The Singularity of Bipartite Graph

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Abstract
A graph G is said to be singular if and only if its adjacency matrix is singular. A graph $G = (V, E)$ is said to be bipartite graph if and only if we can write its vertex set as $V(G) = V_1 \cup V_2$, and each edge has exactly one end point in V_1 and other end point in V_2. In this work, we will use graphic permutation to find the determinant of adjacency matrix of bipartite graph. After that, we will determine the conditions that the bipartite graph is singular or non-singular.

Keyword: adjacency matrix, bipartite graph, Singular Bipartite graph.

وحدانية البيان ثنائي التجزئة

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الخلاصة
يقال للبيان بيان وحيد اذا كانت مصفوفته المتجاورة وحيدة وكذلك يقال للبيان بيان ثنائي التجزئة اذا استطعنا كتابة مجموعة رؤوس البيان بشكل مجموعتين منفصلتين وأن كل ضلع من اضلاعه له رأس في المجموعة الاولى من الرؤوس البيان ورأس في المجموعة الثانية من الرؤوس البيان في هذا البحث سوف نستخدم البيان التبديلي لحساب المحدد للمصفوفة المتجاورة للبيان ثنائي التجزئة وبعد ذلك سوف نحدد الشروط لوحدانية او عدم وحدانية البيان ثنائي التجزئة.

1.Introduction

A graph G is said to be singular if and only if its adjacency matrix is singular; otherwise G is said to be non-singular [١]. In this paper, we will deal with a special type of graph is called a bipartite graph with no loops and multiple edges and it is define as ((a graph $G = (V, E)$ is said to be a bipartite graph if and only if we can write its vertex set as $V(G) = V_1 \cup V_2$, and each edge has exactly one end point in V_1 and other end point in V_2[٢]. The degree of vertex is the number of vertices adjacent to it, on the other words that, is the number of edges incident with it [٢]. Therefore, for each graph there is a matrix is called adjacency matrix. The adjacency matrix of a graph on n vertices is n by n matrix such that

$$a_{ij} = \begin{cases} 1 & if\ i\ is\ adjacent\ to\ j \\ 0 & otherwise \end{cases}$$

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We usually write the adjacency matrix of a graph G as $A(G)$[2]. So, the adjacency matrix of bipartite graph on $2n$ vertices is

$$A(G) = \begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix}$$

Where B is n by n matrix[3].

In this work, we will use the perfect matching to decide that there is a graphic permutation in a bipartite graph. Let $G = (V, E)$ is a bipartite graph on n vertices labelled as $1,2,\ldots,n$ with no loops and multiple edges and assume that

$$P = \begin{pmatrix} 1 & 2 \ldots & n \\ p(1) & p(2) \ldots & p(n) \end{pmatrix}$$

Be a permutation of the numbers $1,2,\ldots,n$ so p is graphic permutation of G if all vertex pairs $(i, p(i)), i = 1,2,\ldots n,$ are adjacent in G [3].Subsequently, we will use the graphic permutation to compute the determinant of adjacency matrix of bipartite graph. After that, we will determine the conditions that a bipartite graph is singular or non-singular. Therefore, we will use the following theorem to determine the singularity of bipartite graph.

Theorem 1.1[4, p.95]

A square matrix A is invertible (non-singular) if and only if $det\,(A) \neq 0$.

2. Matching

A matching M in a graph G is a set of edges of G that no two of which are incident with each other. Note, if a vertex is incident with an edge of matching, we say matched vertex (with respect to M), on the other words, the edges of M cover the vertex. Therefore, a matching which covers every vertex of a graph G is said to be a perfect matching [2].

Lemma 2.1 [2]

Suppose a bipartite graph G, with vertex partition $V(G) = V_1 \cup V_2$ has a perfect matching M ,then $|V_1| = |V_2| = |M|$.

Proof

Suppose G is a bipartite graph with vertex partition $V(G) = V_1 \cup V_2$ has a perfect matching M. So, each edge of M has exactly one end point in V_1 and other end point in V_2. The edges of M are covered every vertex of G (by the definition of perfect matching). Therefore, $|V_1| = |V_2| = |M|$. Since, the edges of M are not incident with each other. ∎

Definition 2.2 (The Hall's condition) [5]

A bipartite graph G with vertex partition $V(G) = V_1 \cup V_2$ is said to satisfy the hall condition if and only if for every $A \subseteq V_1$ we have defining $\Gamma(A) = \{v_2 \in V_2 : v_2 \sim a \text{ for some } a \in A\}$ such that $|\Gamma(A)| \geq |A|$.

Theorem 2.3 (Hall's theorem) [5]

A bipartite graph G with vertex partition $V(G) = V_1 \cup V_2$, has a perfect matching if and only if it satisfies the hall's condition.

Proof

Suppose that G has a perfect matching M and let $A \subseteq V_1$. Assume (for contradiction) that $|\Gamma(A)| < |A|$. Since, G has a perfect matching M, so the edges of M cover every vertex in G. Therefore, each vertex in V_1 has one neighbour in V_2 but $|\Gamma(A)| < |A|$, so there is some vertices in A has no neighbour (this is a contradiction) by the definition of perfect matching.

To prove the other direction, suppose H be a spanning subgraph of a graph G satisfies the Hall's condition with further property that any edge removed from H, it would not be longer satisfies the Hall's condition. Let $A \subseteq V_1$ and $A = \{a\}$, we get $d_H(a) \geq 1 \forall a \in V_1$. Therefore, it is enough to show that $d_H(a) = 1$. Assume (for contradiction) that, some $a \in A$ has two neighbours say b_1 and b_2. Let H_1 and H_2 are two subgraph of H and obtained from removing ab_1 and ab_2 respectively from H. So, they are both fail to satisfy the Hall's condition. Thus, $\forall i = 1,2$ there are $A_i \subseteq V_1$ such

that $|\Gamma_{H_i}(A)| < |A_i|$, both these sets must contains a. So
$|\Gamma_H(A_1 \cap A_2 \backslash \{a\})| \leq |\Gamma_{H_1}(A_1) \cap \Gamma_{H_2}(A_2)|$
Note, that if $v_2$is a neighbour of some vertex $u \in A_1 \cap A_2$other than a, the edge uv is still present in H_1 and H_2. Therefore,
$|\Gamma_{H_1}(A_1) \cap \Gamma_{H_2}(A_2)| =$
$|\Gamma_{H_1}(A_1)| + |\Gamma_{H_2}(A_2)| - |\Gamma_{H_1}(A_1) \cup \Gamma_{H_2}(A_2)|$
$= |\Gamma_{H_1}(A_1)| + |\Gamma_{H_2}(A_2)| - |\Gamma_H(A_1 \cup A_2)| \leq (|A_1| - 1) + (|A_1| - 1) - |A_1 \cup A_2|$
Note: $|\Gamma_{H_i}(A_i)| < |A_i|$
$=$
$|A_1| + |A_2| - |A_1 \cup A_2| - 2 = |A_1 \cap A_2| - 2 = |A_1 \cap A_2 \backslash \{a\}| - 1.$
Thus $|\Gamma_H(A_1 \cap A_2 \backslash \{a\})| \leq |A_1 \cap A_2 \backslash \{a\}| - 1.$
So, H does not satisfy the Hall's condition. ■

3.The determinant of adjacency matrix of bipartite graph
The determinant of a square matrix of order *n* is

$$\det(A) = \sum_p (-1)^{\pi(p)} \prod_{i=1}^{n} a_{ip(i)} \qquad (1)$$

where $(-1)^p$ is +1 for even permutation and -1 for odd permutations[3].

Lemma 3.1 [3]
For a bipartite graph with n vertices

$$\det(A) = (-1)^{\frac{n}{2}} \sum_p (-1)^{f(p)} \qquad (2)$$

Where A the adjacency matrix of bipartite graph is, $f(p)$ is the number of even cycles and divisible by 4 in the permutation graph of the bipartite graph.

Proof
From equation (1) $a_{ip(i)} = 1\ \forall i = 1,2,\ldots,n$ if and only if $(i, p(i))$ are adjacent in G (by the definition of Graphic permutation)
So, equation (1) becomes

$$det(A) = \sum_p (-1)^{\pi(p)} \qquad (3)$$

Note, $(p) \equiv e(p) mod\, 2$, when $e(p)$ is the number of even connected component (with the respect to the number of vertices) in the permutation graph.
Therefore, equation (3) becomes

$$det(A) = \sum_p (-1)^{e(p)} \qquad (4)$$

Let $f(p)$ is number of even cycles and it is divisible by 4. Thus,
$e(p) = \sum_{i\geq 1} c_{2i}(p)$
$f(p) = \sum_{i\geq 1} c_{4i}(p)$
Note, $c_i(p) = 0$ when *i* is odd for all a graphic permutation *p* of a bipartite graph G by lemma (a cycle is a bipartite graph if and only if it is of even length [2])
Therefore,

$$\sum_{i\geq 1} 2ic_{2i}(p) = n. \qquad (5)$$

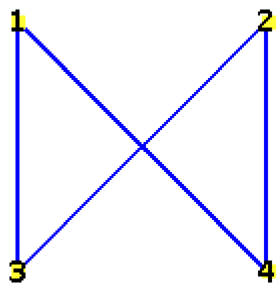
Then

$$\frac{n}{2} + f(p) = \sum_{i\geq 1} ic_{2i}(p) + \sum_{i\geq 1} c_{4i}(p)$$
$$= \left[\sum_{i\geq 1} 2ic_{4i}(p) + \sum_{i\geq 0} (2i+1)c_{4i+2}(p)\right] + \sum_{i\geq 1} c_{4i}(p)$$
$$= \sum_{i\geq 1} (2i+1)c_{4i}(p) + \sum_{i\geq 0} (2i+1)c_{4i+2}(p)$$
$$\equiv \left[\sum_{i\geq 1} c_{4i}(p) + \sum_{i\geq 0} c_{4i+2}(p)\right] (mod\ 2)$$
$$= e(p)(mod 2)$$

Then

$$\frac{n}{2} + f(p) \equiv e(p)(mod\ 2) \ ■$$

The proof of this theorem was found in [3] but the difference that *e (p)* is the number of even cycles while in this proof *e (p)* is the number of connected component. The reason of this difference that, if we take the bipartite graph G



Suppose, this permutation
1 2 3 4
3 4 1 2
So it is a graphic permutation of G with no cycle only single edges. So, equation (5) which is a count the number of vertices, $\sum_{i\geq 1} 2ic_{2i}(p) = 0$, if $e(p)$ is the number of even cycles. The second reason that, in this paper we deal with a bipartite graph with no loops and

multiple edges. So, $c_{2i}(p), i \geq 1$ is single edge when i=1 (not cycle)

Theorem 3.2[3]

The number of graphic permutation in a bipartite graph *G* is equal to m^2, where *m* is the number of perfect matching of *G*.

Corollary 3.3 [3]

A bipartite graph is contained a graphic permutation if and only if it possesses a perfect matching.

Note, theorem (3.1) deals with a bipartite graph with cycle. Therefore, if bipartite graph that is connected and has no cycle so it is a tree. Thus, we can find the determinant of adjacency matrix of tree by this theorem.

Theorem 3.4 [1]

let T be tree of *n* vertices .then

$$\det(A(T)) = \begin{cases} 1 & \text{if } T \text{ has a perfect matching} \\ & \text{and } n = 4k \text{ for som } k\epsilon z^+ \\ -1 & \text{if } T \text{ has a perfect matching} \\ & \text{and } n = 4k + 2 \text{ for some } k\epsilon z^+ \\ 0 & \text{Otherwise} \end{cases}$$

Thus, from above theorems, it is clear that a bipartite graph is singular if it has no perfect matching and it is non-singular otherwise.

Theorem 3.5

Let *G* be a bipartite graph with vertex partition $V(G) = V_1 \cup V_2$ is singular if and only if one of these three conditions is hold

1- *G* does not satisfy the Hall's condition.

2- $|V_1| \neq |V_2|, \forall x, y \in V(G)$

3- - $N(x) = N(y) \text{ where } x, y \in V(G)$

Note, $N(x)$ is the number of vertices adjacent to it.

Proof

Suppose *G* is a bipartite graph with vertex partition $V(G) = V_1 \cup V_2$ and it is singular so by (the definition of singular graph) its adjacency matrix is singular. So, by theorem (1.1) the determinant of its adjacency matrix is equal to zero. Therefore, by lemma (3.1) *G* has no Graphic permutation. Thus, by corollary (3.3) G does not possesses a perfect matching, so by (Hall's theorem) the first condition is hold and by Lemma (2.1) the second condition is also hold. Otherwise, if *G* has a perfect matching and the determinant of its adjacency matrix is equal to zero so, by the properties of the determinant of matrix there are at least two rows of the adjacency matrix are equal [6], so the third condition is satisfied

For the second direction, we will suppose that the first condition is hold. So, by Hall's theorem *G* has no perfect matching. Therefore by corollary (3.3), *G* does not contain a graphic permutation. Thus by lemma (3.1) the determinant of its adjacency matrix is equal to zero. Therefore by theorem (1.1) *G* is singular graph. ■

Since, from theorem (3.5), it is clear that a complete bipartite graph is singular because it satisfies the third condition of theorem (3.5).While, a regular bipartite graph is non-singular by theorem (3.5) if it does not satisfy the third condition.

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