Stability Analysis of A stage Structure Prey-Predator Model with Hollimg Type IV Functional Response

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Abstract
In this paper a stage structure prey-predator model with Hollimg type IV functional response is proposed and analyzed. The local stability analysis of the system is carried out. The occurrence of a simple Hopf bifurcation and local bifurcation are investigated. The global dynamics of the system is investigated with the help of the Lyapunov function. Finally, the analytical obtained results are supported with numerical simulation and the effects of parameters system are discussed. It is observed that, the system has either stable point or periodic dynamics.

Keywords: Holling type IV functional response, equilibrium points, stability, local bifurcation and Hopf bifurcation.

تحليل الاستقرارية لنظام الفريسة–المفترس ذي المراحل العمرية المركبةوالمتضمن دالة الاستجاب لهولنك من النوع الرابع

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الخلاصة
في هدا البحث تم اقتراح وتحليل نظام الفريسة –المفترس ذي المراحل العمرية المركبة مع دالة الاستجابة لهولنك ذات النوع الرابع. السلوك الديناميكي المحلي للنظام درس . ناقشنا امكانية حدوث كل من تفرع هوبف البسيط والتفرع المحلي، كما تم مناقشة السلوك الديناميكي الشامل بمساعدة دالة اليابانوف، واخيرا تم تدعيم النتائج التحليلية الناتجة باستخدام المحاكاة العددية كما تمت مناقشة تأثير معلمات النظام. لوحظ بان النظام يمتلك اما نقطة استقرار او ديناميكية دورية.

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1. Introduction.

Ecology relates to the study of living beings in relation to their living styles. Research in the area of the theoretical ecology was initiated by Lotka (1925) and by Volterra (1926). Since then many mathematicians and ecologists contributed to the growth of this area of knowledge. Consequently, several mathematical models deal with the dynamics of prey predator models involving different types of functional responses have been proposed and studied, see for example [1-7] and the references therein.

On the other hand, recently some mathematical models of stage structured population growth have appeared in the literature, in which the populations consist of immature and mature individuals. This seems reasonable for a number of mammals while they proceed from birth to death. Several predator–prey models based on age-structure of prey-predator models with or without delay are developed and studied by many authors, see for example [8–15] and the references therein. Recently, Naji et al [16] has been proposed and analyzed a stage structure prey-predator model with Beddington-DeAngelis type of functional response.

Keeping the above in view, in this paper a stage structure prey-predator model with Holling type-IV functional response have been proposed and studied.

2. The mathematical models.

In the following a Holling type IV prey-predator model with stage structure in predator is proposed and analyzed. Consequently the predator population is divided into two groups immature and mature. Further, it is assumed that only the mature predator is capable to consume the prey according to Holling type IV functional response and reproductive, while the immature predator does not attack prey and has no reproductive ability. The dynamics of such prey-predator model can be represented mathematically by the following system of differential equations:

$$\frac{dx}{dt} = x\left(a - bx - \frac{\alpha\gamma y_2}{x^2 + \gamma x + \gamma\beta}\right) = f_1(x, y_1, y_2)$$
$$\frac{dy_1}{dt} = -(d_1 + D)y_1 + \frac{e\alpha\gamma x y_2}{x^2 + \gamma x + \gamma\beta} = f_2(x, y_1, y_2) \quad (1)$$
$$\frac{dy_2}{dt} = D y_1 - d_2 y_2 = f_3(x, y_1, y_2)$$

Here x, y_1, y_2 represent the densities of the prey, immature and mature predators at time t respectively. Note that all the parameters of system (1) are assumed to be positive constants and can be described as following:

The parameter a represents the intrinsic growth rate of the prey in the absence of predator, while the parameter b is the strength of intra-specific competition among the prey species; the constants d_1, d_2 are the death rates of immature predator and the mature predator respectively ; the parameter β can be interpreted as the half-saturation constant: the parameter γ is a direct measure of the predator immunity from the prey; α is the maximum attack rate of the prey by a predator; e represents the conversion rate; and finally the constant D denotes to the rate at which an immature predator becomes mature predator.

Obviously, due to biological meaning of the variables x, y_1 and y_2, system (1) has the following domain $\Re_+^3 = \{(x, y_1, y_2) \in \Re^3 : x \geq 0, y_1 \geq 0, y_2 \geq 0\}$. Further the interaction functions in the right hand side of system (1) are continuous and have continuous partial derivatives on the state space $\Re_+^3$.

Hence they are Liptshizain. Therefore for any given initial values belong to $\Re_+^3$, system (1) has a unique solution.

In addition to the above all the solutions of system (1) with non-negative initial value are uniformly bounded as shown in the following theorem.

Theorem (1): All the solutions of system (1) which initiate in $\Re_+^3$ are uniformly bounded.

Proof: Let $(x(t), y_1(t), y_2(t))$ be any solution of the system (1) with non negative initial condition (x_0, y_{01}, y_{02}). From the first equation of system (1) we have that

$$\frac{dx}{dt} \leq x(a - bx) .$$

Then by solving this differential inequality we obtain that

$$x(t) \leq \frac{a x_0}{a e^{-at} + b x_0 - x_0 b e^{-at}}$$

Thus $x(t) \leq M$ where $M = \max\left\{\frac{a}{b}, x_0\right\}$ for all value of t. Define the function

$W(t)=x(t)+\frac{1}{e}y_1(t)+\frac{1}{e}y_2(t)$. So the time derivative of $W(t)$ along the solution curve of the system (1) can be written as:

$$\frac{dW}{dt}=\frac{dx}{dt}+\frac{1}{e}\frac{dy}{dt}+\frac{1}{e}\frac{dy_2}{dt}$$
$$\frac{dW}{dt}\le(a+1)x-d(x+\frac{1}{e}y_1+\frac{1}{e}y_2)$$
$$\frac{dW}{dt}+dW\le(a+1).M$$

Again by solving the above linear differential inequality we get

$$W(t)\le\frac{(a+1)}{d}M+W(0)e^{-dt}-\frac{(a+1)M}{d}e^{-dt}$$

Consequently, for $t\to\infty$ we have

$$W(t)\le\frac{(a+1)M}{d}$$

Hence all solutions of system (1) enter to the region

$$\Omega=\left\{(x(t),y_1(t),y_2(t))\in\Re_+^3:x(t)\le M \text{ and } W(t)\le\frac{(a+1)M}{d}\right\}$$

Which refer to uniformly bounded of them. ■

3. Existence and local stability analysis of system (1).

In this section, the existence and local stability analysis of all possible non –negative equilibrium points of system (1) are investigated. There are three non-negative equilibrium points of system (1) the existence and the stability analysis for each of them are given below:

(1) The trivial equilibrium point $E_0=(0,0,0)$ always exists.

(2) It is well known that, the prey population grows to the carrying capacity $\frac{a}{b}$ in the absence of predator, while the predator population dies in the absence of the prey, then the axial equilibrium point $E_1=\left(\frac{a}{b},0,0\right)$ always exists.

(3) There is no equilibrium point in the $y_1y_2-plane$ as the predator population dies in the absence of its prey.

(4) The positive equilibrium point $E_2=(x^*,y_1^*,y_2^*)$ exists in the $Int.\Re_+^3$ if there is a positive solution to the following set of algebraic equations:

$$\frac{f(x,y_1,y_2)}{x}=a-bx-\frac{\alpha\gamma y_2}{(x^2+\gamma x+\gamma\beta)}=0 \qquad (2a)$$

$$f_2(x,y_1,y_2)=-(d_1+D)y_1+\frac{e\alpha\gamma xy_2}{(x^2+\gamma x+\gamma\beta)}=0 \qquad (2b)$$

$$f_3(x,y_1,y_2)=Dy_1-d_2y_2=0 \qquad (2c)$$

From (2a) we have

$$y_2=\frac{(a-bx)(x^2+\gamma x+\gamma\beta)}{\alpha\gamma} \qquad (3)$$

Substituting in (2c) we get

$$y_1=d_2\left(\frac{(a-bx)(x^2+\gamma x+\gamma\beta}{\alpha\gamma D}\right) \qquad (4)$$

Now by substituting in (2b) we obtain the following third order polynomial equation.

$$B_3x^3+B_2x^2+B_1x+B_0=0 \qquad (5)$$

here

$$B_3=bd_2(d_1+D)\ >0$$
$$B_2=(-a+b\gamma)(d_1d_2+Dd_2)-be\alpha\gamma D$$
$$B_1=d_1d_2\gamma(-a+b\beta)+\gamma aD(e\alpha-d_2)+bDd_2\gamma\beta$$
$$B_0=-ad_2\gamma\beta(d_1+D)<0$$

Consequently, due to Descarte rule [17] Eq. (5) has a unique positive root, say x^*, provided that at least one of the following two conditions holds.

$$\begin{aligned}&d_2\gamma(d_1+D)>\frac{a}{b}d_2(d_1+D)+e\alpha\gamma D\\&\frac{b}{a}d_2\gamma\beta(d_1+D)+e\alpha\gamma D<d_2\gamma(d_1+D)\end{aligned} \qquad (6)$$

Therefore,

$E_2=(x^*,y_1^*,y_2^*), where\ y_1^*=y_1(x^*)$ and $y_2^*=y_2(x^*)$,exists uniquely in $Int\Re_+^3$ provided that condition (6) holds with $x^*<\frac{a}{b}$.

Now, in order to discuss the local dynamical behavior of system (1) near the above equilibrium points, the varitional matrix of system (1) at each of these points is computed and then the eigenvalues of the resulting varitional matrix are determined.

Assume that $J(x,y_1,y_2)$ denoted the varitional matrix at the point (x,y_1,y_2) which can be written as follows,

$$J(x,y_1,y_2)=\begin{pmatrix} x\frac{\partial g}{\partial x}+g & x\frac{\partial g}{\partial y_1} & x\frac{\partial g}{\partial y_2} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial y_2} \end{pmatrix} \quad \ldots (7)$$

where $g=\frac{f_1(x,y_1,y_2)}{x}$ and $f_i(x,y_1,y_2)\,; i=1,2,3$ are given in system (1) with

$\frac{\partial g}{\partial x}=-b+\frac{\alpha\gamma y_2(2x+\gamma)}{(x^2+\gamma x+\gamma\beta)^2}$, $\frac{\partial g}{\partial y_1}=0$,

$\frac{\partial g}{\partial y_2}=\frac{-\alpha\gamma}{(x^2+\gamma x+\gamma\beta)}$, $\frac{\partial f_2}{\partial x}=\frac{e\alpha\gamma y_2(\gamma\beta-x^2)}{(x^2+\gamma x+\gamma\beta)^2}$,

$\frac{\partial f_2}{\partial y_1}=-(d_1+D)$, $\frac{\partial f_2}{\partial y_2}=\frac{e\alpha\gamma x}{(x^2+\gamma x+\gamma\beta)}$, $\frac{\partial f_3}{\partial x}=0$,

$\frac{\partial f_3}{\partial y_1}=D$, $\frac{\partial f_3}{\partial y_2}=-d_2$.

Accordingly, by substituting the equilibrium points $E_i, i=0,1,2$ in (7) and then computing the eigenvalues, for $J(E_i), i=0,1,2$ respectively the following results are obtained.

The varitional matrix of system (1) at E_0 is

$$J(E_0)=\begin{pmatrix} a & 0 & 0 \\ 0 & -(d_1+D) & 0 \\ 0 & 0 & -d_2 \end{pmatrix}$$

Therefore, the eiganvalues of $J(E_0)$ are given by $\lambda_{0x}=a>0$, $\lambda_{0y_1}=-(d_1+D)<0$ and $\lambda_{0y_2}=-d_2<0$.

Here $\lambda_{0u}, u=x,y_1,y_2$ represent the eigenvalues of the varitional matrix at E_0 which describes the dynamics in the $u-direction$. Thus E_0 is a saddle point with locally stable manifold in the $y_1y_2-plane$ and with locally unstable manifold in the $x-direction$.

However the varitional matrix at the point $E_1=\left(\frac{a}{b},0,0\right)$ can be written as:

$$J(E_1)=\begin{pmatrix} -a & 0 & \frac{-\alpha\gamma ab}{(a^2+ab\gamma+b^2\gamma\beta)} \\ 0 & -(d_1+D) & \frac{e\alpha\gamma ab}{(a^2+ab\gamma+b^2\gamma\beta)} \\ 0 & D & -d_2 \end{pmatrix}$$

The eigenvalues of the $J(E_1)$ are given by

$$\lambda_{1x}=-a<0 \qquad (8)$$

$$\lambda_{1y_1}+\lambda_{1y_2}=-(d_1+d_2+D)<0 \qquad (9)$$

$$\lambda_{1y_1}\lambda_{1y_2}=d_2(d_1+D)-\frac{e\alpha\gamma abD}{(a^2+ab\gamma+b^2\gamma\beta)} \qquad (10)$$

Clearly E_1 is locally asymptotically stable provided that the following condition holds

$$d_2(d_1+D)>\frac{e\alpha\gamma D\hat{x}}{\hat{R}} \qquad (11)$$

where

$$\hat{R}=\hat{x}^2+\gamma\hat{x}+\gamma\beta.$$

However E_1 is a saddle point with locally stable manifold of dimension two and with locally unstable manifold of dimensions one if we reversed condition (11). Finally the varitional matrix at $E_2=(x^*,y_1^*,y_2^*)$ can be determined as follows.

$$J(x^*,y_1^*,y_2^*)=\begin{pmatrix} -bx^*+\frac{\alpha\gamma x^* y_2^*(2x^*+\gamma)}{R_*^2} & 0 & \frac{-\alpha\gamma x^*}{R_*} \\ \frac{e\alpha\gamma y_2^*(\gamma\beta-x^{*2})}{R_*^2} & -(d_1+D) & \frac{e\alpha\gamma x^*}{R_*} \\ 0 & D & -d_2 \end{pmatrix} = (a_{ij})_{3\times 3}$$

With $R_*=(x^*+\gamma x^*+\gamma\beta)$. Therefore the characteristic equation of $J(x^*,y_1^*,y_2^*)$ is

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0 \qquad (12)$$

where

$$A_1=-(a_{11}+a_{22}+a_{33})$$
$$A_2=a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}-a_{23}a_{32}$$
$$A_3=a_{11}(a_{23}a_{32}-a_{22}a_{33})-a_{13}a_{21}a_{32}$$

Consequently, it is easy to verify that $A_i>0$ for $i=1,3$ if the following conditions are satisfied:

$$\gamma\beta>x^{*2} \qquad (13)$$

$$e\alpha\gamma x^* D<(d_1+D)d_2R_* \qquad (14)$$

Moreover, we have

$$\begin{aligned}\Delta &= A_1A_2 - A_3\\ &= -a_{11}a_{22}(a_{11}+a_{22}) - a_{11}a_{33}(a_{11}+a_{33})\\ &\quad -2a_{11}a_{22}a_{33} - (a_{22}+a_{33})\big[a_{22}a_{33}\\ &\quad - a_{23}a_{32}\big] + a_{13}a_{21}a_{32}\end{aligned} \quad \dots (15)$$

Then $\Delta > 0$ if the following condition holds

$$b > \frac{\alpha\gamma y_2^*(2x^*+\gamma)}{R_*^2} + \frac{e\alpha^2\gamma^2 y_2^* D(\gamma\beta - x^{*^2})}{R_*^3} \quad \dots (16)$$

Therefore, due to Routh-Hurwitz criterion, E_2 is locally asymptotically stable in the $Int\Re_+^3$ provided that conditions (13), (14) and (16) hold.

4. Global dynamical behavior of system (1).

In this section the global dynamics of system (1) near the equilibrium points E_1 and E_2 are investigated with the help of Lyapunov function as shown below.

In the following theorem the global stability condition of $E_1 = (\hat{x},0,0)$ with $\hat{x} = \frac{a}{b}$ is established.

Theorem (2): Suppose that the equilibrium point $E_1 = (\hat{x},0,0)$ is locally asymptotically stable and let the following condition holds.

$$\frac{(d_1+D)}{De} d_2 > \frac{\alpha}{\beta}\hat{x} \qquad (17)$$

Then it is a globally asymptotically stable point.

Proof: Consider the following function about $(\hat{x},0,0)$

$$V(x,y_1,y_2) = c_1\left(x - \hat{x} - \hat{x}\ln\frac{x}{\hat{x}}\right) + c_2y_1 + c_3y_2$$

Where c_1, c_2 and c_3 are positive constants to be determined. Obviously $V:\Re_+^3 \to \Re$ such that $V(E_1)=0$ and $V(x,y_1,y_2)>0$ for all $(x,y_1,y_2)\in\Re_+^3$. and $(x,y_1,y_2)\neq E_1$. Hence V is a positive definite function.

$$\begin{aligned}\frac{dV}{dt} &= c_1\left(\frac{x-\hat{x}}{x}\right)\frac{dx}{dt} + c_2\frac{dy_1}{dt} + c_3\frac{dy_2}{dt}\\ &= -c_1b(x-\hat{x})^2 - \frac{\alpha\gamma x y_2}{\hat{R}}(c_1 - ec_2)\\ &\quad -(c_3d_2 - \frac{c_1\alpha\gamma\hat{x}}{\hat{R}})y_2 - (c_2(d_1+D) - c_3D)y_1\end{aligned}$$

Where $\hat{R} = \hat{x}^2 + \gamma\hat{x} + \gamma\beta$ and $\hat{x} = \frac{a}{b}$, then by Choosing $c_1 = 1$, $c_2 = \frac{1}{e}$, $c_3 = \frac{(D+d_1)}{De}$ give that

$$\frac{dV}{dt} = -c_1b(x-\hat{x})^2 - \left(\frac{(D+d_1)}{De}d_2 - \frac{\alpha\gamma\hat{x}}{\hat{R}}\right)y_2$$

Now if condition (17) holds then we get $\frac{dV}{dt} < 0$, hence V is Lyapunov function. Therefore, E_1 is a globally asymptotically stable in the $Int\Re_+^3$. ■

Theorem (3): Assume that $E_2 = (x^*, y_1^*, y_2^*)$ exists and is locally asymptotically stable in the $Int\Re_+^3$ then E_2 is a globally asymptotically stable provided that the following conditions are satisfied.

$$q_{12}^2 < q_{11}q_{22} \qquad (18a)$$

$$q_{13}^2 < q_{11}q_{33} \qquad (18b)$$

$$q_{23}{}^2 < q_{22}q_{33} \qquad (18c)$$

Where

$$q_{11} = \left[b - \frac{\alpha\gamma y_2^*}{RR_*}(\gamma + x + x^*)\right], \qquad q_{22} = \left(\frac{d_1+D}{y_1}\right),$$

$$q_{33} = \left(\frac{d_2}{y_2}\right), \quad q_{12} = \frac{e\alpha\gamma y_2^*(\gamma\beta - xx^*)}{y_1RR_*}, \quad q_{13} = -\left(\frac{\alpha\gamma}{R}\right),$$

$$q_{23} = \left[\frac{e\alpha\gamma x}{y_1R} + \frac{D}{y_2}\right]$$

With $R = x^2 + \gamma x + \gamma\beta$ and $R_* = x^{*^2} + \gamma x^* + \gamma\beta$.

Proof:

Consider the following positive definite function about $E_2 = (x^*, y_1^*, y_2^*)$

$$\begin{aligned}V_1(x,y_1,y_2) &= \left(x - x^* - x^*\ln\frac{x}{x^*}\right)\\ &\quad + \left(y_1 - y_1^* - y_1^*\ln\frac{y_1}{y_1^*}\right)\\ &\quad + \left(y_2 - y_2^* - y_2^*\ln\frac{y_2}{y_2^*}\right)\end{aligned}$$

Obviously $V_1:\Re_+^3 \to \Re$ such that $V_1(E_2)=0$ and $V_1(x,y_1,y_2)>0$ for all $(x,y_1,y_2)\in\Re_+^3$ and $(x,y_1,y_2)\neq E_2$. Hence V_1 is a positive definite function. Now, since

$$\frac{dV_1}{dt}=\left(\frac{x-x^*}{x}\right)\frac{dx}{dt}+\left(\frac{y_1-y_1^*}{y_1}\right)\frac{dy_1}{dt}+\left(\frac{y_2-y_2^*}{y_2}\right)\frac{dy_2}{dt}$$

Then by substituting the values of $\frac{dx}{dt},\frac{dy_1}{dt},\frac{dy_2}{dt}$ and then doing some mathematical manipulation we obtain that

$$\frac{dV_1}{dt}=-\left(b-\frac{\alpha\gamma.y_2^*}{RR_*}(\gamma+x+x^*)\right)(x-x^*)^2-\frac{\alpha\gamma}{R}(x-x^*)(y_2-y_2^*)$$
$$+\left(\frac{e\alpha\gamma.y_2^*}{y_1RR_*}(\gamma\beta-xx^*)\right)(x-x^*)(y_1-y_1^*)-\frac{(d_1+D)}{y_1}(y_1-y_1^*)^2$$
$$+\left(\frac{e\alpha\gamma x}{y_1R}+\frac{D}{y_2}\right)(y_1-y_1^*)(y_2-y_2^*)-\frac{d_2}{y_2}(y_2-y_2^*)^2$$

From which we get

$$\frac{dV_1}{dt}=-\frac{q_{11}}{2}u_1^2+q_{13}u_1u_3-\frac{q_{33}}{2}u_3^2$$
$$-\frac{q_{11}}{2}u_1^2+q_{12}u_1u_2-\frac{q_{22}}{2}u_2^2$$
So
$$-\frac{q_{22}}{2}u_2^2+q_{33}u_2u_3-\frac{q_{33}}{2}u_2^2$$

by using the conditions (18a)-(18c) we obtain that

$$\frac{dV_1}{dt}<-\tfrac{1}{2}\left[\sqrt{q_{11}}u_1-\sqrt{q_{33}}u_3\right]^2$$
$$-\tfrac{1}{2}\left[\sqrt{q_{11}}u_1-\sqrt{q_{22}}u_2\right]^2$$
$$-\tfrac{1}{2}\left[\sqrt{q_{22}}u_2-\sqrt{q_{33}}u_3\right]^2$$

Therefore, we have $\frac{dV_1}{dt}<0$, under the condition (18a) –(18c) and hence V_1 is a Lyapunov function. Therefore, E_2 is a globally asymptotically stable in the $Int\Re_+^3$. ■

5. The Bifurcation of system (1).

In this section the occurrence of a simple Hopf bifurcation and local bifurcation (such as saddle node, pitck fork and a transcritical bifurcation) near the equilibrium points of system (1) are investigated. In the following theorem, we shall establish the condition, at which a simple Hopf bifurcation occurs in the $Int.\Re_+^3$ for the system (1).

Theorem (4): Assume that the positive equilibrium point $E_2=(x^*,y_1^*,y_2^*)$ of system (1) exists and let that condition (13) along with the following condition

$$b>\frac{\alpha\gamma y_2^*(2x^*+\gamma)}{R_*^2} \qquad (19)$$

is satisfied then system (1) has a simple Hopf bifurcation at the point

$$e=e_*\equiv\frac{P(P_1^2-PP_1+P_2)}{\left(\frac{\alpha\gamma x^*D}{R_*}\left(P+\frac{\alpha\gamma y_2^*(\gamma\beta-x^{*2})}{R_*^2}\right)\right)}$$

where

$$P=(d_1+d_2+D)>0,\ P_1=-bx^*+\frac{\alpha\gamma x^*y_2^*(2x^*+\gamma)}{R_*^2}<0$$

$$P_2=(d_1+D)d_2>0,\qquad R_*=x^{*2}+\gamma x^*+\gamma\beta$$

Proof: According to the Liu (1994) [18], a simple Hopf-bifurcation occurs if and only if

$$A_1(e_*)>0\ ,\ A_3(e_*)>0\ ,\Delta(e_*)=0\ \text{and}\ \left.\frac{d\Delta}{de}\right|_{e=e_*}\neq 0$$

where e_* is a critical value of the key parameter and A_i for $i=1,3$ and Δ are given in equations (12) and (15) respectively. Now since, for

$$\Delta=(d_1+D)\left[-bx^*+\frac{\alpha\gamma x^*y_2^*(2x^*+\gamma)}{R_*^2}\right]^2$$
$$-(d_1+D)^2\left[-bx^*+\frac{\alpha\gamma x^*y_2^*(2x^*+\gamma)}{R_*^2}\right]$$

$$+\left[-bx^*+\frac{\alpha\gamma x^*y_2^*(2x^*+\gamma)}{R_*^2}\right]^2d_2-d_2^2\left[-bx^*+\frac{\alpha\gamma x^*y_2^*(2x^*+\gamma)}{R_*^2}\right]$$

$$-2\left[-bx^*+\frac{\alpha\gamma x^*y_2^*(2x^*+\gamma)}{R_*^2}\right](d_1+D)d_2+((d_1+D)+d_2)$$
$$\left[(d_1+D)d_2-\frac{e\alpha\gamma x^*}{R_*}D\right]-\frac{e\alpha^2\gamma^2x^*y_2^*(\gamma\beta-x^{*2})D}{R_*^3}.$$

and then we have

$$\Delta=(d_1+d_2+D)\left[-bx^*+\frac{\alpha\gamma x^* y_2^*(2x^*+\gamma)}{R_*^2}\right]^2$$
$$-\left((d_1+d_2+D)^2\right)\left[-bx^*+\frac{\alpha\gamma x^* y_2^*(2x^*+\gamma)}{R_*^2}\right]$$
$$+(d_1+d_2+D)\left[(d_1+D)d_2-\frac{e\alpha\gamma x^* D}{R_*}\right]$$
$$-\frac{e\alpha^2\gamma^2 x^* y_2^*(\gamma\beta-x^{*^2})D}{R_*^3}.$$

$$=P(P_1^2-PP_1+P_2)-e\left[\frac{\alpha\gamma x^* D}{R_*}\left(P+\frac{\alpha\gamma y_2^*(\gamma\beta-x^{*^2})}{R_*^2}\right)\right]$$

Clearly $\Delta(e_*)=0$. Now we have

$$A_1(e_*)=-\left[\left(-bx^*+\frac{\alpha\gamma x^* y_2^*(2x^*+\gamma)}{R_*^2}\right)+(-(d_1+D)+(-d_2)\right]$$

Obviously $A_1(e_*)>0$ under the condition (19) also we have

$$A_3(e_*)=P_1\left(\frac{e\alpha\gamma x^* D}{R_*}-P_2\right)+\frac{e\alpha^2\gamma^2 x^* y_2^*(\gamma\beta-x^{*^2})D}{R_*^3}$$
$$=-P_1P_2+\frac{\left[P\left(P_1^2-PP_1+P_2\right)\right]\left[P_1R_*^2+\alpha\gamma y_2^*(\gamma\beta-x^{*^2})\right]}{\left(PR_*^2+\alpha\gamma y_2^*(\gamma\beta-x^{*^2})\right)}$$

Clearly, $A_3(e_*)>0$ under the conditions (13) and (19). Further more,

$$\frac{d\Delta}{de}=\frac{\alpha\gamma x^* D}{R_*}\left(-P-\frac{\alpha\gamma y_2^*(\gamma\beta-x^{*^2})}{R_*^2}\right)$$
$$\left.\frac{d\Delta}{de}\right|_{e=e_*}=\frac{\alpha\gamma x^* D}{R_*}\left(-P-\frac{\alpha\gamma y_2^*(\gamma\beta-x^{*^2})}{R_*^2}\right)\neq 0$$

Thus, a simple Hopf-bifurcation occurs in system (1) at $e=e_*$. ■

Theorem (5): Assume that the parameter α passes through the value $\alpha^*=\dfrac{d_2(d_1+D)\hat{R}}{e\gamma D\hat{x}}$, then system (1) near the equilibrium point E_1 has:

1. No saddle-node bifurcation.
2. Atranscritical bifurcation but no Pitck-fork bifurcation can occur provided that the following condition holds:

$$\gamma\beta\neq\hat{x}^2 \qquad (20)$$

3. A Pitch-fork bifurcation otherwise.

where $\hat{R}=\hat{x}^2+\gamma\hat{x}+\gamma\beta$, $\hat{x}=\dfrac{a}{b}$ and $R=x^2+\gamma x+\gamma\beta$

Proof: According to the varitional matrix of system (1) at E_1, it is easy to verify that $J(E_1,\alpha^*)$ has the following eigenvalues:

$$\hat{\lambda}_1=a_{11}=-a<0$$
$$\hat{\lambda}_2=a_{22}+a_{33}=-\left((d_1+d_2+D\right)<0$$
$$\hat{\lambda}_3=0$$

Now, let $\hat{V}=\left[\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3\right]^T$ be the eigenvector corresponding to the $\hat{\lambda}_3=0$ for the $J(E_1,\alpha^*)$ thus $J(E_1,\alpha^*)\hat{V}=\hat{\lambda}_3\hat{V}$ and then

$$\begin{bmatrix}\hat{b}_{11}&0&\hat{b}_{13}\\0&\hat{b}_{22}&\hat{b}_{23}\\0&\hat{b}_{32}&\hat{b}_{33}\end{bmatrix}\begin{bmatrix}\hat{\theta}_1\\\hat{\theta}_2\\\hat{\theta}_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

From which we get that

$$\hat{V}=\left[\frac{-\hat{b}_{13}}{\hat{b}_{11}}\hat{\theta}_3,\frac{-\hat{b}_{23}}{\hat{b}_{22}}\hat{\theta}_3,\hat{\theta}_3\right]^T,$$

where $\hat{\theta}_3$ represents any non zero real number. Now, let $\hat{\Upsilon}=\left[\hat{h}_1,\hat{h}_2,\hat{h}_3\right]^T$ represents the eigenvector corresponding to the eigenvalue $\hat{\lambda}_3=0$ for $J^T(E_1,\alpha^*)$, so we obtain that

$$\hat{\Upsilon}=\left[0,\frac{-\hat{b}_{32}}{\hat{b}_{22}}\hat{h}_3,\hat{h}_3\right]^T,$$ Where $\hat{h}_3$ is any non zero real number. Now since

$$\frac{\partial F}{\partial\alpha}=F_\alpha(x,\alpha\ )=\left[\frac{-\gamma xy_2}{R},\frac{e\gamma xy_2}{R},0\right]^T$$

where $F=(f_1,f_2,f_3)^T$ and f_i ; i =1,2,3 represent the right hand side of system (1). Then we get

$$F_\alpha(E_1,\alpha^*)=\left[0,0,0\right]^T$$

Therefore ,

$$\hat{\Upsilon}^T\left[F_\alpha(E_1,\alpha^*)\right]=\left[0,\frac{-\hat{b}_{32}}{\hat{b}_{22}}\hat{h}3,\hat{h}_3\right]\left[0,0,0\right]^T=0$$

Thus the system (1) at E_1 does not experience any saddle-node bifurcation in view of Sotomayor theorem [19].

Also since

$$\hat{Y}^T\left[DF_\alpha(E_1,\alpha^*)\hat{V}\right]=\left[0,\frac{-\hat{b}_{32}}{\hat{b}_{22}}\hat{h}_3,\hat{h}_3\right]\begin{bmatrix}\dfrac{-\gamma\hat{x}}{\hat{R}}\hat{\theta}_3\\ \dfrac{e\gamma\hat{x}}{\hat{R}}\hat{\theta}_3\\ 0\end{bmatrix}$$

$$=\frac{-\hat{b}_{32}}{\hat{b}_{22}}\frac{e\gamma\hat{x}}{\hat{R}}\hat{\theta}_3\hat{h}_3\neq 0$$

Where $DF_\alpha(E_1,\alpha^*)=\frac{\partial}{\partial x}F_\alpha(x,\alpha)\Big|_{x=E_1,\alpha=\alpha^*}$ and $x=[x,y_1,y_2]^T$ Moreover

$$\hat{Y}^T\left[D^2F(E_1,\alpha^*)(\hat{V},\hat{V})\right]=\left(0,\frac{-\hat{b}_{32}\hat{h}_3}{\hat{b}_{22}},\hat{h}_3\right)$$

$$\begin{bmatrix}-2b\hat{\theta}_1^2-\left(\dfrac{R\alpha\gamma-\alpha\gamma x(2\hat{x}+\gamma)}{\hat{R}^2}\hat{\theta}_3\right)\hat{\theta}_1+\left(\dfrac{\alpha\gamma x(2\hat{x}+\gamma)}{\hat{R}}-\dfrac{\alpha\gamma}{\hat{R}}\right)\hat{\theta}_1\hat{\theta}_3\\ \left(\dfrac{e\alpha\gamma R-e\alpha\gamma x(2\hat{x}+\gamma)}{\hat{R}^2}\right)\hat{\theta}_1\hat{\theta}_3+\dfrac{e\alpha\gamma(\gamma\beta-\hat{x}^{2})}{\hat{R}}\hat{\theta}_1\hat{\theta}_3\\ 0\end{bmatrix}$$

$$=\left(\frac{2De\alpha^2\gamma^2\hat{x}(\gamma\beta-\hat{x}^2)}{a(d_1+D)\hat{R}^3}\right)\hat{\theta}_3^{\,2}\hat{h}_3$$

Where $D^2F(E_1,\alpha^*)=DJ(x,\alpha)\big|_{x=E_1,\alpha=\alpha^*}$. Clearly, $\hat{Y}^T\left[D^2F(E_1,\alpha^*)(\hat{V},\hat{V})\right]\neq 0$ provided that condition (20) holds, and then by Sotomoyor theorem [19], the system (1) possesses a transcritical bifurcation but not pitck fork bifurcation near E_1 where $\alpha=\alpha^*$. However , violate condition (20) gives that $\hat{Y}^T\left[D^2F(E_1,\alpha^*)(\hat{V},\hat{V})\right]=0$, and hence further computation shows

$$\hat{Y}^T\left[D^3F(E_1,\alpha^*)(\hat{V},\hat{V},\hat{V})\right]=\frac{4De^2\alpha^3\gamma^3\hat{x}^2}{a(d_1+D)^2\hat{R}^5}\left([3\hat{x}^2+\gamma(2\hat{x}^2-\beta\hat{x}-\gamma\beta)]\right)\hat{\theta}_3^{\,3}\hat{h}_3\neq 0$$

Therefore system (1) possesses a Pitch-fork bifurcation near E_1 where $\alpha=\alpha^*$. ■

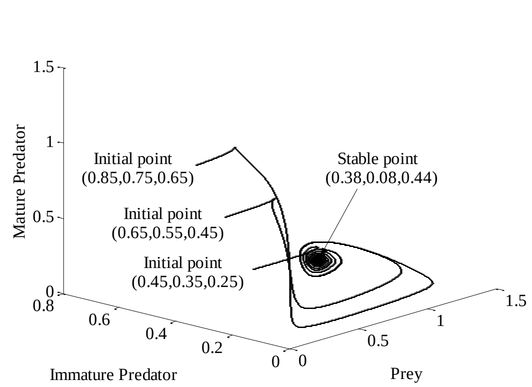
6. Numerical analysis.

In this section the global dynamics of system (1) is studied numerically. The system (1) is solved numerically for different sets of parameters and for different sets of initial condition, and then the attracting sets and their time series are drown as shown below. Now, for the following set of hypothetical parameters

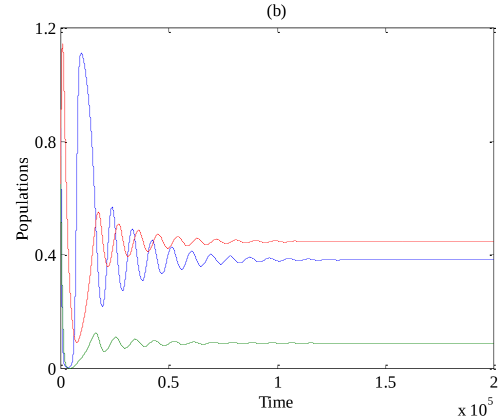
$$\begin{aligned}&a=0.25,\ b=0.2,\ \alpha=1,\ \gamma=0.75,\ \beta=2,\\ &d_1=0.01,\ D=0.25,\ e=0.35,\ d_2=0.05\end{aligned}\qquad(21)$$

The attracting sets along with their time series of system (1) are drown in Figure (1). Note that from now onward, in the time series figures, we will use the following representation: blue color represents the trajectory of the prey, green color represents the trajectory of immature predator and the red color represents the trajectory of the mature predator.

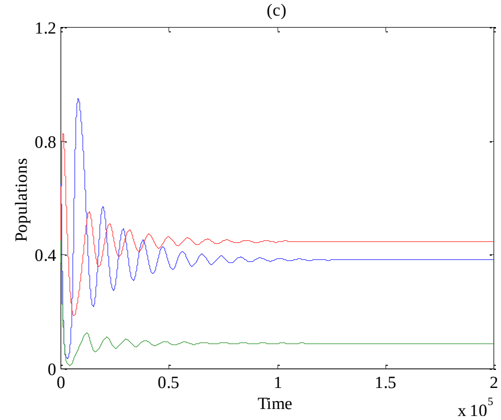
(a)



(b)



(c)



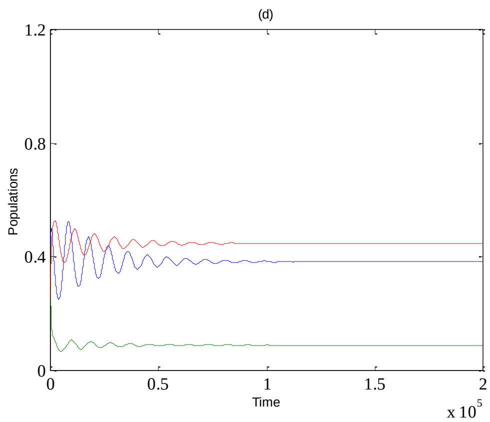
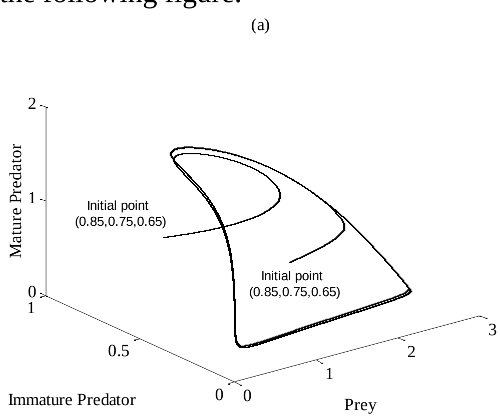


Figure 1- (a) The solution of system (1) approaches asymptotically to the positive equilibrium point starting from different initial values for the data given by Eq. (21). (b) Time series of the attractor in (a) starting at (0.85, 0.75, 0.65). (c) Time series of the attractor in (a) starting at (0.65, 0.55, 0.45). (d) Time series of the attractor in (a) starting at (0.45, 0.35, 0.25).

Clearly, as shown in Figure. (1), the system (1) has a globally stable positive equilibrium point $E_2 = (0.38, 0.08, 0.44)$ in the $Int.\Re^3_+$, hence all the species coexists and the system persists. However, for the parameters values given by Eq. (21) with the intrinsic growth rate $a = 0.5$, system (1) approaches to the periodic dynamics in the $Int.\Re^3_+$, see the following figure.



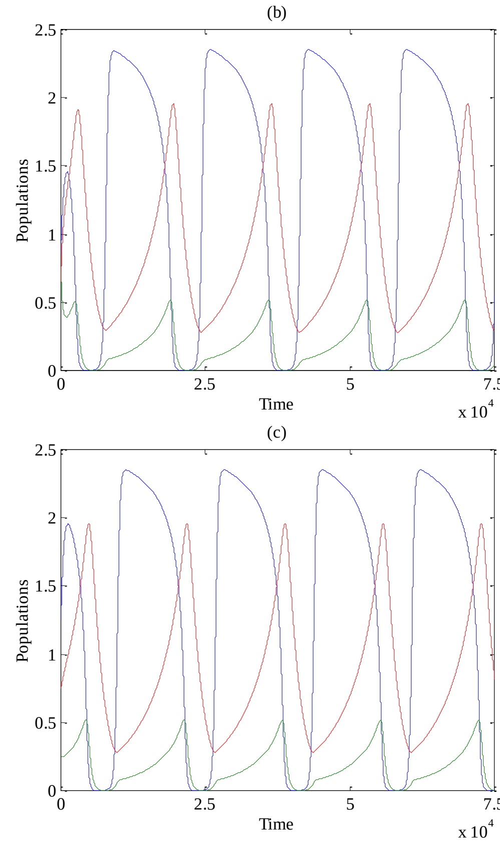


Figure 2- (a) Globally asymptotically stable limit cycle of system (1) starting from different initial values for the data given by Eq. (21) with $a = 0.5$. (b) Time series of the attractor in (a) starting at (0.85, 0.75, 0.65). (c) Time series of the attractor in (a) starting at (1.25, 0.25, 0.75).

Finally, it is observed that, for the parameters values given by Eq. (21) with the intrinsic growth rate in the ranges $a < 0.09$ and $a > 0.78$, system (1) approaches asymptotically to stable point $E_1 = (\frac{a}{b}, 0, 0)$, as shown in the following figure. Further analysis shows that, for the parameter $0.09 \le a \le 0.27$ with the rest of parameters as given in Eq. (21), system (1) has a globally asymptotically stable positive point, while for $0.27 \le a \le 0.78$ the system (1) approaches to periodic dynamic.

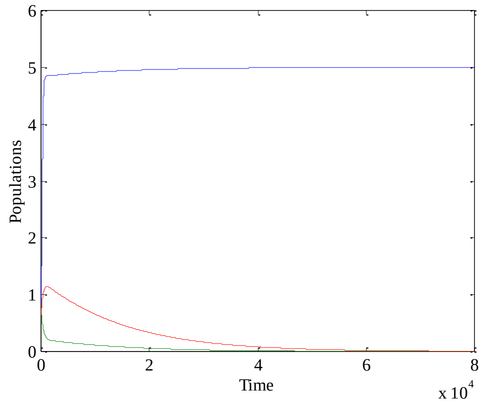
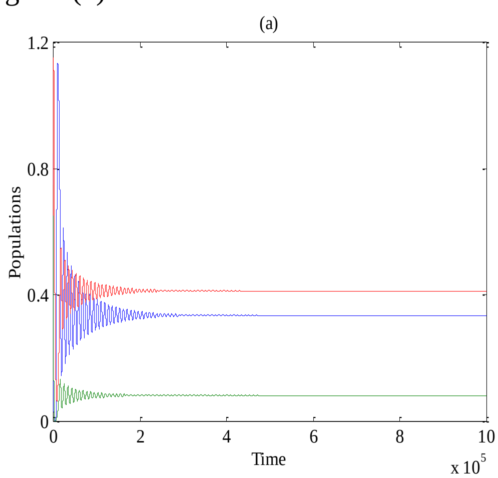
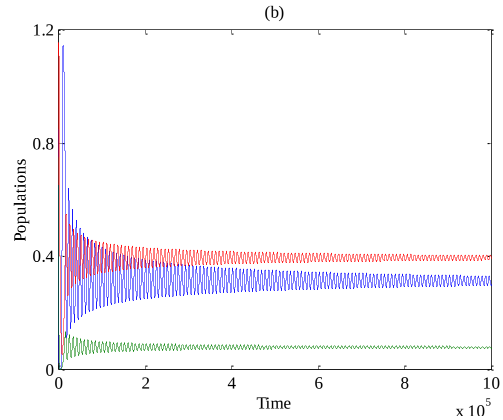


Figure 3-The trajectory of system (1) approaches asymptotically to stable point $E_1 = (5,0,0)$ for $a = 1$ with the rest of parameter as in Eq. (21).

Now, the effect of varying the maximum attack rate (the parameter α) on the dynamical behavior of system (1) that represented by parameter values given in Eq. (21) is studied. It is observed that, decreasing the parameter $\alpha < 0.7$ causes extinction in predator species and the system approaches to the axial equilibrium point $E_1 = (1.25,0,0)$ that means a transcritical bifurcation is occurred, while as the parameter α increases passing through $\alpha = 1.15$ the system transfers from stability at positive equilibrium point to the periodic dynamics in $Int\Re_+^3$ indicating to the occurrence of hopf bifurcation as shown in Figure. (4).





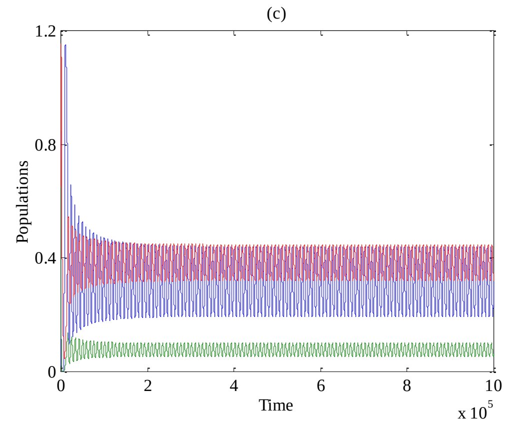
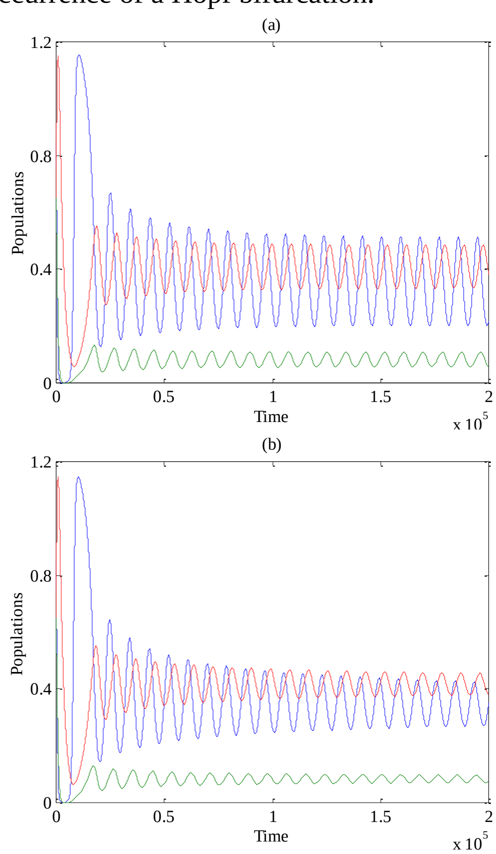


Figure 4-The trajectories of system (1) as a function of time at the data in Eq. (21). (a) Stable positive point $E_2 = (0.33,0.08,0.41)$ for $\alpha = 1.1$. (b) small periodic attractor for $\alpha = 1.16$. (c) Periodic attractor for $\alpha = 1.2$.

The effects of varying the predator immunity against the defensive of the prey (the parameter γ) on the dynamics of system (1) that represented by parameter values given in Eq. (21) is studied. It is observed that, decreasing the parameter $\gamma \le 0.3$ causes extinction in predator species and the system approaches to the axial equilibrium point $E_1 = (1.25,0,0)$ that means a transcritical bifurcation is occurred. Otherwise the system (1) still approaches to the positive equilibrium point. Similar observation as that happened in the parameter γ is obtained as varying the grown up rate of the predator that is means D, in fact the bifurcation occurred at $D < 0.02$.

On contrast to the effect of the above two parameters γ and D, it is observed that, increasing the natural death rate of immature

predator or that of mature predator (d_1 or d_2) causes extinction in predator species and then the solution of system (1) will approaches to axial equilibrium point $E_1 = (1.25,0,0)$.
Now the effects of changing the half saturation constant (the parameter β) on the dynamics of system (1) is also studied and the following result is obtained. Decreasing the value of the parameter $\beta \leq 1.83$ has destabilizing effects on the dynamics of system (1), in fact the system approaches to periodic dynamics in the $Int\Re_+^3$ instead of approaching to the positive equilibrium point indicating to occurrence of a Hopf bifurcation as shown in Fig. (5). Otherwise the system still coexists at the positive equilibrium point. On contrast the effect of the parameter β, it is observed that, increasing the conversion rate (the parameter e) has destabilizing effects on the dynamics of system (1) due to transfer of the dynamics of the system to periodic dynamics in the $Int\Re_+^3$ at the value $e = 0.5$, which means the occurrence of a Hopf bifurcation.



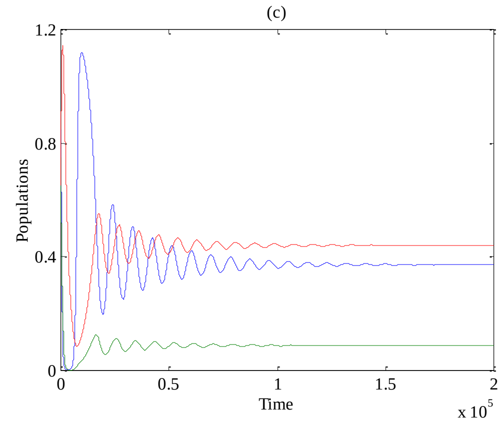


Figure 5-The trajectories of system (1) as a function of time at the data in Eq. (21). (a) Periodic attractor for $\beta = 1.75$. (b) Small periodic attractor for $\beta = 1.8$. (c) Stable positive point $E_2 = (0.37,0.08,0.44)$ for $\beta = 1.95$.

7. Discussion and conclusion:

In this chapter, a mathematical model consisting of a stage structure prey-predator model with Holling type IV functional response has been proposed and analyzed analytically as well as numerically. The local as well as global stability of the proposed system has been studied. The Hopf-bifurcation and the local bifurcation those may occur in system (1) are investigated.
The effect of varying each parameter on the dynamical behavior of system (1) is studied numerically at a set of hypothetical selected data and then the trajectories of the system are drowned. According to these figures the following conclusions are obtained.

1. The intrinsic growth rate of system (1) plays a vital role on the persistence of the system. In fact, for the small values and large values of the parameter a the predator facing extinction. However for suitable choice of this parameter, the system (1) still persists and has either stable point or else periodic dynamics.
2. Although, increasing the maximum attack rate of mature predator to prey keeps the system persists, it works as a destabilizing parameter on the system (1) due to the transferring of the system dynamics from stable point to periodic dynamics. In fact, this parameter has a transcritical

bifurcation and Hopf bifurcation at two different values.

3. Keeping the predator immunity rate against the defensive of the prey and the grown up rate of the predator suitably large will cause the persistence of the system (1) at the positive equilibrium point.
4. On contrast to the parameters in (21), keeping the natural death rates of the immature and mature predator suitably small will cause the persistence of the system (1) at the positive equilibrium point.
5. Although, the half saturation constant of the predator and the conversion rate of the predator represent the Hopf bifurcation parameters of the system (1), they have no effect on the persistence of the system. In fact, decreasing the parameter $\beta \leq 1.83$ or increasing the parameter $e \geq 0.5$ cause destabilizing of the system due to transferring from stability at positive equilibrium point to periodic dynamics.

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