 1)$. Then

$$c(p,q) = \frac{2(1 + e^{\ln(\frac{1}{p}-1)})^2(1 + e^{\ln(\frac{1}{q}-1)})^2}{(1 + e^{\ln(\frac{1}{p}-1)} + e^{\ln(\frac{1}{q}-1)})^3} = \frac{2pq}{(p + q - pq)^3}$$

Now, we need to prove that the obtained copula density from ***Example 3.1.1*** satisfies the two conditions of equations (4.1.)and (4.2), so that we ensure that $C(p,q)$ is simply a copula. Then, implementing those conditions show that c(p, q) is:

1. $c(p,q) \geq 0$ when $0 \leq p,q \leq 1,\ \theta \in [-1,1]$.
2. $\int_0^1 \int_0^1 c(p,q)dp\, dq = \int_0^1 \int_0^1 \frac{2pq}{(p+q-pq)^3}\, dp\, dq$

$$= \int_0^1 2q[(\frac{p}{-2(1-q)}(p + q - pq)^{-2}\mid_0^1 - \frac{1}{2(1-q)^2}(p + q - pq)^{-1}\mid_0^1]dq$$
$$= \int_0^1 2q[(\frac{1}{-2(1-q)}(1 + q - q)^{-2}) - \frac{1}{2(1-q)^2}((1 + q - q)^{-1} - (0 + q - 0)^{-1}]dq$$
$$= \int_0^1 2q[(\frac{1}{-2(1-q)}) - \frac{1}{2(1-q)^2}(1 - (q)^{-1})]dq$$
$$= \int_0^1 2q[(\frac{-q+1}{2q(1-q)})]dq = \int_0^1 \frac{-q+1}{(1-q)}dq$$
$$= -\ln(1 - q)\mid_0^1 + \ln(1 - q)\mid_0^1 + \int_0^1 1\, dq = q\mid_0^1 = 1$$

In ***Example 3.1.2***, and in order to find the density copula, we derive partially a copula with respect to p, q and apply equation (4.3). Then

$$\frac{\partial C(p,q)}{\partial p} = q \text{ , and } \frac{\partial^2 C(p,q)}{\partial p\, \partial q} = 1$$

Therefore, $c(p,q) = 1$

Afterwards, we can find the density within equation (4.4). The marginal distribution functions are $A(s)= (1+e^{-s})^{-1}$ and $B(t)=(1+e^{-t})^{-1}$. Then, the corresponding densities to A(s) and B (t) are the following functions.

$$a(s) = \frac{\partial A}{\partial s} = e^{-s}(1 + e^{-s})^{-2}$$
$$b(t) = \frac{\partial B}{\partial t} = e^{-t}(1 + e^{-t})^{-2}$$

Consequently, we have to determine the joint density function $h(s,t)$. That is

$$\frac{\partial H(s,t)}{\partial s} = \frac{e^{-s}+e^{-s-t}}{(1+e^{-s}+e^{-t}+e^{-s-t})^2}$$
$$\frac{\partial^2 H(s,t)}{\partial s\, \partial t} = \frac{e^{-s-t}+e^{-2s-t}+e^{-s-2t}+e^{-2s-2t}}{(1+e^{-s}+e^{-t}+e^{-s-t})^3} = h(s,t)$$

Now, we substitute the equivalent formulas of the density $h(s,t)$, a(s), and b(t) in equation (4.4). Then, we obtain that

$$c(p,q) = \frac{\frac{e^{-s-t}+e^{-2s-t}+e^{-s-2t}+e^{-2s-2t}}{(1+e^{-s}+e^{-t}+e^{-s-t})^3}}{(e^{-s}(1+e^{-s})^{-2})(e^{-t}(1+e^{-t})^{-2})}$$
$$= \frac{(e^{-s-t}+e^{-2s-t}+e^{-s-2t}+e^{-2s-2t})(1+e^{-s})^2(1+e^{-t})^2}{e^{-s-t}(1+e^{-s}+e^{-t}+e^{-s-t})^3}$$

But s= $-\ln(\frac{1}{p}-1)$ and t= $-\ln(\frac{1}{q}-1)$.
Therefore

$$c(p,q) = \frac{[\frac{1}{pq}+\frac{1}{p^2q}+\frac{1}{pq^2}+\frac{1}{p^2q^2}-\frac{2}{p^2q}-\frac{2}{pq^2}]\frac{1}{p^2q^2}}{\frac{1}{p^3q^3}[\frac{1}{pq}+\frac{1}{p}+\frac{1}{q}+1]} = \frac{\frac{pq-q-p+1}{p^4q^4}}{\frac{1-q-p+pq}{p^4q^4}} = 1$$

where, the obtained density copula function satisfies the first condition $c(p,q) \geq 0$, and also it holds a second condition that is

$\int_0^1 \int_0^1 c(p,q)\, dp\, dq = \int_0^1 \int_0^1 1\, dp\, dq = \int_0^1 p \,|_0^1\, dq = \int_0^1 1\, dq = q|_0^1 = 1.$

One of the most popular copula families is well-known as Farlie-Gumbel-Morgenstern (briefly, FGM). It involves a wide range of copulas, where one can extended the classical form in various types [1]. This property motivated us to discuss the following example that shows some features and conditions related to this copula family.

Example 4.1 Let C_θ be a member of the FGM copula family [1]. We have $\theta \in [-1,1]$. Then $C_\theta(p,q) = pq + \theta pq(1-p)(1-q)$, and by using equation (4.3), we obtain the density copula, that is

$\frac{\partial C}{\partial p} = q + \theta q(1-2p)(1-q)$, implies that $\frac{\partial^2 C}{\partial p\, \partial q} = 1 + \theta(1-2p)(1-2q)$

Hence,

$$c(p,q) = 1 + \theta(1-2p)(1-2q) \quad (4.5)$$

To show that the function in equation (4.5) holds the conditions of being density, we can see that $1 + \theta(1-2p)(1-2q) \geq 0$, for all $0 \leq p,q \leq 1$, and $\theta \in [-1,1]$. While, simple computations via the second condition and supposing that $\theta = 1$ yield that

$$\int_0^1 \int_0^1 c(p,q)dp\, dq = \int_0^1 \int_0^1 1 + \theta(1-2p)(1-2q)dp\, dq$$

$$\int_0^1 \int_0^1 1 + (1-2p)(1-2q)\, dp\, dq = \int_0^1 p + (p - \frac{2p^2}{2})(1-2q)|_0^1\, dq$$
$$= \int_0^1 1\, dq = q\,|_0^1 = 1$$

Similarly, when $\theta = -1$, so we obtain that

$$\int_0^1 \int_0^1 1 - (1-2p)(1-2q)dp\, dq = 1$$

Table-4.1 exhibits the relationships between other important types of copulas that are classified as non-elliptical (Archimedean copulas) and their densities [6, 7].

Table 4.1 Some copula functions with their density functions.

Family of copula	*C(p,q)*	*c(p,q)*
Frank family	$\frac{-1}{\theta}\ln[\frac{(1-e^{-\theta})-(1-e^{-\theta p})(1 -}{(1-e^{-\theta})}$	$\frac{\theta(1-e^{-\theta})e^{-\theta(p+q)}}{[(1-e^{-\theta})-(1-e^{-\theta p})(1-e^{-\theta q})]^2}$
Gumbel family	$K = e^{-[(-\ln p)^\theta+(-\ln q)^\theta]^{\frac{1}{\theta}}}$	$\frac{K\,([(-\ln p)^\theta+(-\ln q)^\theta]^{\frac{1}{\theta}}+\theta-1)(\ln p\ln q)^{\theta-}}{pq^{-1}[(-\ln p)^\theta+(-\ln q)^\theta]^{2-\frac{1}{\theta}}}$
Clayton's family	$(p^{-\theta}+q^{-\theta}-1)^{\frac{-1}{\theta}}$	$(1+\theta)(pq)^{-1-\theta}(p^{-\theta}+q^{-\theta}-1)^{-2-\frac{1}{\theta}}$

Note that, in the same way that associates the joint distribution function to its marginal distribution function, copula can be obtained in association with its density function. Mathematically speaking

$$C(p,q) = \int_0^q \int_0^p c(s,t)dsdt, \ \forall p,q \in [0,1] \quad (4.6)$$

4. Conclusions

We can summrize that this study explained in details almost all the concepts related to the constructions of copulas and the methods utilized to build them. Each spcified method has totally different techinques that led us to gain the final copula formula which has the same form. The

Archimedean method is favorable over the method of inversion because it involves most the powerful copula families. The relationship between each copula and its density were shown in two different techniques. We showed that the first technique depedns on differenation while the second one is related to the rule in equation (4.4). Finally, we beleive that all the intrepretations, details, and the easy explainations within this paper provide a good vision to many readers to have a well understanding to the notions of copulas.

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