New Characterization of Topological Transitivity

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Abstract
Let (X,f) be a dynamical system, (X,f) is said to be topological transitive if for every pair of non-empty open set U,V , there exists $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$. We introduce and investigate a new definition of topological transitive by using the nation N-open subset and we called N-transitive and prove the equivalent definitions of this new definition.

Keyword: Topological transitive ,N-open Subset

MSC:54H20

وصف جديد للانتقالية التبولوجية

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الخلاصة
قدم البحث وصفاً جديدا للدوال الانتقالية وباستخدام مفهوم المجموعة (N–Open) وسمية هذا المفهوم N الانتقالية (N–transitive) ، وتم تقديم عدد من التعريفات المكافئة لهذا المفهوم الجديد .

1.Introduction

The concept of topological transitive goes back to G.D Birkhoff who introduced it in 1920, [1].

A topological transitive dynamical system has points which eventually move under from one arbitrarily small open set to any other. Consequently, such a dynamical system cannot be decomposed into two disjoint sets with non-empty interiors which do not interact under the transformation.

We will consider a discrete dynamical system(X,f) given by a metric space X and continuous map $f: X \to X$.The trajectory of a point $x \in X$ being the sequence $x, f(x), f^2(x), f^3(x), \ldots$,where $f^n(x)$ is the nth iteration of f. The set of points of the trajectory of x under f is called the orbit of x ,denoted by $O_f(x)$. A point $x \in X$ is called non-wandering if for every neighborhood U of x there is a positive integer n such that $f^n(U) \cap U \neq \emptyset$ [2]. The set of non-wandering points of $f$will be denoted by $\Omega(f)$.

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A dynamical system (X,f) is said to be topological transitive (hereafter briefly called transitivity),if for every pair of non-empty open set U,V , there exists $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$, [3].

Recently several researches were conducted to introduce weak forms of open set and obtain some characterization and preserving theorem of topological properties Al Omari A. and Noorani M. in [4] introduce new class of set called N-open sets "A subset A of a space X is said to be an N-open if for every $x \in A$ there exists an open subset U_x in X contains x such that $U_x - A$ is a finite set". They prove that the family of all N-open establishes a topology. Moreover ,they obtain a characterization and preserving theorem of compact space. The objective of this paper is use the new class of N-open set to create an N-transitive function and prove some equivalent definitions.

2. N-Open Set

In 2009 Al. Omari and Noorani[4], introduce the concept of "N-open Set" in a topological space, they prove that the family of all N-open sets establishes a topology. Moreover, they obtain a characterization and preserving theorems of compact spaces.

Definition2.1[4]:

A subset A of a space X is said to be an N-open if for every $x \in A$ there exists an open subset U_x in X contains x such that $U_x - A$ is a finite set.

The complement of N-open set is said to be an N-closed set. Clearly every open set is N-open but the converse is not true. If X is a topological space then the family of all N-open subsets of X is a topological space, [4].

Theorem2.2[4]:

Let (X,T) be the topological space then the family of all N-open subset is a topological space.

The union of all N-open sets of X contained in a subset A is called the N-interior of A and denoted by $A^{\circ N}$[5].

Proposition 2.3 [5]:

A subset A of a space X is N-open if and only if $A = A^{\circ N}$.

Clearly, the interior of A is a subset of $A^{\circ N}$($A^{\circ} \subseteq A^{\circ N}$).The N-neighborhood of a point $x \in X$ is any N-open subset of Xwhich contains x. Now we can introduce the following definition.

Definition 2.4:

A subset A is said to be N-nowhere dense if the closure has empty N-interior.

The function $f: X \to Y$ is said to be continuous function if $f^{-1}(A)$ is an open set in X for every open set A in Y. This definition is equivalent to the following definition see[4], a function $f: X \to Y$ is said to be N-continuous if $f^{-1}(A)$ is an N- open set in X for every open set A in Y , if $f^{-1}(A)$ and $A$is N-open then $f$is said to be N*- continuous, [5],clearly every continuous is an N-continuous function, but the converse is not true in general, and every N*-continuous function is an N-continuous ,but the converse is not true in general, for more details see [5].

The following proposition gives the condition on a continuous function which implies N*-continuous

Proposition 2.5 [5]:

If $f: X \to Y$ be continuous injective ,then $f^{-1}(A)$ is N-open whenever $A$is an N-open subset of Y .

A subset A of topological space X is said to be dense set (or everywhere dense) in$X$,if the closure of A is equal to $X(\bar{A} = X)$ [6]. Equivalently ,A is dense if and only if A intersects every non-empty open set in$X$. Now ,we can prove the relation between dense set and N-open set in a topological space.

Theorem 2.6:

let Xbe a topological space and $A \subset X$ then A is dense in X if and only if $A \cap U \neq \emptyset$ for every non-empty N-open set U.

Proof:

let A be a dense subset of X,then for every non-empty open set $V \subset X$, $A \cap V \neq \emptyset$. Let U be N-open subset in X, suppose $A \cap U = \emptyset$.
$\forall x \in U$, implies $x \notin A$,but U is N-open then there exist N_x is open set in X such that $x \in N_x$,and $N_x - U$ is finite. $N_X \cap A = \emptyset$ this is a contradiction, $A \cap U \neq \emptyset$
Conversely , suppose $A \cap U \neq \emptyset$ for every non-empty N-open set of X , we shall show that A is dense . if not suppose $p \in X$ and $p \notin \bar{A}$, so $p \in X - \bar{A}$.
$X - \bar{A} \cap A = \emptyset$,
which is a contradiction to that fact $X - \bar{A}$ is open set in X,so $X - \bar{A}$ is N-open set in X. Hence $p \in \bar{A}$, therefore $X \subseteq \bar{A} \subseteq X$ then, $\bar{A} = X$(A is dense). □

3. N-Transitive Map

In this section we introduce the following new notion

Definition 3.1.

let X be a compact metric space and $f: X \to X$ a continuous map. The map f is said to be N-transitive if for all non empty N-open sets U, V there exists $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$. Clearly every transitive map is N-transitive but the converse is not true. The following results gives an equivalent condition for N-transitivity.

Theorem 3.2:

Let (X, f) be a dynamical system ,then f is topologically N- transitive if and only if for every non-empty N-open set U in X , $\bigcup_{n=0}^{\infty} f^n(U)$ is dense in X .

Proof : Assume $\bigcup_{n=0}^{\infty} f^n(U)$ is not dense .Then there exists a non-empty N-open set V such that $\bigcup_{n=0}^{\infty} f^n(U) \bigcap V = \phi$ .This implies $f^n(U) \bigcap V = \phi$ for all $n \in \mathrm{N}$. This is a contradiction to the N-transitivity of f . Hence $\bigcup_{n=0}^{\infty} f^n(U)$ is dense in X .
Now, let U and V be two non-empty N-opens sets in X . $\bigcup_{n=0}^{\infty} f^n(U)$ is dense in X so, by Theorem 2.6, we have $\bigcup_{n=0}^{\infty} f^n(U) \bigcap V \neq \phi$ .This implies there exists $m \in \mathrm{N}$ such that $f^m(U) \bigcap V \neq \phi$, hence f is N-transitive . □

Now, if (X, f) is dynamical system and f is N*-continuous then the following condition are equivalent.

Theorem3.3:

let (X, f) be a dynamical system and f is N*-continuous then the following are equivalent:
(i) f is N-transitive.
(ii) For every non-empty N-open set U in X , $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is dense in X .
(iii)If $E \subset X$ is N-closed and $f(E) \subset E$ then $E = X$ or E is N-nowhere dense.
(iv) If $U \subset X$ is N-open and $f^{-1}(U) \subset U$ then $U = \phi$ or U is dense in X

Proof:

(i)$\Rightarrow$(ii) Since f is N*-continuous and the family of N-open set is topological space ,thus $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is N-open and since f is N-transitive , it has to meet every N-open set in X and hence is dense .
(ii) $\Rightarrow$(i) Let U,V be N-open and non-empty sets in X . Then $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is dense in X . As a result $U \bigcap \bigcup_{n=0}^{\infty} f^{-n}(V) \neq \phi$. This implies $\exists m \in \mathrm{N}$ such that $U \bigcap f^{-m}(V) \neq \phi$. We further have $f^m(U \bigcap f^{-m}(V)) = f^m(U) \bigcap V \neq \phi$. Hence f is N-transitive .
(i) $\Rightarrow$(iii) f is N-transitive, $E \subset X$ is N-closed and $f(E) \subset E$. Assume that $E \neq X$

and E has a non-empty N-interior . Define $U = X \setminus E$ .Clearly U is N-open , since E is N-closed. Let $V \subset E$ be N-open since E has a non-empty N-interior .We have $f^n(V) \subset E$ since E is invariant. Then $f^n(V) \bigcap U = \phi$ for all $n \in \mathrm{N}$ .This is a contradiction to N-transitivity . Hence $E = X$ or E is N-nowhere dense .

(iii) $\Rightarrow$(i) Let U be non-empty N-open set in X . Suppose f is not N-transitive ,then from(ii) of this theorem , $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is not dense ,but $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is N-open . Define $E = X \setminus \bigcup_{n=0}^{\infty} f^{-n}(U)$ .Clearly E is N-closed and $E \neq X$ .Claim $f(E) \subset E$.

Suppose $f(E)$ is not a subset of E . This implies $f(E) \bigcap \bigcup_{n=0}^{\infty} f^{-n}(U) \neq \phi$. This further implies

$$f^{-1}\left[f(E) \bigcap \bigcup_{n=0}^{\infty} f^{-n}(U) \right] = E \bigcap \bigcup_{n=o}^{\infty} f^{-n}(U) \neq \phi$$

. This is contradiction to the definition of E , thus $f(E) \subset E$.

Since $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is not dense , there exists a non-empty N-open set V in X such that $\bigcup_{n=0}^{\infty} f^{-n}(U) \bigcap V = \phi$. This implies $V \subset E$,this is contradiction to the fact that E is N-nowhere dense . Hence f is N-transitive .

(i)$\Rightarrow$(iv) f is N-transitive ,$U \subset X$ is N-open and $f^{-1}(U) \subset U$ .Assume that $U \neq \phi$ and U is not dense in X . Then there exists a non-empty N-open set V in X such that $U \bigcap V = \phi$. Further $f^{-n}(U) \bigcap V = \phi$ for all $n \in \mathrm{N}$. This implies $U \bigcap f^n(V) = \phi$ for all $n \in \mathrm{N}$,a contradiction to N-transitivity of f . Hence $U = \phi$ or U is dense in X .

(iv)$\Rightarrow$(i) Suppose f is not N-transitive, for a non –empty N-open set U in X , let $W = \bigcup_{n=0}^{\infty} f^{-n}(U)$ is non-empty ,N-open and not dense. Clearly $f^{-1}(W) \subset W$, this is contradiction since $W \neq \phi$ is dense. This proves that f is N-transitive . □

We can introduce the new definition of non-wandering point.

Definition3.4:

A point $x \in X$ is called N- non-wandering if for every N-neighborhood $N(x)$ of x there is a positive integer n such that $f^n(N(x)) \cap N(x) \neq \emptyset$. The set of all N-non-wandering points of f will be denoted by $\Omega_N(f)$.

Topological transitivity and existence of a dense orbit are two equivalent definition for some space but is not true generally, [7, 3].In the following we will make a connection between the set of N-non wandering points , N-transitive and a dense orbit.

Proposition3.5:

Let $f : X \to X$ be a continuous map on compact metric space , f is N-transitive if and only if $\Omega_N(f) = X$,and f has a dense orbit.

Proof:

Suppose f is N- transitive, clearly has a dense orbit, i.e. ,there exits $x_\circ \in X$,such that $O_f(x_\circ)$ is dense in X, if $\Omega_N(f) \neq X$ then there exist a non-empty N-open subset U such that $\{f^n(U)\ n > 0\}$ are pairwise disjoint set. Since $O_f(x_\circ)$ is dense orbit, for some $n_\circ \geq 0, f^{n_\circ}(x_\circ) \in U$, $f^{n+n_\circ}(x_\circ) \in f^n(U),\ n \geq 0$,which contradiction with $f^n(U)$ is pairwise disjoint set. Therefore $\Omega_N(f) = X$.

Now, suppose f has a dense orbit and $\Omega_N(f) = X$,let U, V be two non-empty N-open subset of X . let $x \in X$ have a dense orbit, thus the orbit of x will enter both U and V. Let m and n be the least integers such that $f^m(x) \in U$ and $f^n(x) \in V$.

Assume $m < n$ and set $k = n - m$.
Then obviously $f^k(U) \cap V \neq \emptyset$. □

Let $f: A \to A$ and $g: B \to B$ be two maps ,f, g are said to be topologically conjugate, if there exists a homeomorphism $h: A \to B$ such that $hof = goh$, [2]. Mapping which are conjugate are completely equivalent in terms of their dynamics .Now we can prove the following lemma:

Lemma 3.6:
let f, g be two conjugate function , then if f is N-transitive the g is N-transitive also.

Proof: let $U\&V$ be two N-open subset of B, since h is continuous and one to one ,then h is N*-continuous (Theorem3.3), Thus $h^{-1}(U)$ and $h^{-1}(V)$ are N-open in A .

Since $f$is N-transitive then there exists $k > 0$ such that $f^k\left(h^{-1}(U)\right) \cap h^{-1}(V) \neq \emptyset$,i.e.,

$hof^k(h^{-1}(U)) \cap V \neq \emptyset$

$g^k oh(h^{-1}(U)) \cap V \neq \emptyset$

$g^k(U) \cap V \neq \emptyset$. Thus g is N-transitive function.

□

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