Systems Reliability Estimations of Models Using Exponentiated Exponential Distribution

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Abstract
This article deals with estimations of system Reliability for one component, two and s-out-of-k stress-strength system models with non-identical component strengths which are subjected to a common stress, using Exponentiated Exponential distribution with common scale parameter. Based on simulation, comparison studies are made between the ML, PC and LS estimators of these system reliabilities when scale parameter is known.
Keywords: Reliability, Exponential Distribution, s out of k stress strain system.

تقديرات معولية أنظمة نماذج باستخدام توزيع أس الأسي

ندى صباح كرم
قسم الرياضيات، كلية التربيه، الجامعه المستنصريه، بغداد، العراق.

الخلاصة
يدرس البحث تقدير معولية انظمة مكونه واحده، مكونتين و S-out of-K من المكونات لنماذج الاجهاد–المتانه، مع كون المتانه غير متماثل وتخضع لاجهاد متماثل، وذلك باستخدام توزيع اس الاسي بمعلمة قياس مشترك . وبالاستناد على المحاكاه، تم اجراء دراسه مقارنه بين مقدرات ML ، PC و LS لمعولية هذه الانظمه في حالة معلمة القياس معلومه.

1. Introduction.

The reliability of a system is the probability that when operating under stated environmental conditions, the system will perform its intended function adequately. For stress-strength models both the strength of the system and the stress, imposed on it by its operating environments are considered to be random variable. The reliability, R, of the system is the probability that the system is strong enough to overcome the stress imposed on it. Several authors have studied the problem of estimating R, including, [1-5].

We consider the reliability, R, when the strength and stress random variables are independent but not identically distributed Exponentiated Exponential (EE) random variables. The Exponentiated Exponential distribution is one of the cumulative distribution functions which was used during the first half of the nineteenth century by Gompertz [6] to compare known human mortality tables and to represent population growth, as follows: [7]

$$G(t) = \left(1 - \rho e^{t\lambda}\right)^{\alpha} \quad \text{for} \quad t > \frac{1}{\lambda} \ln \rho$$

Where ρ, α and λ are all positive real numbers (see [8-10]).

The Exponentiated Exponential distribution, also known as the generalized exponential distribution when ρ=1, is defined as a particular case of Gompestz – Verhulst distribution function.

Therefore, X is a two –parameter EE r. Variable with following distribution function;

$$F(x;\alpha,\lambda) = \left(1-e^{-\lambda x}\right)^{\alpha} \quad x>0\ ,\ \alpha,\lambda>0 \qquad (1)$$

Here α and λ are the shape and scale parameters respectively when $\alpha = 1$ it represent the exponential family. The survival function corresponding with EE density is given as; [9]

$$S(x;\alpha,\lambda) = 1-\left(1-e^{-\lambda x}\right)^{\alpha}$$

and the hazard function is; [3]

$$h(x;\alpha,\lambda) = \frac{\alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1}e^{-\lambda x}}{1-(1-e^{-\lambda x})^{\alpha}}$$

The different moments of this distribution is;

$$E\left(X^k\right) = \alpha\lambda\int_0^\infty x^k\left(1-e^{-\lambda x}\right)^{\alpha-1}e^{-\lambda x}\,dx$$

Now since $0<e^{-\lambda x}<1$, for $\lambda>0$ and $x>0$,therefore by using the series representation (finite or in finite) of $\left(1-e^{-\lambda x}\right)^{\alpha-1}$,we obtain:

$$E\left(X^k\right) = \frac{\alpha\Gamma(k+1)}{\lambda^k}\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\frac{1}{(i+1)^{k+1}}$$

The main aim of this article is estimating the reliability of one, two and multicomponent stress-strength model of an s-out of –k system. Assuming both stress and strength are independently distributed as EE with common and known scale parameter λ. This problem is studied when the strengths of the components are independently but not all identically distributed. Maximum likelihood Estimator (MLE), Percentile estimator (PCE) and Least Square estimator (LS) are obtained. Mont Carlo simulation is performed for comparing the different methods of estimation.

2. Reliability Expressions.

2.1. One Component Reliability.

In the context of reliability the stress-strength model describes the life of a component which has a random strength X_2 and is subjected to random stress X_1. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $X_1<X_2$ thus $R=p(X_1<X_2)$.

We shall assume throughout this paper that $X_1\,and\,X_2$ are continuous and independent r. variabiles let $f_i\ and\ F_i$ denote, respectively, the probability density function (pdf) and the Cumulative distribution function (cdf) of X_i with this notation, one can write;

$$R_1 = p(X_1<X_2) = \int_{x_2=0}^{\infty} f(x_2)\,F_{X_1}(x_2)\,dx_2$$

using eq. (1), then;

$$R_1 = \int_0^\infty \alpha_2\ \lambda\left(1-e^{-\lambda x_2}\right)^{\alpha_2-1}e^{-\lambda x_2}\left(1-e^{-\lambda x_2}\right)^{\alpha_1}dx_2$$

$$= \alpha_2\lambda\int_0^\infty e^{-\lambda x_2}\left(1-e^{-\lambda x_2}\right)^{\alpha_1+\alpha_2-1}dx_2$$

Let $t=\left(1-e^{-\lambda x_2}\right)^{\alpha_2}$ then

$x_2 = \frac{-1}{\lambda}\,ln\left(1-t^{\frac{1}{\alpha_2}}\right),\ dx_2 = \frac{1}{\alpha_2\lambda}\,\frac{t^{\frac{1}{\alpha_2}-1}}{\left(1-t^{\frac{1}{\alpha_2}}\right)}dt$ so;

$$R_1 = \alpha_2\lambda\int_0^1\left(1-t^{\frac{1}{\alpha_2}}\right)t^{\frac{\alpha_1+\alpha_2-1}{\alpha_2}}\frac{1}{\alpha_2\lambda}\frac{t^{\frac{1}{\alpha_2}}}{\left(1-t^{\frac{1}{\alpha_2}}\right)}dt$$

$$= \int_0^1 t^{\frac{\alpha_1}{\alpha_2}}\,dt = \frac{\alpha_2}{\alpha_1+\alpha_2} \qquad \ldots(2)$$

Note that R_1 is independent from parameter λ.

2.2. Two Component Parallel system Reliability.

In this section, we consider the reliability of a parallel system with two components, assuming the strengths of two components are subjected to a common stress which is independent of the strengths of the components. If (X_2,X_3) are independent but not identically distributed strengths of two components subjected to a common random stress X_1 , then the reliability of a system or system reliability (R_2)is given by: [5]

$$R_2 = p[X_1<max(X_2,X_3)]$$

We first obtain the distribution of $Z=max(X_1,X_2)$

$$H(z) = P[Z<z].$$
$$= P[X_2<z].P[X_3<z]$$
$$= F_2(z)\,F_3(z)$$
$$= \left(1-e^{-\lambda z}\right)^{\alpha_2}\left(1-e^{-\lambda z}\right)^{\alpha_3}$$
$$=\left(1-e^{-\lambda z}\right)^{\alpha_2+\alpha_3}$$

$$\therefore \bar{H}(z) = 1-H(z) = 1-\left(1-e^{-\lambda z}\right)^{\alpha_2+\alpha_3}$$

Now the system reliability R is given by
$R_2 = \int_0^\infty \bar{H}(x_1)\ dF_1(x_1)$

$= \int_0^\infty \left[1-\left(1-e^{-\lambda x_1}\right)^{\alpha_2+\alpha_3}\right] f(x_1;\alpha_1,\lambda)dx_1$
$= 1-\alpha_1\lambda \int_0^\infty \left(1-e^{-\lambda x_1}\right)^{\alpha_2+\alpha_3}$
$\left(1-e^{-\lambda x_1}\right)^{\alpha_1-1} e^{-\lambda x_1}\, dx_1$
$= 1-\alpha_1\lambda \int_0^\infty \left(1-e^{-\lambda x_1}\right)^{\alpha_1+\alpha_2+\alpha_3-1}$
$e^{-\lambda x_1}\, dx_1$

Using $t = \left(1-e^{-\lambda x_1}\right)^{\alpha_3}$, by simple transformation we get;

$$R_2 = 1-\int_0^1 t^{\frac{\alpha_1+\alpha_2}{\alpha_3}}\ dt = \frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+\alpha_3} \qquad (3)$$

2.3. Multi – Component s-out of-k System Reliability.

Consider a system made up of k non-identical Component out of these k Components k_1 are of one category and their strengths $y_1,\dots,y_{k_1}$ are iid random variables distributed as EE (β_1,λ) with common cdf $F(y_1,\beta_1,\lambda)$. The remaining components k_2=k-k_1 are of different category and their strengths $y_{k_1+1},\dots,y_k$ are iid random variables distributed as EE (β_2,λ) with common cdf $F(y_2;\beta_2,\lambda)$ this system is subjected to a common stress X. Which is independent r.v. distributed as EE (α,λ) with cdf $F(x;\alpha,\lambda)$.

The system operates successfully if at least s out of k components withstands the stress. According to Johnson [3], the system reliability with non-identical strengths $R_{(s,k)}$, is given by;

$$R_{(s,k)} = \sum_{j_1=s_1}^{k_1} C_{j_1}^{k_1}\, C_{j_2}^{k_2}$$
$$\int_{-\infty}^{\infty} [1-F_{y_1}(x)]^{j_1}\ [F_{y_1}(x)]^{k_1-j_1}$$
$$[1-F_{y_2}(x)]^{j_2}\ [f_x(x)]^{k-j_2}\, dF_x(x)$$

Where the summation is over all possible pair (j_1,j_2) with $0 \le j_1 \le k_1$ and $0 \le j_2 \le k_2$ such that $s \le j_1+j_2 \le k$. Then the reliability can be computed as;

$R_{(s,k)} = \sum_{j_1=s_1}^{k_1}\sum_{j_2=s_2}^{k_2} C_{j_1}^{k_1}\, C_{j_2}^{k_2}$
$\int_0^\infty \left[1-\left(1-e^{-\lambda x}\right)^{\beta_1}\right]^{j_1} \left[\left(1-e^{-\lambda x}\right)^{\beta_1}\right]^{k_1-j_1}$
$\left[1-\left(1-e^{-\lambda n}\right)^{\beta_1}\right]^{j_2} \left[\left(1-e^{-\lambda x}\right)^{\beta_2}\right]^{k_2-j_2}$
$\alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1} e^{-\lambda x}\, dx$

$R_{(s,k)} = \alpha\ \ \lambda \sum_{j_1=s_1}^{k_1}\sum_{j_2=s_2}^{k_2} C_{j_1}^{k_1}\, C_{j_2}^{k_2}$
$\int_0^\infty \left[1-\left(1-e^{-\lambda x}\right)^{\beta_1}\right]^{j_1} \left[1-\left(1-\right.\right.$
$\left.\left. e^{-\lambda x}\right)\beta_2\right]^{j_2}$
$\left(1-e^{-\lambda x}\right)^{\beta_1(k_1-j_1)+\beta_2(k_2-j_2)+\alpha-1} \lambda\, e^{-\lambda x}\ dx$

Using transformation $t = \left(1-e^{-\lambda x}\right)^{\alpha}$, we get:

$$R_{(s,k)} = \sum_{j_1=s_1}^{k_1}\sum_{j_2=s_2}^{k_2} C_{j_1}^{k_1}\, C_{j_2}^{k_2}\ \int_0^1 \left[1-t^{\frac{\beta_1}{\alpha}}\right]^{j_1} \left[1-t^{\frac{\beta_2}{\alpha}}\right]^{j_2} t^{\frac{\beta_1}{\alpha}(k_1-j_1)+\frac{\beta_2}{\alpha}(k_2-j_2)}\ dt$$

Let $v_1 = \frac{\beta_1}{\alpha}$, $v_2 = \frac{\beta_2}{\alpha}$ and simplifying by using the Binomial series expansion, $(x+a)^n = \sum_{k=0}^{n} C_k^n\ x^k\ a^{n-k}$, then $R_{(s,k)}$ can be rewritten as;

$$R_{(s,k)} = \sum_{j_1=s_1}^{k_1} C_{j_1}^{k_1}\ \sum_{j_2=s_2}^{k_2} C_{j_2}^{k_2}\ \sum_{i=0}^{j_1} C_i^{j_1}\,(-1)^{j_1} \sum_{l=0}^{j_2} C_l^{j_2}$$
$$(-1)^{j_2}[v_1(i+k_1-j_1)+v_2(l+k_2-j_2)+1]^{-1} \qquad (4)$$

3. Reliability Estimation.

3.1. Maximum Likelihood System Reliability Estimation.

By the invariance property for the Maximum likelihood estimation we estimate the parameters $(\alpha_1,\alpha_2,\alpha_3,\lambda)$ and substitute them in the systems Reliability, $(R_1, R_2\ and\ R_{(s,k)})$ to give the MLE for R_1, say $\hat{R}_{1(\mathrm{MLE})}$, where from eq. (2)

$$\hat{R}_{1(\mathrm{MLE})} = \frac{\hat{\alpha}_{2(MLE)}}{\hat{\alpha}_{1(MLE)}+\hat{\alpha}_{2(MLE)}}$$

The MLE for R_2, say $\hat{R}_{2(\mathrm{MLE})}$, from eq. (3)

$$\hat{R}_{2(\mathrm{MLE})} = \frac{\hat{\alpha}_{1(MLE)}+\hat{\alpha}_{2(MLE)}}{\hat{\alpha}_{1(MLE)}+\hat{\alpha}_{2(MLE)}+\hat{\alpha}_{3(MLE)}}$$

and the MLE for $R_{(s,k)}$, say $\hat{R}_{(s,k)(MLE)}$, from eq.(4)

$\hat{R}_{(s,k)(MLE)} =$
$\sum_{j_1=s_1}^{k_1} C_{j_1}^{k_1} \sum_{j_2=s_2}^{k_2} C_{j_2}^{k_2} \sum_{i=0}^{j_1} C_i^{j_1} (-1)^{j_1}$

.
$\sum_{l=0}^{j_2} C_l^{j_2} (-1)^{j_2} [\hat{v}_1(i+k_1-j_1) + \hat{v}_2(l+k_2-j_2)+1]^{-1}$

Where $\hat{v}_1 = \frac{\hat{\beta}_{1(MLE)}}{\hat{\alpha}_{(MLE)}}$ and $\hat{v}_2 = \frac{\hat{\beta}_{2(MLE)}}{\hat{\alpha}_{(MLE)}}$

The parameters estimation for $\hat{R}_{1(MLE)}$ is by the log- likelihood function:

Let $x_{11}, x_{12}, \ldots x_{1m}$ and $x_{21}, x_{22}, \ldots x_{2n_1}$ be two ordered random samples of size m and n_1, respectively, on strength and stress variants following EE distribution with shape parameters α_1 and α_2 and common scale parameter λ. then:

$$L(\alpha_1,\alpha_2,\lambda) = m\ln\alpha_1 + (m+n_1)\ln\lambda - \lambda\left(\sum_{i=1}^{m} x_{1i} + \sum_{j=1}^{n1} x_{2j}\right) + n_1\ln\alpha_2 + (\alpha_1-1)\sum_{i=1}^{m} ln\left(1-e^{-\lambda x_{1i}}\right) + (\alpha_2-1)\sum_{j=1}^{n1} \ln\left(1-e^{-\lambda x_{2j}}\right)$$

The first derivatives are:

$$\frac{\partial \ln L}{\partial \alpha_1} = \frac{m}{\alpha_1} + \sum_{i=1}^{m} \ln\left(1-e^{-\lambda x_{1i}}\right) \Rightarrow 0$$

$$\frac{\partial \ln L}{\partial \alpha_2} = \frac{n1}{\alpha_2} + \sum_{j=1}^{n_1} \ln\left(1-e^{-\lambda x_{2j}}\right) \Rightarrow 0$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{m+n_1}{\lambda} - \left(\sum_{i=1}^{m} x_{1i} + \sum_{j=1}^{n1} x_{2j}\right) + \frac{(\alpha_1-1)\sum_{i=1}^{m} x_{1i}\, e^{-\lambda x_{1i}}}{\sum_{i=1}^{m} \ln\left(1-e^{-\lambda x_{1i}}\right)} + (\alpha_2-1)\frac{\sum_{j=1}^{n_1} x_{2j}\, e^{-\lambda x_{2j}}}{\sum_{j=1}^{n_1} \ln\left(1-e^{-\lambda x_{2j}}\right)} \Rightarrow 0$$

The solutions of the above equations are:

$$\hat{\alpha}_{1(MLE)} = \frac{-m}{\sum_{i=1}^{m} \ln\left(1-e^{-\lambda x_{1i}}\right)},$$

$$\hat{\alpha}_{2(MLE)} = \frac{-n_1}{\sum_{j=1}^{n_1} \ln\left(1-e^{-\lambda x_{2j}}\right)}$$

And if λ is unknown, then $\hat{\lambda}$ is a simple iterative solution of the non-linear equation $g_1(\lambda) = 0$, where

$g_1(\lambda) = 0$

$$\Rightarrow \frac{m+n_1}{\lambda} - \left(\sum_{i=1}^{m} x_{1i} + \sum_{j=1}^{n1} x_{2j}\right) + \frac{(\hat{\alpha}_1-1)\sum_{i=1}^{m} x_{1i}\, e^{-\lambda x_{1i}}}{\sum_{i=1}^{m} \ln\left(1-e^{-\lambda x_{1i}}\right)} + \frac{(\hat{\alpha}_2-1)\sum_{j=1}^{n_1} x_{2j}\, e^{-\lambda x_{2j}}}{\sum_{j=1}^{n_1} \ln\left(1-e^{-\lambda x_{2j}}\right)} = 0$$

Now for $\hat{R}_{2(MLE)}$, let $x_{31}, x_{32}, \ldots x_{3n_2}$ be ordered random sample of size n_2, following EE distribution with (α_3, λ), then;

$$\ln L(\alpha_1,\alpha_2,\alpha_3,\lambda) = m\ln\alpha_1 + n_1\ln\alpha_2 + n_2\ln\alpha_3 + (m+n_1+n_2)\,ln\,\lambda - \lambda\left(\sum_{i=1}^{m} x_{1i} + \sum_{j=1}^{n1} x_{2j} + \sum_{l=1}^{n2} x_{3l}\right) + (\alpha_1-1)\sum_{i=1}^{m} ln\left(1-e^{-\lambda x_{1i}}\right) + (\alpha_2-1)\sum_{j=1}^{n1} \ln\left(1-e^{-\lambda x_{2j}}\right) + (\alpha_3-1)\sum_{l=1}^{n2} \ln\left(1-e^{-\lambda x_{3l}}\right)$$

The MLE for α_3, say $\hat{\alpha}_{3(MLE)}$, which can be used to estimate R_2 is given as;

$$\hat{\alpha}_{3(MLE)} = \frac{-n_2}{\sum_{l=1}^{n_2} \ln\left(1-e^{-\lambda x_{3l}}\right)}$$

And for unknown λ, then $\hat{\lambda}$ is a simple iterative solution of the non-linear equation $g_2(\lambda) = 0$, where

$g_2(\lambda) = 0$

$$\Rightarrow \frac{m+n_1+n_1}{\lambda}\left(\sum_{i=1}^{m} x_{1i} + \sum_{j=1}^{n1} x_{2j} + \sum_{l=1}^{n2} x_{3l}\right) + \frac{(\hat{\alpha}_1-1)\sum_{i=1}^{m} x_{1i}\, e^{-\lambda x_{1i}}}{\sum_{i=1}^{m} \ln\left(1-e^{-\lambda x_{1i}}\right)} + \frac{(\hat{\alpha}_2-1)\sum_{j=1}^{n1} x_{2j}\, e^{-\lambda x_{2j}}}{\sum_{j=1}^{n1} \ln\left(1-e^{-\lambda x_{2j}}\right)} + \frac{(\hat{\alpha}_3-1)\sum_{l=1}^{n2} x_{3l}\, e^{-\lambda x_{3l}}}{\sum_{l=1}^{n2} \ln\left(1-e^{-\lambda x_{3l}}\right)} = 0$$

For $R_{(s,k)}$, let $Y_{11}, Y_{12}, \ldots, Y_{1n_1}$, be a random sample of size n1 drawn from EE(β_1, λ), then $Y_{1(1)} < Y_{1(2)} < \cdots < Y_{1(n_1)}$ denotes the corresponding order statistic sample. Let $Y_{21}, Y_{22}, \ldots, Y_{2n_2}$, be a random sample of size n2 drawn from EE(β_2, λ), then $Y_{2(1)} < Y_{2(2)} < \cdots < Y_{2(n_2)}$ denotes the corresponding order statistic sample. Let $X_1, X_2, \ldots, X_m$, be a random sample of size m drawn from EE(α, λ), then $X_{(1)} < X_{(2)} < \cdots < X_{(m)}$ denotes the corresponding order statistic sample. Then the likelihood function is given by;

$$L(\alpha,\beta_1,\beta_2,\lambda) = \alpha^m \beta_1^{n1} \beta_2^{n2} \lambda^{m+n1+n2} \exp\left[-\lambda\left(\sum_i^m x_i + \sum_j^{n1} y_{1j} + \sum_l^{n2} y_{2l}\right)\right] \prod_{i=1}^{m}\left(1-e^{-\lambda x_i}\right)^{\alpha-1} \prod_{j=1}^{n1}\left(1-e^{-\lambda y_{1j}}\right)^{\beta_1-1} \prod_{l=1}^{n2}\left(1-e^{-\lambda y_{2l}}\right)^{\beta_2-1}$$

then the ML estimators of α, β_1 ,β_2, denoted by $\hat{\alpha}_{(MLE)}$, $\hat{\beta}_{1(MLE)}$ and $\hat{\beta}_{2(MLE)}$, respectively, are;

$$\hat{\alpha}_{(MLE)} = \frac{-m}{\sum_{i=1}^{m} \ln(1-e^{-\lambda x_i})},$$

$$\hat{\beta}_{1(MLE)} = \frac{-n_1}{\sum_{j=1}^{n_1} \ln(1-e^{-\lambda y_{1j}})}$$

and $$\hat{\beta}_{2(MLE)} = \frac{-n_2}{\sum_{j=1}^{n_1} \ln(1-e^{-\lambda y_{2l}})}$$

So for unknown λ, then $\hat{\lambda}$ here is a simple iterative solution of the non-linear equation $g_3(\lambda) = 0$, where $g_3(\lambda) = 0$

$$\Rightarrow \frac{m+n_1+n_2}{\lambda} - \left(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n1} y_{1j} + \sum_{l=1}^{n2} y_{2l}\right) + \frac{(\hat{\alpha}-1)\sum_{i=1}^{m} x_i\, e^{-\lambda x_i}}{\sum_{i=1}^{m} \ln(1-e^{-\lambda x_i})} + \frac{(\hat{\beta}_1-1)\sum_{j=1}^{n_1} y_{1j}\, e^{-\lambda y_{1j}}}{\sum_{j=1}^{n_1} \ln(1-e^{-\lambda y_{1j}})} + \frac{(\hat{\beta}_2-1)\sum_{l=1}^{n2} y_{2l}\, e^{-\lambda y_{2l}}}{\sum_{l=1}^{n2} \ln(1-e^{-\lambda y_{2l}})} = 0$$

3.2 Percentile System Reliability Estimators.

When the method of estimation of unknown parameter is changed from ML to any other traditional method, the invariance principle does not hold good to estimate the parametric function. However such a case is attempted in different situations by different authors [5].

The percentile estimator can be obtained by equating the sample percentile points with the population percentile points, if the data comes from a distribution function which has a closed form, and then the unknown parameters can be estimates by fitting a straight line to the theoretical points obtained from the distribution function and the sample percentile points.

Because of the structure of EE distribution function, it is possible to use the same concept to obtain the estimators based on the percentiles.

Several estimators of p_i, p_j and p_l can be used here [see 9], where they are the samples percentile $F(x_{1(i)}; \alpha_1, \lambda)$, $F(x_{2(j)}; \theta_2, \lambda)$ and $F(x_{3(l)}; \alpha_3, \lambda)$. The following formulas will be considered in this work:

$p_i = \frac{i}{m+1}; i = 1 \ldots m$, $p_j = \frac{j}{n_1+1}; j = 1 \ldots n_1$ and $p_l = \frac{l}{n_2+1}; l = 1 \ldots n_2$ which are expected values of $F(x_{1(i)})$, $F(x_{2(j)})$ and $F(x_{3(l)})$, respectively.

Then the PCE's of α_1 , α_2 and α_3 can be obtained by minimizing the following equations with respect to α_1 , α_2 , α_3 and λ, respectively,

$$\sum_{i=1}^{m}\left[\ln(p_i) - \alpha_1 \ln\left(1-e^{-\lambda x_{1(i)}}\right)\right]^2,$$

$$\sum_{j=1}^{n_1}\left[\ln(p_j) - \alpha_2 \ln\left(1-e^{-\lambda x_{2(j)}}\right)\right]^2,$$

$$\sum_{l=1}^{n_2}\left[\ln(p_l) - \alpha_3 \ln\left(1-e^{-\lambda x_{3(l)}}\right)\right]^2$$

Then the PCE's say, $\hat{\alpha}_{1(PCE)}$, $\hat{\alpha}_{2(PCE)}$ and $\hat{\alpha}_{3(PCE)}$ take the forms;

$$\hat{\alpha}_{1(PCE)} = \frac{\sum_{i=1}^{m} \ln pi\ \ln\left(1-e^{-\lambda x_{1(i)}}\right)}{\sum_{i=1}^{m}\left[\ln\left(1-e^{-\lambda x_{1(i)}}\right)\right]^2},$$

$$\hat{\alpha}_{2(PCE)} = \frac{\sum_{j=1}^{n_1} \ln pj\ \ln\left(1-e^{-\lambda x_{2(j)}}\right)}{\sum_{j=1}^{n_1}\left[\ln\left(1-e^{-\lambda x_{2(j)}}\right)\right]^2},$$

$$\hat{\alpha}_{3(PCE)} = \frac{\sum_{l=1}^{n_2} \ln pl\ \ln\left(1-e^{-\lambda x_{3(l)}}\right)}{\sum_{l=1}^{n_2}\left[\ln\left(1-e^{-\lambda x_{3(l)}}\right)\right]^2}.$$

The PCE of R_1 and R_2, say $\ddot{R}_{1(PCE)}$ and $\ddot{R}_{2(PCE)}$ are obtained by substitute $\hat{\alpha}_{1(PCE)}, \hat{\alpha}_{2(PCE)}$ and $\hat{\alpha}_{3(PCE)}$ in equations (2) and (3).

For $R_{(s,k)}$ the PCE 's of α_1, β_1 and β_2, denoted by $\hat{\alpha}_{(PCE)}, \hat{\beta}_{1(PCE)}$ and $\hat{\beta}_{2(PCE)}$, respectively and take the forms:

$$\hat{\alpha}_{(PCE)} = \frac{\sum_{i=1}^{m} \ln pi\ \ln\left(1-e^{-\lambda x_{(i)}}\right)}{\sum_{i=1}^{m}\left[\ln\left(1-e^{-\lambda x_{(i)}}\right)\right]^2}$$

$$,\hat{\beta}_{1(PCE)} = \frac{\sum_{j=1}^{n_1} \ln pj\ \ln\left(1-e^{-\lambda y_{1(j)}}\right)}{\sum_{j=1}^{n_1}\left[\ln\left(1-e^{-\lambda y_{1(j)}}\right)\right]^2},$$

$$\hat{\beta}_{2(PCE)} = \frac{\sum_{l=1}^{n_2} \ln pl\ \ln\left(1-e^{-\lambda y_{2(l)}}\right)}{\sum_{l=1}^{n_2}\left[\ln\left(1-e^{-\lambda y_{2(l)}}\right)\right]^2}$$

And the PCE for $R_{(s,k)}$, denoted by $\ddot{R}_{(s,k)(PCE)}$, is by replacing $\hat{\alpha}_{(PCE)}, \hat{\beta}_{1(PCE)}$ and $\hat{\beta}_{2(PCE)}$ in eq(4).

3.3. Least squares system in Reliability Estimators.

Least square estimators are obtained by minimizing the sum of squared errors between the value and it's expected value. It provide the regression based method estimators of unknown parameters.

According to Johnson [10], we have;

$E[F(x_{1(i)})] = p_{1i}$, $i = 1 \ldots m$, ,

$E[F(x_{2(j)})] = p_{1j}$, $j = 1 \ldots n_1$,

$E[F(x_{3(l)})] = p_{3l}$, $l = 1 \ldots n_2$.

The LS estimators of α_1, α_2 and α_3, denoted by $\hat{\alpha}_{1(LSE)}, \hat{\alpha}_{2(LSE)}$ and $\hat{\alpha}_{3(LSE)}$, respectively, can be obtained by minimizing the following equations with respect

$$\sum_{i=1}^{m}\left[F(x_{(i)}) - E\left(F(x_{1(i)})\right)\right]^2 = \sum_{i=1}^{m}\left[\left(1-e^{-\lambda x_{1(i)}}\right)^{\alpha_1} - \frac{i}{m+1}\right]^2,$$

$$\sum_{j=1}^{n_1}\left[F(x_{2(j)}) - E\left(F(x_{2(j)})\right)\right]^2 = \sum_{i=1}^{n_1}\left[\left(1-e^{-\lambda x_{2(j)}}\right)^{\alpha_2} - \frac{j}{n_1+1}\right]^2,$$

$$\sum_{l=1}^{n_2}\left[F(x_{3(l)}) - E\left(F(x_{3(l)})\right)\right]^2 = \sum_{i=1}^{n_2}\left[\left(1-e^{-\lambda x_{3(l)}}\right)^{\alpha_3} - \frac{l}{n_2+1}\right]^2.$$

The LSE of R_1 and R_2 are denoted by $\hat{R}_{1(LSE)}$ and $\hat{R}_{2(LSE)}$ respectively, are obtained by substitute $\hat{\alpha}_{1(LSE)}, \hat{\alpha}_{2(LSE)}$ and $\hat{\alpha}_{3(LSE)}$ in equations (2) and (3).

Then for $R_{(s,k)}$, denoted as $\ddot{R}_{(s,k)(LSE)}$, is by substituting the LSE of parameters in eq(4), which denoted as $\hat{\alpha}_{(LSE)}, \hat{\beta}_{(LSE)}$ and $\hat{\beta}_{2(LSE)}$ the LSE' s of α, β_1 and β_2, respectively, obtained by minimizing the following equations with respect to α, β_1, β_2 and λ.

$$\sum_{i=1}^{m}\left[F(x_{(i)}) - E\left(F(x_{(i)})\right)\right]^2,$$

$$\sum_{i=1}^{n_1}\left[F(y_{1(j)}) - E\left(F(y_{1(j)})\right)\right]^2,$$

$$\sum_{i=1}^{n_2}\left[F(y_{2(l)}) - E\left(F(y_{2(l)})\right)\right]^2.$$

4. Numerical Simulation Results and Conclusions.

In this section, a numerical experiment will be represented to compare the performance of the three estimators of three system Reliabilities R_1, R_2 and $R_{(s,k)}$ proposed in the previous subsections when λ is known with respect to their mean squared errors (MSE). Note that the generatition of EE(α, λ) is very simple, if U follows uniform distribution in [0, 1], then X=((-ln(1-$U^{(1/\alpha)}$)/λ) follows EE(α, λ), Monte carlo simulation is applied for different sample sizes, different parameters values and for two different s-out of-k systems.

We consider different sample sizes ranging from very small to large (5, 10, 30, 50). Since λ is the scale parameter and all the estimators are scale invariant, we take λ=3 in all our computations and we consider different values of the shape parameters [(α_1, α_2, α_3)=(2, 1.5, 0.5),(0.5, 1.5, 2)] and the selected values for s-out of-k systems are [(s_1,k_1,s_2,k_2)=(1, 2, 1, 2),(1, 4, 1, 4)]for [(α, β_1, β_2)=(2.5, 1.3, 1.6)]. The MSE's are computed for the different estimators over (1000) replications. The results are reported in tables (1), (2) and (3), and the conclusions can be summarized as follows:

1. The value of R_1 increases and of R_2 decreases as the value of α_1 decreases (2, 0.5) and the value of α_3 increases(0.5, 2) with fixed value of α_2 (1.5).
2. For different values of (m, n_1), for R_1, the MSEs for all methods decreases as (m, n_1) increases.
3. For all methods it is clear that when (m=n_1=n_2), for R_2, then the MSEs decreases, except for (m=n_1=n_2=50) is increases. And when (m=n_1) fixed and n_2 increases the MSEs decreases, also for decreases (m=n_1) and n_2 is fixed.
4. The value of $R_{(s,k)}$ decreases as (k) value increases, and for (m=n_1) and n_2 increases the MSEs for all methods decreases.
5. In general, the MLE works better estimator for R_1, R_2 and $R_{(s,k)}$ than the other two methods.

Table 1- The MSE values when λ=3, α_1=0.5, α_2=1.5, α_3=2, then R_1=0.75 and R_2=0.5.

m, n_1	R_1			m,n_1,n_2	R_2		
	MLE	PC	LS		MLE	PC	LS
5,5	1.53E-02	1.63E-01	2.28E-02	5,5,5	1.99E-02	2.13E-02	2.92E-02
5,10	1.25E-02	1.27E-02	1.84E-02	5,5,10	1.50E-02	1.62E-02	2.23E-02
5,30	1.19E-02	1.07E-02	1.47E-02	5,5,30	1.11E-02	1.11E-02	1.55E-01
5,50	9.56E-03	8.37E-03	1.23E-01	5,5,50	1.09E-02	1.06E-02	1.48E-02
10,5	9.91E-03	1.18E-02	1.64E-02	10,10,5	1.58E-02	1.70E-02	2.26E-01
10,10	7.73E-03	9.16E-03	1.28E-02	10,10,10	1.00E-02	1.12E-02	1.54E-02
10,30	6.63E-03	5.73E-03	8.05E-02	10,10,30	6.17E-03	7.10E-03	1.01E-02
10,50	4.83E-03	4.94E-03	6.95E-03	10,10,50	5.40E-03	6.36E-03	9.12E-03
30,5	7.60E-03	9.98E-03	1.42E-02	30,30,5	1.42E-02	1.51E-02	2.01E-02
30,10	4.72E-03	6.33E-03	9.35E-03	30,30,10	7.21E-03	8.28E-03	1.18E-02
30,30	2.36E-03	2.87E-03	4.30E-03	30,30,30	3.43E-03	4.12E-03	6.01E-03
30,50	1.84E-03	2.21E-03	3.38E-03	30,30,50	2.60E-03	3.38E-03	5.19E-03
50,5	6.77E-03	1.01E-02	9.67E-03	50,50,5	1.51E-02	1.55E-02	2.03E-02
50,10	4.08E-03	5.85E-02	8.99E-03	50,50,10	6.45E-03	7.57E-03	1.06E-02
50,30	1.88E-03	2.55E-03	4.06E-03	50,50,30	3.06E-03	3.74E-03	5.51E-03
50,50	1.42E-03	1.79E-03	2.75E-03	50,50,50	2.30E-03	2.54E-03	3.89E-03

Table 2-The MSE values when λ=3, α_1=2, α_2=1.5, α_3=0.5, then R_1=0.4286 and R_2=0.875.

m, n_1	R_1			M,n1,n2	R_2		
	MLE	PC	LS		MLE	PC	LS
5,5	2.24E-02	2.36E-02	3.31E-02	5,5,5	4.58E-03	4.79E-03	7.25E-03
5,10	1.74E-02	1.93E-02	2.57E-02	5,5,10	2.71E-03	3.42E-03	4.95E-03
5,30	1.42E-02	1.62E-02	2.18E-02	5,5,30	1.72E-03	2.26E-03	3.28E-03
5,50	1.17E-02	1.36E-02	1.93E-02	5,5,50	1.57E-03	2.08E-03	2.90E-03
10,5	1.74E-02	1.84E-02	2.58E-02	10,10,5	4.54E-03	4.42E-03	6.81E-03
10,10	1.11E-02	1.28E-02	1.78E-02	10,10,10	2.20E-03	2.50E-03	3.67E-03
10,30	8.11E-03	9.77E-03	1.40E-02	10,10,30	1.07E-03	1.49E-03	2.27E-03
10,50	6.74E-03	8.45E-02	1.22E-02	10,10,50	8.78E-04	1.35E-03	2.09E-03
30,5	1.51E-02	1.52E-02	2.02E-02	30,30,5	4.65E-03	3.79E-03	5.66E-03
30,10	8.45E-03	9.62E-03	1.33E-02	30,30,10	1.64E-03	1.60E-03	2.40E-03
30,30	3.83E-03	4.75E-03	7.13E-03	30,30,30	6.68E-04	8.08E-04	1.20E-03
30,50	2.99E-03	3.80E-03	5.81E-03	30,30,50	4.59E-04	6.22E-04	9.75E-04
50,5	1.29E-02	1.38E-02	1.97E-02	50,50,5	5.40E-03	4.03E-03	5.54E-03
50,10	7.36E-03	8.41E-03	1.22E-02	50,50,10	1.62E-03	1.52E-03	2.13E-03
50,30	3.19E-03	4.10E-03	6.35E-03	50,50,30	5.84E-04	6.74E-04	9.88E-04
50,50	2.39E-03	3.05E-03	4.65E-03	50,50,50	3.72E-04	4.83E-04	7.56E-04

Table 3-The MSE values for $R_{(s,k)}$.

m,n1,n2	s_1=1,k_1=2, α=2.5	s_2=1,k_2=2 β_1=1.3 **$R_{(s,k)}$=0.30120**	β_2=1.6	s_1=1,k_1=4, α=2.5	s_2=1,k_2=4 β_1=1.3 **$R_{(s,k)}$=0.1773**	β_2=1.6
	MLE	PC	LS	MLE	PC	LS
5,5,5	4.5E-03	1.49E-02	2.03E-02	1.99E-02	8.23E-03	1.16E-02
5,5,10	2.71E-03	1.35E-02	1.82E-02	1.50E-02	7.24E-03	1.01E-02
5,5,30	1.72E-03	1.44E-02	1.90E-02	1.11E-02	8.07E-03	1.10E-02
5,5,50	1.57E-03	1.26E-02	1.75E-02	1.09E-02	6.85E-03	9.99E-03
10,10,5	4.54E-03	8.43E-03	1.21E-02	1.58E-02	4.33E-03	6.35E-03
10,10,10	2.20E-03	7.95E-03	1.14E-02	1.00E-02	4.15E-03	6.32E-03
10,10,30	1.07E-03	6.68E-03	9.15E-03	6.17E-03	3.39E-03	4.64E-03
10,10,50	8.78E-04	5.92E-03	8.67E-02	5.40E-03	2.95E-03	4.44E-03
30,30,5	4.65E-03	5.12E-03	7.53E-03	1.42E-02	2.57E-03	3.81E-03
30,30,10	1.64E-03	3.78E-03	5.50E-03	7.20E-03	1.89E-03	2.79E-03
30,30,30	6.68E-04	2.55E-03	3.96E-03	3.43E-03	1.25E-03	2.00E-03
30,30,50	4.59E-04	2.57E-03	3.68E-03	2.60E-03	1.20E-03	1.85E-03
50,50,5	5.40E-03	4.54E-03	6.53E-03	1.51E-02	2.24E-03	3.24E-03
50,50,10	1.62E-03	2.97E-03	4.38E-03	6.45E-03	1.48E-03	2.23E-03
50,50,30	5.84E-04	1.80E-03	2.79E-03	3.06E-03	8.84E-04	1.39E-03
50,50,50	3.72E-04	1.66E-03	2.49E-03	2.03E-03	8.12E-04	1.22E-03

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